On the Arthur-Barbasch-Vogan conjecture*

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(August 3, 2024)

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Contents

1 Real reductive groups: background

G: real reductive Lie group. For example, $\mathrm{GL}_n(\mathbb{R}), \mathrm{O}_{p,q}, \mathrm{Sp}_{2n}(\mathbb{R})$.

- The fundamental algebraic objects: $(\mathfrak{g}, K)$-modules, where $\mathfrak{g}$ is the complexified Lie algebra of G, and K is a maximal compact subgroup.
  - The good ones: admissible $(\mathfrak{g}, K)$-modules of finite length, called Harish-Chandra modules.
- The fundamental analytic objects: the canonical globalization of Harish-Chandra modules, called Casselman-Wallach representations.
  - Key requirements: smooth, Fréchet and of moderate growth.

Two fundamental invariants:

- Infinitesimal character χ: $\mathcal{Z}(\mathfrak{g}) \to \mathbb{C}$.
  - An irreducible representation has an infinitesimal character.
  - Harish-Chandra isomorphism: An infinitesimal character χ is represented by (Weyl group orbit of) an element $\lambda \in \mathfrak{h}^*$.
- Complex associated variety $\mathrm{AV}_{\mathbb{C}}(X) = \mathrm{V}(\mathrm{Ann}(X))$.
  - This is the variety of the zeroes of the graded ideal $\mathrm{Gr}(\mathrm{Ann}(X))$.
  - It is contained in

$$\mathrm{Nil}(\mathfrak{g}^*) = \{\lambda \in \mathfrak{g}^* \mid p(\lambda) = 0, \forall\, p \in S^+(\mathfrak{g})^G\}.$$

More refined invariants:

- associated variety $\mathrm{AV}(X)$, associated cycle $\mathrm{AC}(X)$ (Vogan).
- wavefront set = asymptotic support (Howe, Barbasch-Vogan).

Two fundamental results

- Harish-Chandra: for any fixed infinitesimal character χ,

$$\sharp(\mathrm{Irr}_\chi(G)) < \infty.$$

- Borho-Brylinski, Joseph:
  - If X is irreducible,

$$\mathrm{V}(\mathrm{Ann}(X)) = \bar{\mathcal{O}}.$$

  - In words, the associated variety of a primitive ideal of $\mathcal{U}(\mathfrak{g})$ is the closure of single nilpotent $\mathrm{Ad}(\mathfrak{g})$-orbit in $\mathfrak{g}^*$.

2 Special unipotent representations: Arthur-Barbasch-Vogan

The problem:

- Determine all special unipotent representations (definition to follow) and show in particular that they are unitary.
  - The unitarity assertion: Arthur-Barbasch-Vogan conjecture
    - Arthur's conjecture on L^2- automorphic forms
- We solve the classification problem (for all real classical groups) by
  - counting, construction, distinguishing,

  with unitarity as a direct consequence.

Arthur-Barbasch-Vogan conjecture:

- Complex classical groups: Barbasch (1989);
- Real classical groups (including the metaplectic groups and the spin groups): Barbasch-Ma-Sun-Z;
- Quasi-split real classical groups: Adams-Arancibia-Mezo;
- Exceptional groups: Miller, Adams-Van Leeuwen-Miller-Vogan.

- Given a $\check{G}$-orbit $\check{\mathcal{O}}$ in $\mathrm{Nil}(\check{\mathfrak{g}})$, one attaches an infinitesimal character $\chi_{\check{\mathcal{O}}}$, represented by $\lambda_{\check{\mathcal{O}}} \in \mathfrak{h}^*$ (via an $\mathfrak{sl}_2$-triple containing $\check{\mathcal{O}}$).
- By a theorem of Duflo, there exists a unique maximal G-stable ideal $I_{\check{\mathcal{O}}}$ of $\mathcal{U}(\mathfrak{g})$ that contains the kernel of $\chi_{\check{\mathcal{O}}}$.
- The associated variety of $I_{\check{\mathcal{O}}}$ is the closure of a nilpotent $\mathrm{Ad}(\mathfrak{g})$-orbit $\mathcal{O}$ in $\mathfrak{g}^*$.
  - $\mathcal{O}$ is called the Barbasch-Vogan dual of $\check{\mathcal{O}}$.
  - $\mathcal{O}$ is special in the sense of Lusztig.

Definition: (Barbasch-Vogan, 1985)

An irreducible Casselman-Wallach representation π of G is said to be special unipotent attached to $\check{\mathcal{O}}$ if $I_{\check{\mathcal{O}}}$ annihilates π.

Equivalent conditions:

- π has infinitesimal character $\chi_{\check{\mathcal{O}}}$, and $\mathrm{AV}_{\mathbb{C}}(\pi) \subseteq \overline{\mathcal{O}}$.

Notation: $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$, the set of equivalent classes of irreducible Casselman-Wallach representations of G that are special unipotent attached to $\check{\mathcal{O}}$, now known as the weak ABV packet (attached to $\check{\mathcal{O}}$).

Arthur-Barbasch-Vogan conjecture: (1980's)

- All representations in $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ are unitarizable.

3 Counting representations

Problem: count the set $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$.

- **Main tool**: coherent continuation
  - Every irreducible representation can be placed inside a coherent family of (virtual) representations.
  - The space of all coherent families carries a representation of the integral Weyl group, called the coherent continuation representation.
  - The coherent continuation representation can be analyzed in great detail via Kazhdan-Lusztig theory (primitive ideas, left cells, double cells, Springer correspondence, ...).

The coherent continuation representation: (Jantzen, Schmid, Zuckerman, Speh-Vogan)

- $\mathcal{K}(G)$: the Grothendieck group of the category of Casselman-Wallach representations of G.
- $\mathcal{K}_\nu(G)$: the subgroup of $\mathcal{K}(G)$ generated by $\mathrm{Irr}_\nu(G)$, $\nu \in \mathfrak{h}^*$.
- $\Lambda = \nu + \mathrm{P} \subset \mathfrak{h}^*$: a coset of the weight lattice P for G.

- A $\mathcal{K}(G)$-valued coherent family on Λ is a map $\Psi\colon \Lambda \to \mathcal{K}(G)$ such that, for all $\nu \in \Lambda$,
  - $\Psi(\nu) \in \mathcal{K}_\nu(G)$, and
  - for any finite-dimensional representation F of G,

$$\Psi(\nu) \otimes \mathrm{F} = \sum_{\mu} \Psi(\nu + \mu),$$

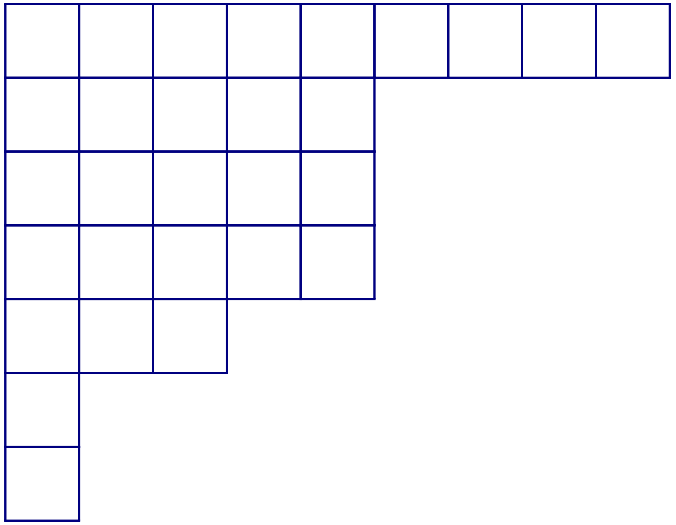
    where μ runs over the set of all weights (counting multiplicities) of F.

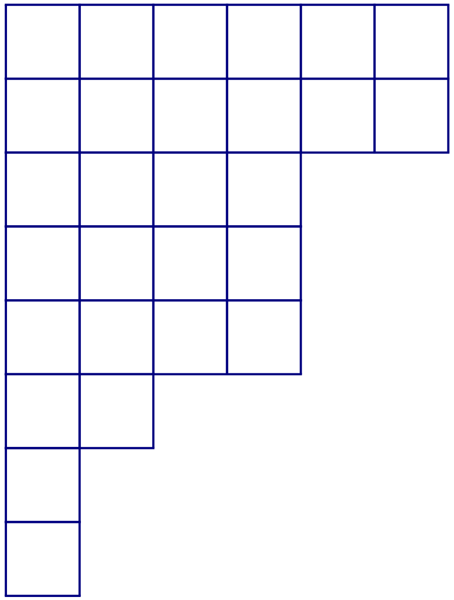
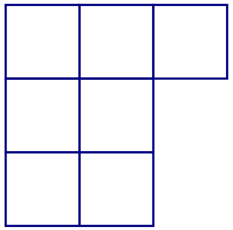
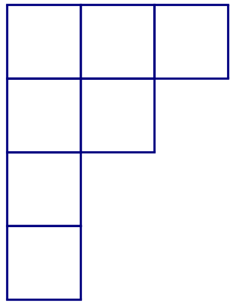
- **Theorem** (Barbasch-Ma-Sun-Z, arXiv:2205.05266): If $\check{\mathcal{O}}$ has good parity in the sense of Mœglin, then

$$\sharp \mathrm{Unip}_{\check{\mathcal{O}}}(G) = \begin{cases} 2^{\sharp \mathrm{PP}_\star(\check{\mathcal{O}})} \cdot \sharp \mathrm{PBP}_G(\check{\mathcal{O}}), & \text{if } \star = C, \tilde{C}; \\ 2 \cdot 2^{\sharp \mathrm{PP}_\star(\check{\mathcal{O}})} \cdot \sharp \mathrm{PBP}_G(\check{\mathcal{O}}), & \text{if } \star = B, D. \end{cases}$$

  - $\mathrm{PBP}_G(\check{\mathcal{O}})$: set of painted bipartitions attached to $(G, \check{\mathcal{O}})$ (painting rules depends on the group G);
  - $2^{\sharp \mathrm{PP}_\star(\check{\mathcal{O}})}$: size of Lusztig's canonical quotient.

Example: $G = \mathrm{Sp}(28, \mathbb{R})$, $\check{G} = \mathrm{O}(29, \mathbb{C})$.

$\check{\mathcal{O}} =$ 

$\mathcal{O} =$  $\xrightarrow{\text{Springer}} (\imath, \jmath) =$  $\times$ 

- $\mathrm{PP}_\star(\check{\mathcal{O}}) = \{(1,2),(5,6)\}$.
- $\sharp\mathrm{PBP}_G(\check{\mathcal{O}}) = 80$.
  - e.g. of a painted bipartition, with symbols $\bullet, s, r, c, d$:

$$\begin{array}{|c|c|c|}\hline \bullet & \bullet & r \\ \hline \bullet & \bullet & \multicolumn{1}{c}{} \\ \cline{1-2} c & d & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} \quad \times \quad \begin{array}{|c|c|c|}\hline \bullet & \bullet & s \\ \hline \bullet & \bullet & \multicolumn{1}{c}{} \\ \cline{1-2} s & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} s & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

- $\sharp\mathrm{Unip}_{\check{\mathcal{O}}}(G) = 320$.

4 Constructing representations

Main tool: theta correspondence

Definition: (Howe, 1979)

- W: a finite-dimensional real symplectic vector space.
- (G, G'): a reductive dual pair in $\mathrm{Sp}(W)$, i.e., a pair of subgroups such that
  - G and G' are mutual centralizers of each other;
  - G and G' act reductively on W.

Irreducible reductive dual pairs (seven families):

- Type II: correspond to a division algebra D

$$(\mathrm{GL}_m(\mathbb{R}), \mathrm{GL}_n(\mathbb{R})) \subseteq \mathrm{Sp}_{2mn}(\mathbb{R})$$

$$(\mathrm{GL}_m(\mathbb{C}), \mathrm{GL}_n(\mathbb{C})) \subseteq \mathrm{Sp}_{4mn}(\mathbb{R})$$

$$(\mathrm{GL}_m(\mathbb{H}), \mathrm{GL}_n(\mathbb{H})) \subseteq \mathrm{Sp}_{8mn}(\mathbb{R})$$

- Type I: correspond to a division algebra D with involution $\natural$

$$(\mathrm{O}_{p,q}, \mathrm{Sp}_{2n}(\mathbb{R})) \subseteq \mathrm{Sp}_{2(p+q)n}(\mathbb{R})$$

$$(\mathrm{O}_p(\mathbb{C}), \mathrm{Sp}_{2n}(\mathbb{C})) \subseteq \mathrm{Sp}_{4pn}(\mathbb{R})$$

$$(\mathrm{U}_{p,q}, \mathrm{U}_{r,s}) \subseteq \mathrm{Sp}_{2(p+q)(r+s)}(\mathbb{R})$$

$$(\mathrm{Sp}_{p,q}, \mathrm{O}^*_{2n}) \subseteq \mathrm{Sp}_{4(p+q)n}(\mathbb{R})$$

(G, G'): a reductive dual pair in $\mathrm{Sp}(W)$.

- Fix an oscillator (or Weil) representation $\widehat{\omega}$ (by fixing a nontrivial unitary character on $\mathbb{R}$). This is a unitary representation of $\widetilde{\mathrm{Sp}}(W)$ (the real metaplectic group), constructed by Segal, Shale and Weil.
  - The existence of $\widehat{\omega}$ (essentially) amounts to the uniqueness of the canonical commutation relations (CCR).
- Let ω be the associated smooth representation, called a smooth oscillator representation.
- For a reductive subgroup E of $\mathrm{Sp}(W)$, denote by $\widetilde{E}$ its inverse image in $\widetilde{\mathrm{Sp}}(W)$, and
  - $\mathrm{Irr}(\widetilde{E}, \omega)$: the subset of $\mathrm{Irr}(\widetilde{E})$ which are realizable as quotients by $\omega(\widetilde{E})$-invariant closed subspaces of ω.

- **Howe duality theorem**: The set $\mathrm{Irr}(\widetilde{G} \cdot \widetilde{G'}, \omega)$ is the graph of a bijection between $\mathrm{Irr}(\widetilde{G}, \omega)$ and $\mathrm{Irr}(\widetilde{G'}, \omega)$. Moreover any element $\pi \otimes \pi'$ of $\mathrm{Irr}(\widetilde{G} \cdot \widetilde{G'}, \omega)$ occurs as a quotient of ω in a unique way.
  - The correspondence $\pi \leftrightarrow \pi'$ is defined by the condition

$$\mathrm{Hom}_{\widetilde{G} \times \widetilde{G'}}(\omega, \pi \otimes \pi') \neq 0.$$

  - Companion statement: (multiplicity-1)

$$\dim \mathrm{Hom}_{\widetilde{G} \times \widetilde{G'}}(\omega, \pi \otimes \pi') \leq 1.$$

- Howe duality also holds true for p-adic local fields:
  - works of Waldspurger, Minguez, Gan-Takeda, Gan-Sun

An important question is to describe first the domain of theta correspondence, and then

- theta correspondence in terms of the Langlands parameters.
  - Many works, but still no full answer.

Another important question is to understand how unitarity behaves under theta correspondence:

- Li, He, Barbasch-Ma-Sun-Z (via integration of matrix coefficients)

Construction:

- repeatedly apply theta lifting, starting from the trivial representation, and possibly twisting by quadratic characters of orthogonal groups.
- The construction is guided by the descent structure of combinatorial parameters of special unipotent representations.

Descent of combinatorial parameter:

$$
\begin{array}{lll} \bullet & \bullet & r \\ \bullet & \bullet & \\ c & d & \end{array}
\times
\begin{array}{lll} \bullet & \bullet & s \\ \bullet & \bullet & \\ s & & \\ s & & \end{array}
\times C \xrightarrow{\nabla}
\begin{array}{lll} \bullet & \bullet & r \\ \bullet & s & \\ c & d & \end{array}
\times
\begin{array}{ll} \bullet & \bullet \\ \bullet & \end{array}
\times D
$$

$$
\xrightarrow{\nabla}
\begin{array}{ll} \bullet & r \\ \bullet & \\ d & \end{array}
\times
\begin{array}{ll} \bullet & s \\ \bullet & \end{array}
\times C \xrightarrow{\nabla}
\begin{array}{ll} \bullet & r \\ s & \\ d & \end{array}
\times
\begin{array}{l} \bullet \end{array}
\times D
$$

$$
\xrightarrow{\nabla} \boxed{r} \times \boxed{s} \times C \xrightarrow{\nabla} \boxed{r} \times \emptyset \times D \xrightarrow{\nabla} \emptyset \times \emptyset \times C.
$$

Corresponding Lie groups:

$$
\begin{array}{rl}
 & \mathrm{Sp}(28,\mathbb{R}) \rightarrow \mathrm{O}(10,10) \\
\rightarrow & \mathrm{Sp}(14,\mathbb{R}) \rightarrow \mathrm{O}(5,5) \\
\rightarrow & \mathrm{Sp}(4,\mathbb{R}) \rightarrow \mathrm{O}(2,0) \rightarrow \mathrm{Sp}(0,\mathbb{R}).
\end{array}
$$

5 Distinguishing the representations

Main tool: associated cycle

- Write $\mathcal{K}_{\mathcal{O}}(G)$ for the Grothedieck group of the category of Casselman-Wallach representations π of G such that

$$\mathrm{AV}_{\mathbb{C}}(\pi) \subset \overline{\mathcal{O}}.$$

  – We say π is $\mathcal{O}$-bounded.

- $\mathscr{O} \subset \mathcal{O} \cap \mathfrak{p}$: a $\mathbf{K}$-orbit. ($\mathbf{K} = K_{\mathbb{C}}$, the complexification of K)
- $\mathcal{K}_{\mathscr{O}}(\mathbf{K})$: the Grothedieck group of the category of $\mathbf{K}$-equivariant algebraic vector bundles on $\mathscr{O}$.
-
$$\mathcal{K}_{\mathcal{O}}(\mathbf{K}) := \bigoplus_{\mathscr{O} \text{ is a } \mathbf{K}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}} \mathcal{K}_{\mathscr{O}}(\mathbf{K}).$$
- There is a canonical homomorphism: (Vogan, 1989)
$$\mathrm{AC}_{\mathcal{O}} : \mathcal{K}_{\mathcal{O}}(G) \to \mathcal{K}_{\mathcal{O}}(\mathbf{K}).$$
  - $\mathrm{AC}_{\mathcal{O}}(\pi)$ is called the associated cycle of π.

An important question is to understand how associated cycle behaves under theta correspondence.

- Tool: geometry of moment maps

$$\begin{array}{ccccc} \mathfrak{p} & \xleftarrow{M} & \mathcal{X} & \xrightarrow{M'} & \mathfrak{p}', \\ \phi^*\phi & \longleftarrow\!\!\!| & \phi & |\!\!\!\longrightarrow & \phi\phi^* \end{array}$$

$\rightsquigarrow$ notion of the descent of a nilpotent $\mathbf{K}$-orbit:

$$\mathscr{O} \mapsto \mathscr{O}' =: \nabla(\mathscr{O}).$$

$\rightsquigarrow$ notion of the geometric theta lift:

$$\check{\vartheta}_{\mathscr{O}'}^{\mathscr{O}} : \mathcal{K}(\mathscr{O}') \to \mathcal{K}(\mathscr{O}).$$

- Result: the associated cycles of all constructed representations.

6 More on the Arthur-Barbasch-Vogan conjecture

- $G_{\mathbb{C}}$: connected reductive complex Lie group;
- G: a real form of $G_{\mathbb{C}}$.

Arthur-Barbasch-Vogan conjecture:

- All representations in $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ are unitarizable.

It suffices to consider the case:

- $G_{\mathbb{C}}$ is simply connected, and $\mathrm{Lie}(G)$ is simple.

Type A:

- $G_{\mathbb{C}}$: $\mathrm{SL}_n(\mathbb{C})$ or $\mathrm{SL}_n(\mathbb{C}) \times \mathrm{SL}_n(\mathbb{C})$
- $G = \mathrm{SL}_n(\mathbb{R}),\ \mathrm{SU}(p, q)\ (p + q = n),\ \mathrm{SL}_{\frac{n}{2}}(\mathbb{H})\ (n \text{ is even}),\ \text{or } \mathrm{SL}_n(\mathbb{C})$

[BMSZ4]: (easy)

- Special unipotent representations of simple linear groups of type A, Acta Math. Sin. (2024).

Type B, D: (genuine)

- $G_{\mathbb{C}}$: $\mathrm{Spin}(m, \mathbb{C})$ or $\mathrm{Spin}(m, \mathbb{C}) \times \mathrm{Spin}(m, \mathbb{C})$
- $G = \mathrm{Spin}(p, q)$ $(p + q = m)$, $\mathrm{Spin}^*(2n)$ $(m = 2n)$, or $\mathrm{Spin}(m, \mathbb{C})$

[BMSZ3]: (moderate)

- Genuine special unipotent representations of spin groups, Kobayashi Festscrhift (2024).

Type B, D: (classical)

- $G_{\mathbb{C}}$: $\mathrm{SO}(m, \mathbb{C})$ or $\mathrm{SO}(m, \mathbb{C}) \times \mathrm{SO}(m, \mathbb{C})$
- $G = \mathrm{SO}(p, q)$ $(p + q = m)$, $\mathrm{SO}^*(2n)$ $(m = 2n)$, or $\mathrm{SO}(m, \mathbb{C})$

Type C: (classical)

- $G_{\mathbb{C}}$: $\mathrm{Sp}(2n, \mathbb{C})$ or $\mathrm{Sp}(2n, \mathbb{C}) \times \mathrm{Sp}(2n, \mathbb{C})$
- $G = \mathrm{Sp}(p, q)$ $(p + q = n)$, $\mathrm{Sp}(2n, \mathbb{R})$, or $\mathrm{Sp}(2n, \mathbb{C})$

[BMSZ1] and **[BMSZ2]**: (difficult)

- Special unipotent representations of real classical groups: counting and reduction
- Special unipotent representations of real classical groups: construction and unitarity

Theorem: (Barbasch-Ma-Sun-Z, arXiv:1712.05552)

- Let $G_{\mathbb{C}}$ be a connected reductive complex Lie group, and G a real form of $G_{\mathbb{C}}$. Assume that every simple factor of the Lie algebra $\mathfrak{g}$ of $G_{\mathbb{C}}$ is of a classical type. Let $\check{\mathcal{O}}$ be a nilpotent $\check{G}$-orbit in $\check{\mathfrak{g}}$. Then all representations in $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ are unitarizable.

Remark:

- The same result holds for the real metaplectic group. There is an analogous notion of metaplectic Barbasch-Vogan duality, and the corresponding representations are called metaplectic special.

Thank you !