Dirac index, translation principle and associated cycles I

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S. Mehdi, P. Pandžić, D. Vogan, R. Zierau, *Dirac Index and associated cycles of Harish-Chandra modules*, Adv. Math. **361** (2020), 106917, 34 pp.

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We denote by G and K the complexifications of $G_{\mathbb{R}}$ and $K_{\mathbb{R}}$. They will be important later.

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To study algebraic properties of representations, it is convenient to introduce their algebraic analogs, $(\mathfrak{g}, K_{\mathbb{R}})$-modules (or equivalently, $(\mathfrak{g}, K)$-modules).

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$\mathfrak{g} = (\mathfrak{g}_0)_{\mathbb{C}}$ also acts, and V_K is a $(\mathfrak{g}, K_{\mathbb{R}})$-module, or a $(\mathfrak{g}, K)$-module.

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(If K is disconnected, require also that the action $\mathfrak{g} \otimes V \to V$ is K-equivariant).

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Such M can be decomposed under K as

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Equivalently, the $(\mathfrak{g}, K)$-module M has finite length.

Dirac operator

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Let $C(\mathfrak{p})$ be the Clifford algebra of $\mathfrak{p}$ with respect to B:

the associative algebra with 1, generated by $\mathfrak{p}$, with relations

$$xy + yx + 2B(x, y) = 0.$$

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D^2 is the "spin Laplacean" (Parthasarathy):

$$D^2 = -\operatorname{Cas}_{\mathfrak{g}} \otimes 1 + \operatorname{Cas}_{\mathfrak{k}_\Delta} + \|\rho_{\mathfrak{k}}\|^2 - \|\rho_{\mathfrak{g}}\|^2.$$

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$\mathfrak{k}_\Delta$ is the diagonal copy of $\mathfrak{k}$ in $U(\mathfrak{g}) \otimes C(\mathfrak{p})$,
defined by $\mathfrak{k} \hookrightarrow U(\mathfrak{g})$ and $\mathfrak{k} \to \mathfrak{so}(\mathfrak{p}) \hookrightarrow C(\mathfrak{p})$.

Dirac cohomology

Let M be an admissible $(\mathfrak{g}, K)$-module. Let S be a spin module for $C(\mathfrak{p})$; it is constructed as $S = \bigwedge \mathfrak{p}^+$ for $\mathfrak{p}^+ \subset \mathfrak{p}$ max isotropic.

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$H_D(M)$ is a module for the spin double cover $\widetilde{K}$ of K. It is finite-dimensional if M is of finite length.

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where the bottom arrow is defined by the adjoint action, and the right vertical arrow is the usual double covering map.

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If $H_D(M) \neq 0$, the infinitesimal character of M can be read off from $H_D(M)$:

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Then the infinitesimal character of M is $\gamma + \rho_{\mathfrak{k}}$ up to Weyl group $W_{\mathfrak{g}}$.

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3. Study various applications.

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- also fd modules - Kostant, Huang-Kang-P.

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- H_D is related to classical topics like generalized Weyl character formula, generalized Bott-Borel-Weil Theorem, construction of discrete series, multiplicities of automorphic forms
- H_D is related to $\mathfrak{n}$-cohomology in some cases (Huang-P.-Renard)

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- can construct reps with $H_D \neq 0$ via "algebraic Dirac induction" (P.-Renard; Prlić)
- H_D is related to K-characters and branching problems (Huang-P.-Zhu)

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- Barbasch-Ciubotaru-Trapa: graded affine Hecke algebras and p-adic groups.

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To define the Dirac index, assume $\operatorname{rank}\mathfrak{g} = \operatorname{rank}\mathfrak{k}$. Then $\dim\mathfrak{p}$ is even, so the $C(\mathfrak{p})$-module S is graded:

$$S = S^+ \oplus S^- \quad (= \textstyle\bigwedge^{\mathrm{even}}\mathfrak{p}^+ \oplus \bigwedge^{\mathrm{odd}}\mathfrak{p}^+).$$

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The additivity of I works on modules with infinitesimal character:

Proposition

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Then there is an equality of virtual $\widetilde{K}$-modules

$$X \otimes S^{+} - X \otimes S^{-} = I(X).$$

Proof

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for any eigenvalue λ of D^2.

Since D commutes with D^2, it preserves each eigenspace.

Proof - continued

Since D also switches parity, we see that D defines maps

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Since D is a differential on $\operatorname{Ker} D^2$, and the cohomology of this differential is exactly $H_D(M)$, the statement now follows from the Euler-Poincaré principle. $\square$

Corollary

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be a short exact sequence of finite length $(\mathfrak{g}, K)$-modules.

Assume that V has infinitesimal character (so that U and W must have the same infinitesimal character as V).

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$$I(V) = I(U) + I(W).$$

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Namely, the left hand side of that formula clearly satisfies the additivity property. $\square$

Modules with generalized infinitesimal character

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Therefore, the above proposition and corollary are not sufficient for our purposes, because they apply only to modules with infinitesimal character.

Namely, if M is of finite length and has infinitesimal character, then $M \otimes F$ is of finite length, but it typically cannot be written as a direct sum of modules with infinitesimal character.

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Recall that $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ is the generalized infinitesimal character of a $\mathfrak{g}$-module M if there is a positive integer N such that

$$(z - \chi(z))^N = 0 \quad \text{on } M, \qquad \text{for every } z \in Z(\mathfrak{g}),$$

where $Z(\mathfrak{g})$ denotes the center of $U(\mathfrak{g})$.

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Here is an example (taken from P.-Somberg) showing that the above proposition and corollary can fail for modules with generalized infinitesimal character.

Example

Let $G_{\mathbb{R}} = SU(1,1) \cong SL(2,\mathbb{R})$, so that
$K_{\mathbb{R}} = S(U(1) \times U(1)) \cong U(1)$, and $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$.

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Then there is an indecomposable $(\mathfrak{g}, K)$-module P fitting into the short exact sequence

$$0 \to V_0 \to P \to V_{-2} \to 0, \tag{1}$$

where V_0 is the (reducible) Verma module with highest weight 0, and V_{-2} is the (irreducible) Verma module with highest weight -2.

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v_0 \arrow[d, bend right, "1"'] & \\
v_{-2} \arrow[u, bend right, "0"'] \arrow[d, bend right, "2"'] & w_{-2} \arrow[ul, bend right, "1"'] \arrow[d, bend right, "2"'] \\
v_{-4} \arrow[u, bend right, "-1"'] \arrow[d, bend right, "3"'] & w_{-4} \arrow[ul, bend right, "\frac{1}{2}"'] \arrow[u, bend right, "-1"'] \arrow[d, bend right, "3"'] \\
v_{-6} \arrow[u, bend right, "-2"'] & w_{-6} \arrow[ul, bend right, "\frac{1}{3}"'] \arrow[u, bend right, "-2"'] \\
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In this picture, the action of e is represented by upward arrows, the action of f by downward arrows, and the numbers by the arrows represent the coefficients in the action computed in the basis.

Example

It is easy to check that this indeed defines an $\mathfrak{sl}(2, \mathbb{C})$-module. (The only thing that needs to be checked is $ef - fe = h$ on each basis vector, and that is seen by a straightforward computation using the above formulas for the action.)

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It is moreover clear that the v_k span a submodule isomorphic to V_0, and that the quotient is isomorphic to V_{-2}, spanned by the classes of the w_ks. In other words, P indeed fits into the short exact sequence (1).

Example

One checks by a direct calculation that for the index defined by $I(M) = H_D(M)^+ - H_D(M)^-$ the following holds:

$$I(P) = -\mathbb{C}_1; \qquad I(V_0) = -\mathbb{C}_1; \qquad I(V_{-2}) = -\mathbb{C}_{-1},$$

where $\mathbb{C}_1$ respectively $\mathbb{C}_{-1}$ is the one-dimensional $\widetilde{K}$-module of weight 1 respectively -1.

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So the above corollary fails for P. It follows that the above proposition must also fail.

Higher Dirac cohomology

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It is defined as $H(M) = \bigoplus_{k\in\mathbb{Z}_+} H^k(M)$, where

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(Note that $H^k(M)$ consists of the bottom $\widetilde{K}$-types of the Jordan cells for D of size $2k+1$.)

It is easy to see that for M with infinitesimal character, $H(M) = H^0(M)$ and it is equal to the old notion $H_D(M)$.

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Let $H(M)^{\pm}$ be the even and odd parts of $H(M)$, and (re)define the Dirac index as

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In other words, I is well defined on the Grothendieck group of the Abelian category of finite length $(\mathfrak{g}, K)$-modules.

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In any case, from now on we work with Dirac index $I(M)$, defined for any virtual $(\mathfrak{g}, K)$-module M, and satisfying $I(M) = M \otimes S^+ - M \otimes S^-$.

Dirac index and tensoring

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This is clear from $I(X) = X \otimes S^+ - X \otimes S^-$ applied to $X = M \otimes F$ and $X = M$.

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A family of virtual $(\mathfrak{g}, K)$-modules X_λ is called coherent if

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2. For any finite-dimensional $(\mathfrak{g}, K)$-module F, and for any $\lambda \in \Lambda$,

   $$X_\lambda \otimes F = \sum_{\mu \in \Delta(F)} X_{\lambda+\mu},$$

   where $\Delta(F)$ denotes the multiset of all weights of F.

Theorem

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Let

$$I(X_\lambda) = \sum_{w \in W_\mathfrak{g}} a_w \tilde{E}_{w\lambda},$$

where $\tilde{E}$ denotes the coherent family of finite-dimensional $\widetilde{K}$-modules as above, and a_w are integers.

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where $\tilde{E}$ denotes the coherent family of finite-dimensional $\widetilde{K}$-modules as above, and a_w are integers.

Then for any weight ν,

$$I(X_{\lambda+\nu}) = \sum_{w \in W_\mathfrak{g}} a_w \tilde{E}_{w(\lambda+\nu)},$$

with the same coefficients a_w.