Segal-Shale-Weil Representations and Universal Fourier Transfroms

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Descartes 笛卡尔

« La Géométrie » (1637)

Fourier 傅里叶

« Théorie analytique de la cheleur » (1822)

Stone - Von Neumann Theorem (1931)

Oscillator

Discrete Series   Segal-Shale-Weil Repn   Orbit Method

Reductive Groups

Unitary dual

unipotent repns

Deformed

Dirac series   Dual Pair Correspondence   Quantization

Fourier Series & Fourier Transform

$$L^2(S^1) = \bigoplus_{n\in\mathbb{Z}} V_n . \qquad V_n = \mathbb{C}\{e^{in\theta}\}$$

$$\mathcal{F}(f) = \hat{f}(y) = \int_{\mathbb{R}} e^{-ix\cdot y} f(x)\,dx$$

G locally cpt abelian group, $\hat{G}$ dual of G

$$\hat{f}(\gamma) = \int_G \overline{\gamma(t)}\, f(t)\, d\mu(t)$$

Fourier Series : $\quad L^2(S^1) \xrightarrow{\mathcal{F}} \ell^2(\mathbb{Z})$

Stone - von Neumann Theorem

All pairs of 1-parameter unitary groups satisfying the canonical commutation relation $[Q, P] = i$ are unitarily equivalent.

$$e^{itQ} e^{isP} = e^{-ist} e^{isP} e^{itQ}$$

$\exists$ unitary operator $A : L^2(\mathbb{R}) \longrightarrow H$

$$A^* U(t) A = e^{itQ} , \quad A V(s) A^* = e^{isP}$$

<u>Thm</u> H_{2n+1} = Heisenberg group

For any $h \in \mathbb{R}$ $(h \neq 0)$, $\exists !$ W_h acting on $L^2(\mathbb{R}^n)$

$$W_h(M(a,b,c))\psi(x) = e^{i(bx+hc)}\psi(x+ha).$$

Segal - Shale - Weil representations

(V, ω) symplectic space

Heisenberg(V, ω) , $Sp(V, \omega)$ symplectic group

$\forall g \in Sp(V, \omega)$, $\exists$ U_g unitary operator

$$W(g \cdot v) = U_g W(v) U_g^{-1}$$

$g \longmapsto U_g$ defines a repn of $Mp(V, \omega)$

Fourier Transforms

$p+q=n$

$$\mathcal{F}^{p,q}(f)(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\langle x,y\rangle_{p,q}} f(x)\,dx$$

$\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ schwarz space

$$e = r^2 = \Sigma x_i^2, \quad \Delta = \Sigma \partial_i^2 = -f, \quad \Sigma x_i \partial_i + \frac{n}{2} = +h$$

$\{\frac{1}{2} r^2,\ h,\ -\frac{1}{2}\Delta\}$ is a $\mathfrak{sl}_2$-triple $\subseteq \mathfrak{sp}(2n,\mathbb{R})$

Prop. $$\boxed{\mathcal{F}^{p,q} = i^{\frac{n}{2}}\ e^{-\frac{\pi}{2} i(e+f)}} = i^{n/2} e^{-\frac{\pi}{2} k_{p,q}}$$

Proof. (Hermite polynomials) $\cdot\ e^{-\frac{1}{2}r^2}$ form a basis of $\mathcal{S}(\mathbb{R}^n)$ //.

$$\sigma = \mathrm{Ad}(k) \quad \boxed{\pi(k)\,\underline{\pi(g)} = \underset{\pi(kgk^{-1})}{\underline{\pi(g^k)}}\,\pi(k)}$$

$\mathcal{F}^{p,q}$ and $(\overset{(p,q)}{O(n)}, \mathfrak{sl}_2(\mathbb{R}))$-duality

$$S(\mathbb{R}^n) = \sum_{k=0}^{\infty} \mathcal{H}_k(\mathbb{R}^n) \otimes V_{k+\frac{n}{2}} \leftarrow \rho_k \text{ lowest weight module}$$

$$\mathcal{F} \simeq \sum_{k=0}^{\infty} \mathrm{Id}_{\mathcal{H}_k} \otimes T_k .$$

$$\boxed{T_k = \rho_k\left(i^{\frac{n}{2}} e^{-\frac{\pi}{2}\mathfrak{k}_{p,q}}\right)}$$

$$\boxed{\begin{array}{l} \mathfrak{k} = i(e+f) \\ \quad \text{in } i\mathfrak{k}_{\mathbb{R}} \end{array}}$$

$(O(n), Sp(2m, \mathbb{R}))$-duality $\subseteq Sp(2nm, \mathbb{R})$

$$\mathcal{F} \simeq \sum_{\tau \in \widehat{O(n)}} \mathrm{Id}_{V_\tau} \otimes \Theta(\tau)\left(\underbrace{i^{\frac{mn}{2}} e^{-\frac{\pi}{2}\mathfrak{k}}}_{}\right)$$

Universal Fourier Transforms

$G_{\mathbb{R}}$ real reductive group, Θ Cartan involution

$K_{\mathbb{R}}$ max'l compact subgroup, B bilinear form on $\mathfrak{g}_{\mathbb{R}}$

Theorem

Attach to $\mathcal{O}_{mod}$

$$\boxed{\mathcal{F} = e^{i\frac{\pi}{2}(e+f)} : V \longrightarrow V_{(\mathfrak{g},K)\text{-module}}}$$

$\{e, h, f\} \subseteq \mathfrak{sl}_2(\mathbb{R})$-triple in $\mathfrak{g}_{\mathbb{R}}$

$\Theta(h) = -h$, $f = +\theta(e)$ (Cayley triple) Standard triple

$H = \underline{i(e+f)}$, $X = \frac{1}{2}(e-f+ih)$, $Y = \frac{1}{2}(e-f-ih)$

$\{X, H, Y\}$ normal triple Kostant-Sekiguchi triple

$H \in \mathfrak{k}_{\mathbb{C}}$

$\mathfrak{sl}_2$ - triples in $\mathfrak{g}$ (semisimple / $\mathbb{C}$)

$\{X, H, Y\} \subseteq \mathfrak{g}$ $\quad [H, X] = 2X, \ [H, Y] = -2Y, \ [X, Y] = H$

$$\{\mathfrak{sl}_2\text{-triples}\}/\sim \ \xleftrightarrow{\text{Jacobson–Morozov Thm}} \ \left\{\begin{matrix}\text{nilpotent} \\ \text{orbits } \mathcal{O} \neq 0\end{matrix}\right\}$$

Principal $\quad \mathcal{O}_{\text{Prin}} = G \cdot \sum_{\alpha \in \Pi} X_\alpha \quad \Pi$ simple roots $\quad$ Kostant

Kostant-Rallis

Minimal $\quad \mathcal{O}_{\min} = G \cdot X_\beta \quad \beta$ highest root $\quad$ Joseph ideal

Vogan

Model $\quad \mathcal{O}_{\text{mod}} = G \cdot \sum_{\alpha \in S} X_\alpha \quad S \subseteq \Pi$

orthogonal subset

containing a short root

Max'l Primitive Ideal attached to $\mathcal{O}_{mod}$

McGovern: $\quad \mathcal{U}(\mathfrak{g})/J = \bigoplus_{\mu \in \Lambda_r^d} V_\mu$

$J = J_{\max}(\frac{1}{2}\rho)$ $\quad \Lambda_r^d$ root lattice. $V_\mu \simeq V_\mu^*$

Loke-Savin: $\quad \mathcal{U}(\mathfrak{k})^K \xrightarrow{\simeq} \mathcal{U}(\mathfrak{g})^K / J^K$ algebra isomorphism

G/K split $\quad V$ $(\mathfrak{g}, K)$-module $\quad \mathrm{Ann}\, V = J$

$\Rightarrow$ V is K-multiplicity free

The Boundary of $\mathcal{O}_{mod}$ $\quad \partial \mathcal{O}_{mod} \supseteq \mathcal{O} \quad AV(\mathrm{Ann}(V)) = \overline{\mathcal{O}}$

$\mathcal{U}(\mathfrak{k})^K \longrightarrow \mathcal{U}(\mathfrak{g})^K / J^K$ is surjective

$\Rightarrow$ V is K-multiplicity free

Real reductive groups

The *Cartan involution* for $GL(n, \mathbb{R})$ is the automorphism

$$\theta(g) = {}^t g^{-1}.$$

Definition. A Lie group G (having finitely many components) is called *reductive*, if there is a homomorphism $\eta \colon G \to GL(n, \mathbb{R})$, s.t.

1) $\operatorname{Ker} \eta$ is finite;

2) $\operatorname{Im} \eta$ is θ-stable.

We say G is *semisimple* if it is reductive and the center of the connected identity component G_0 is finite.

The unique lift of θ to G which is trivial on $\operatorname{Ker} \eta$ is defined to be the Cartan involution for G.

$\mathfrak{g}_0 =$Lie(G) and $\mathfrak{g}$ for the complexification.

Let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ be the Cartan decompositions.

Unitary duals

A central problems in representation theory is the classification of the irreducible unitary representations of G.
The orbit method suggests a correspondence between irreducible unitary representations of G and orbits for G in $\mathfrak{g}_0^*$

$$\{G - \text{orbits in } \mathfrak{g}_0^*\} \leftrightsquigarrow \{\text{Irreducible unitary repns of } G\}$$

- One expects a finite set of irreducible unitary representations of G corresponding to the nilpotent co-adjoint G-orbits.
- They have a name—'*unipotent representations*'—but not yet a good definition.
- Properly defined unipotent representations form the building blocks of all irreducible unitary representations.

Reference. Vogan's 1986 Hermann Weyl Lectures notes published by Annals of Mathematical Studies.

Unipotent representations for $G(\mathbb{F}_q)$

Let $\mathbb{F}_q$ be the finite field with q elements, let G be a connected reductive algebraic group defined over $\mathbb{F}_q$, and let $G(\mathbb{F}_q)$ be its $\mathbb{F}_q$-rational points.

In 1976, Deligne and Lusztig defined the notion of a unipotent representation of $G(\mathbb{F}_q)$ (geometric and case-free).

In 1984, Lusztig completed the classification of irreducible finite-dimensional representations of $G(\mathbb{F}_q)$, in particular,

1. The classification of all irreducible finite-dimensional representations of $G(\mathbb{F}_q)$ can be reduced to the classification of the unipotent representations, and
2. The unipotent representations are classified by certain geometric data related to the nilpotent co-adjoint orbits for the complex group associated to G.

Unipotent representations for G

There is a rich analogy between the finite-dimensional representations of finite groups of Lie type and the unitary representations of real reductive Lie groups.

This analogy suggests that the unitary dual of a real reductive Lie group should contain a finite set of building blocks parameterized by nilpotent co-adjoint orbits.

- Classifying the irreducible unitary representations of real reductive groups is one of the most important unsolved problems in representation theory.
- Its solution would have major implications for the Langlands program.
- The problem of correctly defining and classifying unipotent representations is one of central importance in the subject.

Coadjoint orbits for reductive groups

Use the trace form to identify $\mathfrak{g}_0^*$ with $\mathfrak{g}_0$.

The map $f \mapsto X(f)$ is given by $f(Y) = \langle X(f), Y \rangle$.

Proposition. Suppose G is are real reductive group, and X is in $\mathfrak{g}_0$.

1) The Jordan components X_h, X_e, X_n are in $\mathfrak{g}_0$.

2) If X is hyperbolic, then it is conjugate to an element in $\mathfrak{s}_0$.

3) If X is elliptic, then it is conjugate to an element in $\mathfrak{k}_0$.

Definition. (Jordan Decomposition)

Let $X(f) = X(f)_h + X(f)_e + X(f)_n$ be the Jordan decomposition.

Then the corresponding

$$f = f_h + f_e + f_n$$

is defined to be the Jordan decomposition of f.

Orbit method for reductive groups (Vogan)

Suppose that $f \in \mathfrak{g}_0^*$.

$G(f) =$ centralizer of $X(f)$ in G, $\mathfrak{g}_0(f) = \{Y \in \mathfrak{g}_0 \mid [X(f), Y] = 0\}$.

$$\theta f_h = -f_h,\ \theta f_e = f_e, \text{ and}$$

$G(f_h)$, $G(f_e)$ and $G(f_s) = G(f_h) \cap G(f_e)$ are preserved by θ.
Since X_e and X_n commute with X_h, and so belong to $\mathfrak{g}(f_h)$, we can identify f_e and f_n (by restriction) with elements of $\mathfrak{g}(f_h)^*$. Thus,

$$G(f_h) \supset [G(f_h)](f_e) \supset \{[G(f_h)](f_e)\}(f_n);$$

these are the same groups as

$$G(f_h) \supset G(f_s) \supset G(f).$$

$$\widehat{G(f)} \to \widehat{G(f_s)} \to \widehat{G(f_h)} \to \widehat{G}.$$

Model unipotent ideals

Proposition. (Losev, Mason-Brown and Matvieievsky)
Let $\mathbf{G} = Sp(2n, \mathbb{C})$. *Then*

(i) *There is one unipotent Harish-Chandra bimodule attached to* O_{mod}. *It is parabolically induced from the trivial representation of the Segal parabolic.*

(ii) *There are two unipotent Harish-Chandra bimodules attached to* $\widetilde{O}_{mod}$. *One (the spherical) is the midpoint of the complementary series. The other (the anti-spherical) is unitarily induced from a nontrivial character of the Segal parabolic.*

Model unipotent ideals

G	$\lambda_0(O_{\mathrm{mod}})$	$\lambda_0(\widetilde{O}_{\mathrm{mod}})$
A_{2n-1}	$\frac{1}{2}(n-1,n-1,n-3,n-3,\ldots,1-n,1-n)$	$\frac{1}{2}\rho$
A_{2n}	$\frac{1}{2}\rho$	no cover
B_{2n}	$(n, n-1, n-1, \ldots, 1, 1, 0)$	$(n, n-1, n-1, \ldots, 1, 1, 0)$
B_{2n+1}	$(n, n, n-1, n-1, \ldots, 1, 1, 0)$	no cover
C_{2n}	$\frac{1}{2}(2n-1,2n-1,2n-3,2n-3, \ldots,1,1)$	$(n, n-1, n-1, \ldots, 1, 1, 0)$
C_{2n+1}	$(n, n, n-1, n-1, \ldots, 1, 1, 0)$	$\frac{1}{2}(2n+1,2n-1,2n-1,\ldots,1,1)$
D_n	$\frac{1}{2}\rho$	$\frac{1}{2}\rho$

Model unipotent representations

Definition. *A unipotent representation of G attached to O_{mod} is an irreducible representation M of G such that*

(i) *M is unitary.*

(ii) *The annihilator of M is one of the unipotent ideals $J_0(\widetilde{O}_{mod})$.*

If G is a nonlinear covering, we require that M is genuine.

We now focus on the groups $Sp(2n, \mathbb{R})$ and $Mp(2n, \mathbb{R})$.

Note there are two unipotent ideals for these groups.

Model unipotent representations

Theorem. (Huang and Mason-Brown) *The following are true:*

(i) *If n even, there are exactly $4n$ model unipotent representations of $Sp(2n, \mathbb{R})$ with annihilator $J_0(\widetilde{O}_{mod})$. All irreducible representations of $Sp(2n, \mathbb{R})$ with this annihilator are unitary and are obtained as theta-lifts of finite-dimensional unitary chbaracters of $O(p, q)$ with $p + q = n$.*

(ii) *If n is odd, there are no model unipotent representations of $Sp(2n, \mathbb{R})$ with annihilator $J_0(\widetilde{O}_{mod})$. There are exactly $4n$ model unipotent representations of $Mp(2n, \mathbb{R})$ with this annihilator. All irreducible representations of $Mp(2n, \mathbb{R})$ with this annihilator are unitary and are obtained as theta-lifts of unitary characters of $O(p, q)$ with $p + q = n$.*

Construction of model unipotent representations

The model unipotent representations of $Sp(2n, \mathbb{R})$ and $Mp(2n, \mathbb{R})$ arise in four different ways:

- By cohomological induction as $A_{\mathfrak{q}}(\lambda)$-modules
- As constituent in as degenerate principal series representations by real parabolic induction
- As theta-lifts of finite-dimensional unitary representations of $O(p, q)$
- By transfer unitary highest weight modules