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The nonexistence of expansive actions of groups with subexponential growth on Suslinian continua

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THE NONEXISTENCE OF EXPANSIVE ACTIONS OF GROUPS WITH SUBEXPONENTIAL GROWTH ON SUSLINIAN CONTINUA

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ABSTRACT. We show that if G is a finitely generated group of subexponential growth and X is a nondegenerate Suslinian continuum, then any continuous action of G on X is not expansive.

1. INTRODUCTION

By a *continuum* we mean a nonempty compact connected metric space. A continuum X is said to be *nondegenerate* if it is not a singleton. By a *curve* we mean a one-dimensional continuum. If X does not contain uncountably many mutually disjoint nondegenerate subcontinua, then it is called *Suslinian* [13]. It is known that Suslinian continua are curves and all rational curves are Suslinian. The Cantor fan is a quick example of a curve but not Suslinian. Cook and Lelek constructed a chainable Suslinian curve that is not rational [3].

Let X be a compact metric space and $\text{Homeo}(X)$ the homeomorphism group on X. By a *continuous action* of a discrete group G on X, written as (X, G, ϕ), $G \curvearrowright X$, or G-action, we mean a group homomorphism $\phi : G \to \text{Homeo}(X)$. For brevity, we shall use gx or $g(x)$ in place of $\phi(g)(x)$.

A continuous action $G \curvearrowright X$ on a compact metric space (X,d) is called *expansive* if there exists $c > 0$ such that $\sup_{g\in G} d(gx, gy) > c$ for any distinct points x and y of X. Such c is called an *expansive constant* for the action $G \curvearrowright X$. Expansivity is closely related to the topological stability of dynamical systems. Walters showed that every expansive $\mathbb{Z}$-action with pseudo-orbit tracing property is topologically stable [18]. Recently Chung and Lee considered the pseudo-orbit tracing property for actions of (finitely generated) countable groups and extended Walter's result [1].

We are interested in the following question.

Question 1.1. *What groups G and continua X admit expansive actions $G \curvearrowright X$ and what groups and continua do not?*

There has been intensively studied around this question. It is well known that the Cantor set, the solenoid, and every compact orientable surface of positive genus admit expansive $\mathbb{Z}$-actions [20, 17]. The unit interval admits an expansive action of some solvable groups (for example, an action of the Baumslag-Solitar group $BS(1,q)$ for $q \geq 2$ on the real line by affine transformations can induce an expansive action on $[0,1] \cong \mathbb{R} \cup \{\pm\infty\}$). On the contrary, the interval, the circle, and the 2-dimensional sphere admit no expansive $\mathbb{Z}$-actions [8]. It is asked by Ward [12] whether the unit circle admits an expansive action of a nilpotent group. This question was implicitly answered by Inaba and Tsuchiya in a

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more general situation of expansive foliations [10]. Connell, Furman, and Hurder gave a more self-contained proof via Ping-pong lemma [2].

For one-dimensional continua, Kato proved the following theorem.

Theorem 1.2. [11] *There are no expansive $\mathbb{Z}$-actions on nondegenerate Suslinian continua.*

The purpose of the paper is to extend this theorem to the actions of groups with subexponential growth. Let H be a finitely generated group with a finite generating set. For each $h \in H$ denote by $|h|$ the word length of h with respect to S. For each $k \in \mathbb{N}$ define

$$\beta(H,S;k) = \#\{h \in H : \ |h| \leq k\},$$

which is called the *growth function* of H with respect to S. If $\lim_{k\to\infty} \sqrt[k]{\beta(H,S;k)} = 1$, H is said to be of *subexponential growth*.

The following is the main result of the paper.

Theorem 1.3. *Let G be a finitely generated group of subexponential growth and X a nondegenerate Suslinian continuum. Then there are no expansive actions of G on X.*

We remark that this theorem can be improved in a continuum-wise expansive setting (see Remark 4.4). On the other hand, by [14], a Suslinian continuum of Theorem 1.3 cannot be changed to a chainable continuum.

By Gromov's theorem [7] on groups of polynomial growth, every finitely generated nilpotent group is of subexponential growth. Thus the following corollary is immediate, which gives a negative answer to [19, Question 1.4].

Corollary 1.4. *Let G be a finitely generated nilpotent group and X a nondegenerate Suslinian continuum. Then G cannot act on X expansively.*

As is already mentioned, there exists an expansive action $G \curvearrowright [0,1]$ of a solvable group G on the unit interval $[0,1]$. Theorem 1.3 implies that

Corollary 1.5. *If there is an expansive action $G \curvearrowright [0,1]$ of a finitely generated solvable group G, then G must be of exponential growth.*

The proof of Theorem 1.3 relies on a comparison of the growth rates between the acting group and the cardinality of pairwise disjoint nondegenerate subcontinua subject to a uniformly expansive scale. A key input is a delicate lemma on expansivity by Meyerovitch and Tsukamoto [15, Lemma 4.4], which is adapted from Fathi's method [5, Section 5]. Taking an advantage of a characterization of Suslinian continua, we then finish the proof following Kato's method for $\mathbb{Z}$-actions.

2. Preliminaries

2.1. Kato's characterization of Suslinian continua.

Definition 2.1. For a continuum X the *hyperspace* $C(X)$ is the set of all subcontinua of X. For $A, B \in C(X)$ define

$$d_H(A,B) = \inf\{\delta > 0 : \ A \subset N_\delta(B) \text{ and } B \subset N_\delta(A)\},$$

where $N_\delta(A)$ denotes the δ-neighborhood of A in X. Then d_H is a metric on $C(X)$ and is called the *Hausdorff metric*. It is known that $(C(X), d_H)$ is a continuum [16, Chapter IV].

In [11], for any subset M of $C(X)$, define

$$\widetilde{M} = \{A \in C(X) \ : \ \text{for any } \varepsilon > 0 \text{ and } k \in \mathbb{N}, \text{ there exist pairwise disjoint nondegenerate subcontinua } A_1, A_2, \ldots, A_k \in M \text{ such that } d_H(A, A_i) < \varepsilon \text{ for every } 1 \leq i \leq k\}.$$

Set $M_0 = M$. Assume that M_β has been defined for every ordinal $\beta < \alpha$. We define $M_\alpha = \widetilde{M_\beta}$ if $\alpha = \beta + 1$, and $M_\alpha = \cap_{\beta<\alpha} M_\beta$ if α is a limit ordinal. By [11, Proposition 3.3], we see that the family $\{M_\alpha\}_\alpha$ is decreasing with respect to α.

Example 2.2. Let X be the topologist's sine curve and $L = \{0\} \times [-1, 1]$. For $M = C(X)$, we have $M_1 = C(L)$, M_2 consists of all singletons of $C(L)$, and $M_3 = \emptyset$.

In [11, Theorem 3.4], Kato gave the following characterization of Suslinian continua.

Theorem 2.3. *Let X be a continuum and $M = C(X)$. Then X is Suslinian if and only if $M_\alpha = \emptyset$ for some countable ordinal α.*

If (X, G, ϕ) is a continuous action on a continuum X, then it naturally induces a continuous action $(C(X), G, \tilde{\phi})$ via $\tilde{\phi}(g)(A) = \phi(g)(A)$ for every $g \in G$ and $A \in C(X)$.

By the definition of M_α, we have

Proposition 2.4. [11, Proposition 3.2] *If $M \subseteq C(X)$ is G-invariant and closed, so is M_α.*

2.2. Pairwise disjoint subcontinua of uniform diameter.

Let X be a compact metric space and A a subset of X. Recall that the *boundary of A in X* is defined by $Bd_X(A) = \overline{A} \cap (\overline{X \setminus A})$. Since all the underlying spaces in the sequel are understood to be X, we shall simply write $Bd(A)$ to denote $Bd_X(A)$.

The following is known as the Boundary Bumping Theorem [16, Theorem 5.4].

Theorem 2.5. *Let U be a nonempty proper open subset of a continuum X. If K is a connected component of $\overline{U}$, then $K \cap Bd(U) \neq \emptyset$.*

A variant version of the Boundary Bumping Theorem is as follows.

Lemma 2.6. [11, Lemma 2.2] *Let X be a compact metric space and let U, V be open subsets of X such that $\overline{V} \subseteq U$. If A is a subcontinuum of X such that $A \cap V \neq \emptyset$ and $A \setminus \overline{U} \neq \emptyset$, then there is a subcontinuum B of $A \cap \overline{U}$ such that $B \cap V \neq \emptyset$ and $B \cap Bd(U) \neq \emptyset$.*

Definition 2.7. Let (X, d) be a compact metric space. For a subset E of X and $\varepsilon > 0$, we say E is *ε-separated* if $d(x, y) \geq \varepsilon$ for any distinct $x, y \in E$. Let $S(X, \varepsilon)$ denote the cardinality of a maximal ε-separated subset of X. The *lower box dimension* of (X, d) is defined as

$$\underline{\dim}_B(X, d) = \liminf_{\varepsilon \to 0} \frac{\log S(X, \varepsilon)}{\log(1/\varepsilon)}.$$

The following lemma says that for a nondegenerate continuum there are plenty of subcontinua with a uniform lower bound on diameters.

Lemma 2.8. *Let X be a nondegenerate continuum. Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, there are more than $1/\sqrt{\varepsilon}$ pairwise disjoint subcontinua of X whose diameters are greater than or equal to $\varepsilon/3$.*

Proof. Since X is nondegenerate and connected, the topological dimension $\dim(X)$ of X is positive [4, Proposition 1.3.3]. Since the lower box dimension bounds above the topological dimension [9, Chapter VII], we obtain that $\underline{\dim}_B(X,d) \geq \dim(X) \geq 1$. Then there exists $\varepsilon_0 > 0$ such that $S(X,\varepsilon) > \frac{1}{\sqrt{\varepsilon}}$ for any $\varepsilon \in (0,\varepsilon_0)$. Choose a maximal ε-separated subset E of X. Then $|E| = S(X,\varepsilon) > \frac{1}{\sqrt{\varepsilon}}$. For each $x \in E$ consider the connected component A_x of the closed ball $\overline{B(x,\varepsilon/3)}$ containing x. By Lemma 2.5, we have $A_x \cap Bd(B(x,\varepsilon/3)) \neq \emptyset$ and hence $\operatorname{diam}(A_x) \geq \frac{\varepsilon}{3}$. It follows that these A_x's satisfy the requirements. □

We also need the following refined version of Lemma 2.8, which will be used in the proof of Theorem 1.3.

Lemma 2.9. *Let X be a nondegenerate continuum and M a subset of $C(X)$. Suppose that M satisfies the following boundary bumping property: for any nondegenerate $C \in M$ and any open sets U,V of X satisfying $\overline{V} \subseteq U$, $C \cap V \neq \emptyset$ and $C \setminus \overline{U} \neq \emptyset$, there exists a nondegenerate subcontinuum D of $C \cap \overline{U}$ such that $D \in M, D \cap \overline{V} \neq \emptyset$, and $D \cap Bd(U) \neq \emptyset$. Then for any nondegenerate $C \in M$, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0,\varepsilon_0)$, there are more than $1/\sqrt{\varepsilon}$ pairwise disjoint subcontinua of C in M whose diameters are all greater than or equal to $\varepsilon/6$.*

Proof. By the proof of Lemma 2.8, there exists $\varepsilon_0 > 0$ such that $S(C,\varepsilon) > \frac{1}{\sqrt{\varepsilon}}$ for any $\varepsilon \in (0,\varepsilon_0)$. Picking a maximal ε-separated subset E of C, we have that $|E| > \frac{1}{\sqrt{\varepsilon}}$, the family $\{\overline{B(x,\varepsilon/3)}\}_{x\in E}$ is pairwise disjoint, and $C \setminus \overline{B(x,\varepsilon/3)} \neq \emptyset$ for every $x \in E$.

For every $x \in E$, applying the boundary bumping property of M to $U = B(x,\varepsilon/3)$ and $V = B(x,\varepsilon/6)$, we obtain a subcontinuum A_x of $C \cap \overline{B(x,\varepsilon/3)}$ such that $A_x \in M, A_x \cap B(x,\varepsilon/6) \neq \emptyset$ and $A_x \cap Bd(B(x,\varepsilon/3)) \neq \emptyset$. It follows that $\operatorname{diam}(A_x) \geq \varepsilon/6$ for every $x \in E$ and the family $\{A_x\}_{x\in E}$ is pairwise disjoint. □

3. Proof of the main theorem

To prove Theorem 1.3 we shall start with the following two lemmas.

Lemma 3.1. [11, Lemma 2.1] *Let (Y,d) be a compact metric pace, $\varepsilon > 0$, and $k \in \mathbb{N}$. Then there exists a positive integer $n = n(\varepsilon,k) \geq k$ satisfying the following. If $y_1, y_2, \ldots, y_n$ are points of Y, then there exists $y \in Y$ and $1 \leq i(1) < i(2) < \cdots < i(k) \leq n$ such that $d(y, y_{i(j)}) < \varepsilon$ for every $j \in \{1,2,\ldots,k\}$.*

The following key lemma is proved for $\mathbb{Z}^k$-actions [15, Lemma 4.4] and the proof works here for any continuous action of finitely generated groups. For readers' convenience, we would add the proof in the appendix.

Lemma 3.2. *Let $G \curvearrowright X$ be an expansive action of a finitely generated group on a compact metric space. Then there exist $a > 1$ and a compatible metric D on X such that for every $n \in \mathbb{N}$ and any $x,y \in X$ satisfying that $D(x,y) \geq a^{-n}$, we have*

$$\max_{g\in G, |g|\leq n} D(gx,gy) \geq \frac{1}{4a}.$$

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. we shall adapt Kato's method [11, Theorem 3.1] into our situation.

Fix a finite generating subset S of G. Assume that $G \curvearrowright X$ is an expansive action. By Lemma 3.2, there exist $a > 1$ and a compatible metric D on X such that for any $n \in \mathbb{N}$ and any $x, y \in X$ satisfying $D(x,y) \geq a^{-n}$, we have

$$\max_{g \in G, |g| \leq n} D(gx, gy) \geq \frac{1}{4a}.$$

We shall use this metric D in the following argument. Since G is of subexponential growth, for sufficiently large n, we have

$$\beta(G,S;n) \leq (\sqrt[3]{a})^n.$$

Set $M = M_0 = C(X)$. For each countable ordinal α, consider the following property:

Property P_α. If $C \in M_\alpha$ is nondegenerate, then for any open sets U, V of X satisfying $\overline{V} \subset U$, $C \cap V \neq \emptyset$, and $C \setminus \overline{U} \neq \emptyset$, there exists a nondegenerate subcontinuum D of $C \cap \overline{U}$ such that $D \in M_\alpha$, $D \cap \overline{V} \neq \emptyset$, and $D \cap Bd(U) \neq \emptyset$.

We shall show that for every countable ordinal λ, the collection M_λ satisfies Property P_λ and contains a nondegenerate subcontinuum A_λ with diam $(A_\lambda) \geq \frac{1}{4a}$. Therefore, by Theorem 2.3, X is not Suslinian, which contradicts to the hypothesis on X.

For $\lambda = 0$, by Lemma 2.6, M_0 satisfies Property P_0. Since X is not degenerate, by Lemma 3.2, M_0 contains an element A_0 with diam $(A_0) \geq \frac{1}{4a}$.

Now assume that for every $\alpha < \lambda$, the collection M_α satisfies Property P_α and contains a nondegenerate subcontinuum A_α with diam $(A_\alpha) \geq \frac{1}{4a}$. We need to show that M_λ satisfies Property P_λ and contains an subcontinuum A_λ with diam $(A_\lambda) \geq \frac{1}{4a}$. We discuss the following two cases.

Case 1. $\lambda = \alpha + 1$.

First we show that there exists $A_\lambda \in M_\lambda$ with $\mathrm{diam}(A_\lambda) \geq \frac{1}{4a}$. Pick an element A_α of M_α with diam $(A_\alpha) \geq \frac{1}{4a}$. Since M_α satisfies Property P_α, by Lemma 2.9, for sufficiently large n, there exists a family $\mathscr{K}_n$ of pairwise disjoint subcontinua of A_α in M_α with $|\mathscr{K}_n| \geq \sqrt{a^n/6}$ and $\mathrm{diam}(K) \geq a^{-n}$ for every $K \in \mathscr{K}_n$. By Lemma 3.2, for every $K \in \mathscr{K}_n$, there exists $g \in G$ with $|g| \leq n$ such that $\mathrm{diam}(gK) \geq \frac{1}{4a}$.

By the pigeonhole principle, we have

Claim. For every sufficiently large n, there exists $g_n \in G$ with $|g_n| \leq n$ such that

$$\#\{K \in \mathscr{K}_n : \mathrm{diam}(g_n K) \geq \frac{1}{4a}\} \geq a_n := \frac{\sqrt{a^n/6}}{(\sqrt[3]{a})^n} = \frac{a^{n/6}}{\sqrt{6}}.$$

Take a decreasing sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ of positive real numbers converging to zero. Consider the compact metric space M_α and ε_k for every $k \in \mathbb{N}$. Then we obtain $n_k = n(\varepsilon_k, k) \geq k$ as in Lemma 3.1.

Since a_n tends to the infinity, we may take an increasing sequence $\{m_k\}_{k\in\mathbb{N}}$ of positive integers such that both $a_{m_k} \geq n_k$ and $m_k \geq n_k$ hold for each $k \in \mathbb{N}$. By Claim, as $k \in \mathbb{N}$ is large enough, we can find $h_k \in G$ with $|h_k| \leq m_k$ and pairwise disjoint nondegenerate subcontinua $B_1, B_2, \ldots, B_{n_k}$ of A_α in $\mathscr{K}_n$ such that for each $i = 1, 2, \ldots, n_k$, we have

$$\mathrm{diam}(h_k(B_i)) \geq \frac{1}{4a}. \tag{3.1}$$

In this way, we obtain n_k pairwise disjoint subcontinua of A_α in $\mathscr{K}_n$ with a uniform lower bound on diameters. On the other hand, by the choice of n_k, there exist $B(k) \in M_\alpha$ and

$1 \le i_1 < i_2 < \cdots < i_k \le n_k$ such that

$$d_H(B(k), h_k(B_{i_j})) < \varepsilon_k, \text{ for each } j = 1,2,\ldots,k. \tag{3.2}$$

By the compactness of M_α, we may assume that $B(k)$ converges to a point $A_{\alpha+1}$ of M_α. By (3.1) and (3.2) we have $A_{\alpha+1} \in \widetilde{M_\alpha} = M_{\alpha+1} = M_\lambda$ and $\operatorname{diam}(A_{\alpha+1}) \ge \frac{1}{4a}$.

Now we show that $M_{\alpha+1}$ satisfies Property $P_{\alpha+1}$. Let $C \in M_{\alpha+1}$ be nondegenerate. Suppose that U and V are open subsets of X such that $\overline{V} \subset U$, $C \cap V \ne \emptyset$ and $C \setminus \overline{U} \ne \emptyset$. We need to find $D \in M_{\alpha+1}$ such that $D \subset C \cap \overline{U}$, $D \cap \overline{V} \ne \emptyset$, and $D \cap Bd(U) \ne \emptyset$.

Since C is an element of $M_{\alpha+1}$, for each $k \in \mathbb{N}$, there exist pairwise disjoint nondegenerate subcontinua $D_1, D_2, \ldots, D_{n_k} \in M_\alpha$ such that

$$d_H(C, D_i) < \varepsilon_k$$

for each $i = 1,2,\ldots,n_k$. Since M_α satisfies Property P_α, as k is large enough, for each $i = 1,2,\ldots,n_k$, there exists $E_i \in M_\alpha$ such that $E_i \subseteq D_i \cap \overline{U}$, $E_i \cap \overline{V} \ne \emptyset$ and $E_i \cap Bd(U) \ne \emptyset$. By Lemma 3.1, there exists $E(k) \in M_\alpha$ and $1 \le i_1 < i_2 < \cdots < i_k \le n_k$ such that

$$d_H(E(k), E_{i_j}) < \varepsilon_k$$

for each $j = 1,2,\ldots,k$. Furthermore, we may assume that $\{E(k)\}$ converges to a point D of M_α. It follows that $D \subset C \cap \overline{U}$, $D \cap \overline{V} \ne \emptyset$ and $D \cap Bd(U) \ne \emptyset$. Since $D \in M_{\alpha+1}$, we have that $M_{\alpha+1}$ satisfies Property $P_{\alpha+1}$.

Case 2. λ is a limit ordinal.

Take a sequence $\alpha_1 < \alpha_2 < \cdots$ of countable ordinals such that $\lim_{i\to\infty} \alpha_i = \lambda$. By the inductive assumption, there exists $A_i \in M_{\alpha_i}$ such that $\operatorname{diam}(A_i) \ge \frac{1}{4a}$ for every $i \in \mathbb{N}$. Furthermore, we may assume that $\{A_i\}$ converges to a point A_λ of $C(X)$. It follows that $A_\lambda \in \bigcap_{\alpha<\lambda} M_\alpha = M_\lambda$ and $\operatorname{diam}(A_\lambda) \ge \frac{1}{4a}$. Running the same argument as in Case 1, applying Lemmas 2.9 and 3.1, we conclude that M_λ satisfies Property P_λ.

□

4. Appendix

In this appendix, following the same argument of [15, Lemma 4.4], we give the proof of Lemma 3.2 in the framework of general group actions. Throughout this section, we fix an expansive action $G \curvearrowright X$ by a finitely generated group G on a compact metric space (X,d). Let $c > 0$ be an expansive constant for the action $G \curvearrowright X$.

Lemma 4.1. *For any $\varepsilon > 0$, there is an integer $n = n(\varepsilon) > 0$ such that for any $x, y \in X$ satisfying $d(x,y) \ge \varepsilon$, we have*

$$\max_{g\in G, |g|\le n} d(gx, gy) \ge c.$$

Proof. Assume the conclusion is false. Then there exist $\varepsilon_0 > 0$ and $x_k, y_k \in X$ for every $k \in \mathbb{N}$ satisfying that

$$d(x_k, y_k) \ge \varepsilon_0, \text{ and } \max_{g\in G, |g|\le k} d(gx_k, gy_k) < c.$$

By the compactness of X, we may assume that x_k and y_k converge to some x and y of X respectively. It follows that $d(x,y) \ge \varepsilon_0$ and $\sup_{g\in G} d(gx, gy) \le c$, which contradicts to the expansivity. □

From Lemma 4.1, there exists an integer $l > 0$ such that for any $x, y \in X$ satisfying $d(x,y) \geq \frac{c}{2}$, we have $\max_{g\in G,|g|\leq l} d(gx,gy) \geq c$. Fix a real number $a > 1$ such that $a^l < 2$.

For $x, y \in X$, consider the following minimal time witnessing the expansivity of x and y defined as

$$n(x,y) = \min\{n \in \mathbb{N} : d(gx,gy) \geq c \text{ for some } g \in G \text{ with } |g| \leq n\},$$

if $x \neq y$ and $n(x,y) = +\infty$ otherwise. It induces a function ρ on $X \times X$ via $\rho(x,y) = a^{-n(x,y)}$.

Lemma 4.2. *The function ρ satisfies the following:*

(1) $\rho(x,y) = \rho(y,x)$ *for any* $x, y \in X$*;*
(2) $\rho(x,y) \geq 0$ *and* $\rho(x,y) = 0$ *if and only if* $x = y$*;*
(3) $\rho(x,z) \leq 2\max\{\rho(x,y),\ \rho(y,z)\}$ *for any* $x, y, z \in X$*;*
(4) *If* $d(x_k,x) \to 0$ *and* $d(y_k,y) \to 0$*, then*

$$\limsup_{k\to\infty} \rho(x_k,y_k) \leq \rho(x,y);$$

(5) ρ *is compatible with the topology of* X*. That is, for all* $r > 0$ *and* $x \in X$*, the balls*

$$B_\rho(x,r) = \{y \in X :\ \rho(x,y) < r\}$$

form an open base of the topology of X*.*

Proof. (1) and (2) are clear. To prove (3), we may assume $x \neq z$. Write $m = n(x,z)$. Then there exists $g \in G$ with $|g| \leq m$ such that $d(gx,gz) \geq c$. By the triangle inequality, we may assume $d(gx,gy) > c/2$. By the choice of l, there exists $h \in G$ with $|h| \leq l$ such that $d(hgx,hgy) \geq c$. Thus $n(x,y) \leq |hg| \leq m+l$. By the choice of a we have

$$\rho(x,y) = a^{-n(x,y)} \geq a^{-m}a^{-l} > \frac{\rho(x,z)}{2}.$$

(4). It suffices to show $\liminf_{k\to\infty} n(x_k,y_k) \geq n(x,y)$. Without loss of generality we may assume that $\liminf_{k\to\infty} n(x_k,y_k) = m < \infty$. Furthermore, we may assume that $n(x_k,y_k) \leq m$ for every $k \in \mathbb{N}$. By definition, this means that for every $k \in \mathbb{N}$, there exists $h_k \in G$ with $|h_k| \leq m$ such that $d(h_kx_k,h_ky_k) \geq c$. Thus there exists $h \in G$ with $|h| \leq m$ and two subsequences $\{x_{k_i}\}_i$ and $\{y_{k_i}\}_i$ satisfying $d(hx_{k_i},hy_{k_i}) \geq c$ for every $i \in \mathbb{N}$.

Since $d(x_k,x) \to 0$ and $d(y_k,y) \to 0$, we have

$$d(hx,hy) = \lim_{i\to\infty} d(hx_{k_i},hy_{k_i}) \geq c.$$

This implies that $n(x,y) \leq m = \liminf_{k\to\infty} n(x_k,y_k)$.

(5). From (4), each $B_\rho(x,r)$ is open under the topology of X. By Lemma 4.1, for any $x \in X$ and $R > 0$ there exists $r > 0$ satisfying $B_\rho(x,r) \subseteq B_d(x,R)$. It follows that those $B_\rho(x,r)$'s form a base of the topology of X. □

We can use the following Frink's metrization lemma to obtain a compatible metric [6, pp.134-135].

Lemma 4.3. [15, Theorem 4.1] *Let ρ be the function as above. Consider the function D defined as*

$$D(x,y) = \inf \sum_{i=0}^{n-1} \rho(x_i,x_{i+1}),$$

for all $x_0, x_1, \ldots, x_n \in X$ with $x_0 = x$ and $x_n = y$. Then for every $x, y \in X$ we have

$$\frac{1}{4}\rho(x,y) \leq D(x,y) \leq \rho(x,y).$$

Now we are ready to prove Lemma 3.2.

Proof of Lemma 3.2. By Lemma 4.2 and Lemma 4.3, we obtain a compatible metric D on X such that

$$\frac{1}{4}\rho(x,y) \leq D(x,y) \leq \rho(x,y).$$

for every $x,y \in X$. Suppose that $D(x,y) \geq a^{-n}$ for some $x,y \in X$ and $n \in \mathbb{N}$. Since $D(x,y) \leq \rho(x,y)$, we have $\rho(x,y) \geq a^{-n}$. By the definition of ρ, there exists $h \in G$ such that $|h| \leq n$ and $d(hx,hy) \geq c$. It follows that $n(hx,hy) = 0$ and hence $\rho(hx,hy) > \frac{1}{a}$. Therefore,

$$\max_{g\in G,|g|\leq n} D(gx,gy) \geq D(hx,hy) \geq \frac{\rho(hx,hy)}{4} > \frac{1}{4a}$$

as desired. □

Remark 4.4. Recall that a continuous action $G \curvearrowright X$ is *continuum-wise expansive* if there exists a constant $c > 0$ such that $\sup_{g\in G} \mathrm{diam}(gK) > c$ for any nondegenerate subcontinuum K of X. Adapting the proof of Lemma 3.2, we can obtain a continuum-wise expansive version of Proposition 3.2. In this way, we can improve Theorem 1.3 into the continuum-wise expansive setting.

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