

Dirac index, translation principle and associated cycles I

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S. Mehdi, P. Pandžić, D. Vogan, R. Zierau, *Dirac Index and associated cycles of Harish-Chandra modules*, Adv. Math. **361** (2020), 106917, 34 pp.

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We denote by G and K the complexifications of $G_{\mathbb{R}}$ and $K_{\mathbb{R}}$. They will be important later.

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To study algebraic properties of representations, it is convenient to introduce their algebraic analogs, $(\mathfrak{g}, K_{\mathbb{R}})$ -modules (or equivalently, (\mathfrak{g}, K) -modules).

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 $\dim \operatorname{Hom}_{K_{\mathbb{R}}}(V_{\delta}, V) < \infty$ for all irreducible $K_{\mathbb{R}}$ -representations V_{δ} .

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$\mathfrak{g} = (\mathfrak{g}_0)_{\mathbb{C}}$ also acts, and V_K is a $(\mathfrak{g}, K_{\mathbb{R}})$ -module, or a
 (\mathfrak{g}, K) -module.

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(If K is disconnected, require also that the action $\mathfrak{g} \otimes V \rightarrow V$ is K -equivariant).

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Equivalently, the (\mathfrak{g}, K) -module M has finite length.

Dirac operator

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Let $C(\mathfrak{p})$ be the Clifford algebra of \mathfrak{p} with respect to B :

the associative algebra with 1, generated by \mathfrak{p} , with relations

$$xy + yx + 2B(x, y) = 0.$$

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D^2 is the “spin Laplacean” (Parthasarathy):

$$D^2 = -\text{Cas}_{\mathfrak{g}} \otimes 1 + \text{Cas}_{\mathfrak{k}_{\Delta}} + \|\rho_{\mathfrak{k}}\|^2 - \|\rho_{\mathfrak{g}}\|^2.$$

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\mathfrak{k}_{Δ} is the diagonal copy of \mathfrak{k} in $U(\mathfrak{g}) \otimes C(\mathfrak{p})$,
defined by $\mathfrak{k} \hookrightarrow U(\mathfrak{g})$ and $\mathfrak{k} \rightarrow \mathfrak{so}(\mathfrak{p}) \hookrightarrow C(\mathfrak{p})$.

Dirac cohomology

Let M be an admissible (\mathfrak{g}, K) -module. Let S be a spin module for $C(\mathfrak{p})$; it is constructed as $S = \bigwedge \mathfrak{p}^+$ for $\mathfrak{p}^+ \subset \mathfrak{p}$ max isotropic.

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$H_D(M)$ is a module for the spin double cover \tilde{K} of K . It is finite-dimensional if M is of finite length.

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where the bottom arrow is defined by the adjoint action, and the right vertical arrow is the usual double covering map.

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and $D^2 \geq 0$ (Parthasarathy's Dirac inequality).

If $H_D(M) \neq 0$, the infinitesimal character of M can be read off from $H_D(M)$:

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Then the infinitesimal character of M is $\gamma + \rho_{\mathfrak{t}}$ up to Weyl group $W_{\mathfrak{g}}$.

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3. Study various applications.

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- ▶ also fd modules - Kostant, Huang-Kang-P.

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- ▶ H_D is related to \mathfrak{n} -cohomology in some cases (Huang-P.-Renard)

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- ▶ can construct reps with $H_D \neq 0$ via “algebraic Dirac induction” (P.-Renard; Prlić)
- ▶ H_D is related to K -characters and branching problems (Huang-P.-Zhu)

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- ▶ Barbasch-Ciubotaru-Trapa: graded affine Hecke algebras and p -adic groups.

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To define the Dirac index, assume $\text{rank } \mathfrak{g} = \text{rank } \mathfrak{k}$. Then $\dim \mathfrak{p}$ is even, so the $C(\mathfrak{p})$ -module S is graded:

$$S = S^+ \oplus S^- \quad (= \bigwedge^{\text{even}} \mathfrak{p}^+ \oplus \bigwedge^{\text{odd}} \mathfrak{p}^+).$$

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The additivity of I works on modules with infinitesimal character:

Proposition

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Then there is an equality of virtual \tilde{K} -modules

$$X \otimes S^+ - X \otimes S^- = I(X).$$

Proof

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Since D commutes with D^2 , it preserves each eigenspace.

Proof - continued

Since D also switches parity, we see that D defines maps

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Since D is a differential on $\text{Ker } D^2$, and the cohomology of this differential is exactly $H_D(M)$, the statement now follows from the Euler-Poincaré principle. □

Corollary

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Assume that V has infinitesimal character (so that U and W must have the same infinitesimal character as V).

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be a short exact sequence of finite length (\mathfrak{g}, K) -modules.

Assume that V has infinitesimal character (so that U and W must have the same infinitesimal character as V).

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$$I(V) = I(U) + I(W).$$

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Namely, the left hand side of that formula clearly satisfies the additivity property. □

Modules with generalized infinitesimal character

To study the translation principle, we need to deal with modules $M \otimes F$, where M is a finite length (\mathfrak{g}, K) -module, and F is a finite-dimensional (\mathfrak{g}, K) -module.

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Therefore, the above proposition and corollary are not sufficient for our purposes, because they apply only to modules with infinitesimal character.

Namely, if M is of finite length and has infinitesimal character, then $M \otimes F$ is of finite length, but it typically cannot be written as a direct sum of modules with infinitesimal character.

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Recall that $\chi : Z(\mathfrak{g}) \rightarrow \mathbb{C}$ is the generalized infinitesimal character of a \mathfrak{g} -module M if there is a positive integer N such that

$$(z - \chi(z))^N = 0 \quad \text{on } M, \quad \text{for every } z \in Z(\mathfrak{g}),$$

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Here is an example (taken from P.-Somberg) showing that the above proposition and corollary can fail for modules with generalized infinitesimal character.

Example

Let $G_{\mathbb{R}} = SU(1, 1) \cong SL(2, \mathbb{R})$, so that
 $K_{\mathbb{R}} = S(U(1) \times U(1)) \cong U(1)$, and $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$.

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Then there is an indecomposable (\mathfrak{g}, K) -module P fitting into the short exact sequence

$$0 \rightarrow V_0 \rightarrow P \rightarrow V_{-2} \rightarrow 0, \quad (1)$$

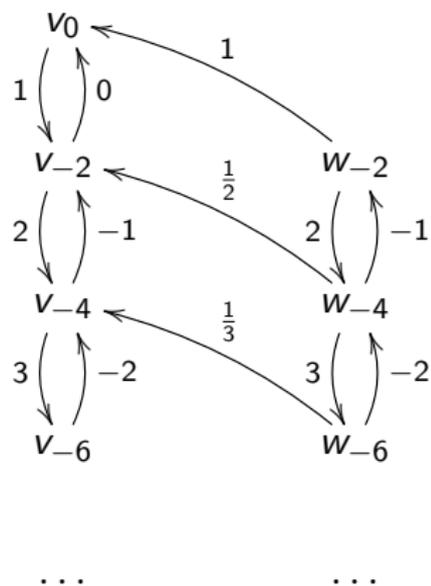
where V_0 is the (reducible) Verma module with highest weight 0, and V_{-2} is the (irreducible) Verma module with highest weight -2.

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Here is a picture describing the module P :

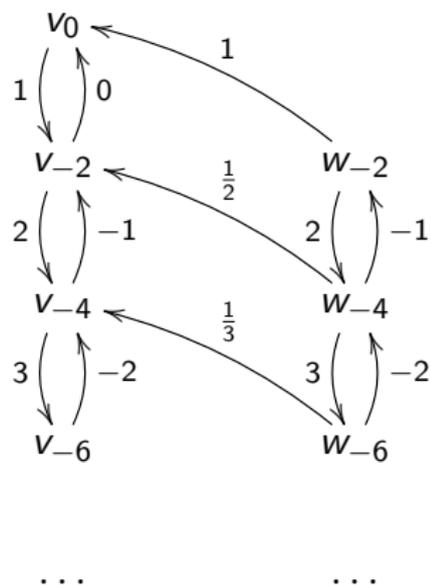
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In this picture, the action of e is represented by upward arrows, the action of f by downward arrows, and the numbers by the arrows represent the coefficients in the action computed in the basis.

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It is easy to check that this indeed defines an $\mathfrak{sl}(2, \mathbb{C})$ -module. (The only thing that needs to be checked is $ef - fe = h$ on each basis vector, and that is seen by a straightforward computation using the above formulas for the action.)

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It is moreover clear that the v_k span a submodule isomorphic to V_0 , and that the quotient is isomorphic to V_{-2} , spanned by the classes of the w_k . In other words, P indeed fits into the short exact sequence (1).

Example

One checks by a direct calculation that for the index defined by $I(M) = H_D(M)^+ - H_D(M)^-$ the following holds:

$$I(P) = -\mathbb{C}_1; \quad I(V_0) = -\mathbb{C}_1; \quad I(V_{-2}) = -\mathbb{C}_{-1},$$

where \mathbb{C}_1 respectively \mathbb{C}_{-1} is the one-dimensional \tilde{K} -module of weight 1 respectively -1 .

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So the above corollary fails for P . It follows that the above proposition must also fail.

Higher Dirac cohomology

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It is defined as $H(M) = \bigoplus_{k \in \mathbb{Z}_+} H^k(M)$, where

$$H^k(M) = \text{Im } D^{2k} \cap \text{Ker } D / \text{Im } D^{2k+1} \cap \text{Ker } D.$$

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(Note that $H^k(M)$ consists of the bottom \tilde{K} -types of the Jordan cells for D of size $2k + 1$.)

It is easy to see that for M with infinitesimal character, $H(M) = H^0(M)$ and it is equal to the old notion $H_D(M)$.

Higher Dirac cohomology

Let $H(M)^\pm$ be the even and odd parts of $H(M)$, and (re)define the Dirac index as

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In other words, I is well defined on the Grothendieck group of the Abelian category of finite length (\mathfrak{g}, K) -modules.

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Another way to deal with the above mentioned problem is to define the index $I(M)$ as $M \otimes S^+ - M \otimes S^-$. (Then one has to prove the finiteness separately.)

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In any case, from now on we work with Dirac index $I(M)$, defined for any virtual (\mathfrak{g}, K) -module M , and satisfying $I(M) = M \otimes S^+ - M \otimes S^-$.

Dirac index and tensoring

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This is clear from $I(X) = X \otimes S^+ - X \otimes S^-$ applied to $X = M \otimes F$ and $X = M$.

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A family of virtual (\mathfrak{g}, K) -modules X_λ is called coherent if

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2. For any finite-dimensional (\mathfrak{g}, K) -module F , and for any $\lambda \in \Lambda$,

$$X_\lambda \otimes F = \sum_{\mu \in \Delta(F)} X_{\lambda+\mu},$$

where $\Delta(F)$ denotes the multiset of all weights of F .

Theorem

Let X_λ be a coherent family of virtual (\mathfrak{g}, K) -modules, with all λ analytically integral for \tilde{K} . Fix a regular $\lambda \in \Lambda$.

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$$I(X_\lambda) = \sum_{w \in W_{\mathfrak{g}}} a_w \tilde{E}_{w\lambda},$$

where \tilde{E} denotes the coherent family of finite-dimensional \tilde{K} -modules as above, and a_w are integers.

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Then for any weight ν ,

$$I(X_{\lambda+\nu}) = \sum_{w \in W_{\mathfrak{g}}} a_w \tilde{E}_{w(\lambda+\nu)},$$

with the same coefficients a_w .