Topology and its Applications 159 (2012) 2662–2668

Contents lists available at SciVerse ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Free subgroups of dendrite homeomorphism group ☆

Enhui Shi

Department of Mathematics, Soochow University, Suzhou, Jiangsu 215006, PR China

ARTICLE INFO

Article history:
Received 28 August 2011
Received in revised form 3 March 2012
Accepted 1 April 2012

MSC:
54F50
37B05

Keywords:
Homeomorphism group
Dendrite
Free group
Ping-pong game
Quasi-Schottky group

ABSTRACT

An action of a group G on a topological space X is called minimal if for every point $x \in X$, the orbit Gx of x is dense in X. A connected and locally connected compact metric space which contains no simple closed curve is called a dendrite. In this paper, it is shown that if a group G acts minimally on a nondegenerate dendrite, then G must contain a free noncommutative subgroup. This is an extension of a Margulis' theorem for minimal group actions on the circle.

1. Introduction

Let X be a topological space, Homeo(X) the homeomorphism group of X and G a group. Recall that a group homomorphism $\varphi : G \to \text{Homeo}(X)$ is called an ***action*** of G on X. The action φ is said to be ***minimal*** if for every point $x \in X$ the orbit $Gx \equiv \{\varphi(g)(x)\colon\ g \in G\}$ is dense in X.

It is proved by G. Margulis that if a group G acts on the circle S^1 minimally, then either G contains a free noncommutative subgroup or there is a G-invariant probability measure on S^1 (see [7, Theorem 2]). This theorem solved a conjecture proposed by É. Ghys which can be viewed as a replacement of the well-known Tits alternative theorem for Homeo(S^1). One may see [10] or [13] for a different proof of this theorem given by Ghys.

Remark. (1) Tits alternative theorem says that if Γ is a finitely generated linear group, then either Γ contains a noncommutative free subgroup or Γ is virtually solvable (see [18]). Nevertheless the exact analogue of the Tits alternative is false for subgroups of the group Homeo(S^1) and false even for subgroups of the group of C^∞-diffeomorphisms of S^1 (see [12]).

(2) Margulis' proof used classical methods of boundary theory first introduced by Furstenberg (see [5]). For minimal actions, it was later shown that, up to a finite cover, the circle is a boundary of the acting group provided it is not Abelian [4].

Recall that a ***Kleinian group*** is a group Γ acting freely and properly discontinuously on hyperbolic 3-space, $\mathbf{H}^3$. It is well known that the action of Γ on the limit set $\Lambda(\Gamma) \subset \partial\mathbf{H}^3$ is minimal (see [14, p. 601]). In some cases, $\Lambda(\Gamma)$ is known to be a dendrite and its Hausdorff dimension has been calculated by some authors (see e.g. [2,8,9]). These facts motivate us to study the question: Does the above theorem proved by Margulis hold for the groups acting on a dendrite minimally?

☆ The work is supported by Qing Lan Project of Jiangsu province and by NSF grants of Jiangsu province (BK2011275).
E-mail address: ehshi@suda.edu.cn.

0166-8641/$ – see front matter
http://dx.doi.org/10.1016/j.topol.2012.04.001

In fact, the author and his collaborators have shown that if a group G acts on a nondegenerate dendrite X minimally, then X admits no G-invariant probability measure (see [17]). In this paper we prove further the following theorem.

Main Theorem. *If a group G acts on a nondegenerate dendrite X minimally, then G must contain a noncommutative free subgroup.*

The strategy of the proof is the same as Margulis' in [7]. Nevertheless, the expanding-contracting behavior of compositions is difficult to handle in this situation. (For the case of the circle, this is rather elementary thanks to the circular order.) So we have to develop some technical ingredients to overcome this difficulty.

We should notice that the homeomorphism group of a dendrite X may have many free noncommutative subgroups even for X being the interval $[0,1]$. Indeed, it can be shown by exactly the same proof as in [13, Prop. 4.5] that, for a generic set of pairs (f,g) of elements of the orientation preserving group $\mathrm{Homeo}_+([0,1])$, the group generated by (f,g) is a free noncommutative group. In addition, there are also some big subgroups of $\mathrm{Homeo}_+([0,1])$ which have no free noncommutative subgroup such as the group $\mathrm{PL}_+([0,1])$ of piecewise linear homeomorphisms of $[0,1]$ (see [3] or [13, Theorem 4.6]). Clearly, this is not contradict to the main theorem in this paper, since any group action on the interval $[0,1]$ cannot be minimal.

For a group action $\varphi: G \to \mathrm{Homeo}(X)$, we often use the symbols gx or $g(x)$ instead of $\varphi(g)(x)$ throughout the paper for convenience.

2. Dendrite

In this section, we will recall and prove some properties of dendrites which will be used in the following.

Recall that a ***continuum*** is a connected compact metric space. If a continuum X is not a single point, then X is called ***nondegenerate***. If X is a locally connected continuum and contains no simple closed curve, then X is called a ***dendrite***. Clearly X is a dendrite if and only if for any two points $x, y \in X$ there is a unique arc $[x,y] \subset X$ connecting x and y. It is known that each point of a dendrite is either a cut point or an endpoint and every subcontinuum of a dendrite is also a dendrite.

The following lemma is taken from [15].

Lemma 2.1. *([15, 10.28]) If X is a nondegenerate dendrite, then X can be written as follows*:

$$X = \mathrm{End}(X) \cup \left(\bigcup_{i=1}^{\infty} A_i \right)$$

where $\mathrm{End}(X)$ *is the endpoint set of X and each A_i is an arc with endpoints p_i and q_i such that*

$$A_{i+1} \cap \left(\bigcup_{j=1}^{i} A_j \right) = \{p_{i+1}\} \quad \textit{for each } i = 1, 2, \ldots$$

and $\mathrm{diam}(A_i) \to 0$ *as* $i \to \infty$.

Let X be a dendrite with metric d'. From Lemma 2.1, we may write $X = \mathrm{End}(X) \cup (\bigcup_{i=1}^{\infty} A_i)$. Fix h_i to be a homeomorphism from the unit interval $[0,1]$ to A_i, for each $i = 1, 2, \ldots$. For $x, y \in [0,1]$, the symbol $\rho([x,y])$ will denote the length of the arc $[x,y]$ under the Euclidean metric on $[0,1]$, i.e., $\rho([x,y]) = |x-y|$. For any $a, b \in X$, define

$$d(a,b) = \sum_{i=1}^{\infty} \frac{1}{2^i} \rho\big(h_i^{-1}\big([a,b] \cap A_i\big)\big). \tag{2.1}$$

Define a probability measure μ on X by

$$\mu(A) = \sum_{i=1}^{\infty} \frac{1}{2^i} \nu\big(h_i^{-1}(A \cap A_i)\big) \tag{2.2}$$

for all Borel subsets A of X, where ν is the Lebesgue measure on $[0,1]$.

Clearly the new metric d on X defined in (2.1) is topologically equivalent to d', since X is compact. The following lemma is easily seen by the definitions of d and μ.

Lemma 2.2. *The probability measure μ on X defined in* (2.2) *satisfies that for each arc $[a,b]$ in X, $\mu([a,b]) = d(a,b)$.*

From Lemma 2.1 and Lemma 2.2, we can easily deduce the following

Proposition 2.3. *Let X be a dendrite with metric d and measure μ defined above and let D_i be a sequence of subdendrites of X. If $\lim_{i\to\infty}\mu(D_i)=0$ then $\lim_{i\to\infty}\operatorname{diam}(D_i)=0$.*

Let us recall some definitions following Mai and the author in [6] or [16]. Let A be an arc, $\operatorname{End}(A)$ the set of two endpoints of A, and $\overset{\circ}{A}=A-\operatorname{End}(A)$.

For a dendrite X and an arc A in X, define

$$X(A)=A\cup\Big(\bigcup\{Y\colon Y \text{ is a component of } X-A, \text{ and } \overline{Y}\cap\overset{\circ}{A}\neq\emptyset\}\Big).$$

$X(A)$ is called the ***subdendrite of*** X ***strung by*** A, and A is called the ***trunk*** of $X(A)$.

The following properties can be deduced directly from the above definitions.

Lemma 2.4. *Let $X(A)$ be a subdendrite of X strung by A.*

(a) *$X(A)-\operatorname{End}(A)$ is open in X.*
(b) *If A' is a subarc of A, then $X(A')\subset X(A)$.*
(c) *If $f: X\to X$ is a homeomorphism, then $f(X(A))=X(f(A))$.*

Lemma 2.5. *Let X be a nondegenerate dendrite with metric d and $[a,b]$ an arc in X. If $\operatorname{diam}(X([a,b]))>\varepsilon$, then there is an arc $[c,d]\subset X(A)$ such that $X([c,d])\subset X([a,b])$ and $d(c,d)=\varepsilon/3$.*

Proof. If $\operatorname{diam}([a,b])>\varepsilon/3$, then we can select $c,d\in[a,b]$ such that $d(c,d)=\varepsilon/3$. Clearly $X([c,d])\subset X([a,b])$. Otherwise, by the triangular inequality, there must be some $x\in X([a,b])$ such that $d(x,y)>\varepsilon/3$ where y is the (unique) point in $[a,b]$ such that $[a,b]\cap[a,x]=[a,y]$. Now select $c,d\in[x,y]$ such that $d(c,d)=\varepsilon/3$, then $X([c,d])\subset X([a,b])$. □

The following lemma is well known in continuum theory and is a direct corollary of [15, 8.30].

Lemma 2.6. *Let X be a dendrite with metric d. Then for any $\varepsilon>0$, there is a $\delta=\delta(\varepsilon)$ such that for any $x,y\in X$ with $0<d(x,y)\leqslant\delta$, the diameter $\operatorname{diam}([x,y])<\varepsilon$.*

Lemma 2.7. *Let X be a nondegenerate dendrite with metric d. Then for every $\varepsilon>0$ there is a $\delta>0$ such that, for any arcs $[a,b]$ and $[c,d]$ in X with $d(a,b)=d(c,d)=\varepsilon$, $d(a,c)<\delta$ and $d(b,d)<\delta$, we have $[a',b']\subset[a,b]\cap[c,d]$, where a' and b' are points in $[a,b]$ such that $d(a',a)=d(b,b')=\varepsilon/3$.*

Proof. For any $\varepsilon>0$, from Lemma 2.6, there is a $\delta>0$ such that $\operatorname{diam}([x,y])<\varepsilon/3$ whenever $d(x,y)<\delta$. So if $[a,b]$ and $[c,d]$ are two arcs such that $d(a,b)=d(c,d)=\varepsilon$, $d(a,c)<\delta$ and $d(b,d)<\delta$, then $\operatorname{diam}([a,c])<\varepsilon/3$ and $\operatorname{diam}([b,d])<\varepsilon/3$. Let $a',b'\in[a,b]$ be such that $d(a',a)=d(b,b')=\varepsilon/3$. By the uniquely arcwise connectedness of X, we have $[a',b']\subset[a,b]\subset[a,c]\cup[c,d]\cup[d,b]$. Since $[a',b']\cap[a,c]=\emptyset$ and $[a',b']\cap[d,b]=\emptyset$, we obtain $[a',b']\subset[c,d]$. Thus the proof is completed. □

3. Contractible neighborhood

First, let us recall some notions which were used by Margulis in [7] and many of the ideas of which are due to Furstenberg. Let a group G act on a compact metric space (X,d) and let $\mathcal{M}(X)$ denote the set of all Borel probability measures on X with the standard weak topology. A set $F\subset X$ is called ***G-contractible*** if there is a sequence $\{g_n\}$ in G such that $\operatorname{diam}(g_nF)\to 0$ as $i\to\infty$. We say that a measure $\mu\in\mathcal{M}(X)$ is ***G-contractible***, if there exist a sequence $\{g_n\}$ in G and $x\in X$ such that $g_n\mu\to\delta_x$, where δ_x is the probability measure with support $\{x\}$. We say that the action of G on X is ***strongly ε-proximal*** if every measure $\mu\in\mathcal{M}(X)$ with $\operatorname{diam}(\operatorname{supp}(\mu))<\varepsilon$ is G-contractible.

The following lemma is clear from the compactness of X.

Lemma 3.1. *If every point $x\in X$ has a G-contractible neighborhood, then the action of G on X is strongly ε-proximal for some $\varepsilon>0$.*

The following lemma is a direct corollary of Proposition 1(ii) in [7].

Lemma 3.2. *Assume that the action of G on X is strongly ε-proximal. Then for any measure $\mu\in\mathcal{M}(X)$ there are a measure $\nu\in\mathcal{M}(X)$ with finite support and a sequence $\{g_n\}$ in G such that $g_n\mu\to\nu$ as $n\to\infty$.*

Recall that the action of G on X is called ***equicontinuous*** if for every $\varepsilon>0$ there is a $\delta>0$ such that $d(gx,gy)<\varepsilon$ for all $g\in G$, whenever $d(x,y)<\delta$. We say the action is ***sensitive*** if there is some $c>0$ such that for any nonempty open subset U of X, there is some $g\in G$ such that $\operatorname{diam}(gU)>c$. Such a constant c is called a ***sensitivity constant*** of G-action.

Now we will prove some dynamical properties of minimal group actions on nondegenerate dendrites.

Lemma 3.3. *Let a group G act on a nondegenerate dendrite (X,d) minimally. Then this action must be sensitive.*

Proof. First this action cannot be equicontinuous, otherwise X will be homogeneous (see [1, Chap. 3, Theorem 6]) and so X is a single point. This is a contradiction, since X is nondegenerate. So there is some $c>0$ such that for any natural number n, there exist $x_n, y_n \in X$ and $g_n \in G$ such that $d(x_n,y_n)<\frac{1}{n}$ and $d(g_nx_n,g_ny_n)>c$. Passing to a subsequence if necessary, we may suppose that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = z$ for some $z\in X$. For any nonempty open subset U of X, there is some $g\in G$ such that $gz\in U$ by the minimality of the action. Thus gx_n and gy_n belong to U for sufficiently large n. It follows that $\operatorname{diam}(g_ng^{-1}U)>c$ for sufficiently large n. So the action is sensitive. □

Proposition 3.4. *Let a group G act minimally on a nondegenerate dendrite X. Then for every $x\in X$ there is a contractible neighborhood U of x.*

Proof. From Lemma 3.3, we may suppose that c is a sensitivity constant of G-action. Now select a nondegenerate arc $[c_1,d_1]\subset X$ such that $\operatorname{diam}(X([c_1,d_1]))<1$. Write $Y_1=X([c_1,d_1])$. By the sensitivity of G-action, there is some $g_1\in G$ such that $\operatorname{diam}(g_1Y_1)>c$. By Lemma 2.5, there is a subarc $[a_1,b_1]\subset g_1Y_1$ such that $X([a_1,b_1])\subset g_1Y_1$ and $d(a_1,b_1)=c/3$. Write $X_1=X([a_1,b_1])$. Select a subarc $[c_2,d_2]\subset X$ such that $\operatorname{diam}(X([c_2,d_2]))<1/2$. Write $Y_2=X([c_2,d_2])$. By the sensitivity, there is a $g_2\in G$ such that $\operatorname{diam}(g_2Y_2)>c$. By Lemma 2.5, there is a subarc $[a_2,b_2]\subset g_2Y_2$ such that $X([a_2,b_2])\subset g_2Y_2$ and $d(a_2,b_2)=c/3$. Write $X_2=X([a_2,b_2])$. Continuing this process, we get a sequence of subdendrites $X_n=X([a_n,b_n])$, Y_n and $g_n\in G$ such that

$$d(a_n,b_n)=c/3,\quad g_n^{-1}(X_n)\subset Y_n \quad\text{and}\quad \operatorname{diam}(Y_n)<1/n. \tag{3.1}$$

For $\varepsilon=c/3$, let $\delta=\delta(\varepsilon)$ be as in Lemma 2.6. By the compactness of X, there is a sequence n_i such that $d(a_{n_i},a_{n_j})<\delta$ and $d(b_{n_i},b_{n_j})<\delta$ for all $i,j=1,2,3,\ldots$. It follows from Lemma 2.7 that $[a'_{n_1},b'_{n_1}]\subset[a_{n_i},b_{n_i}]$ for all $i=1,2,3,\ldots$, where $a'_{n_1},b'_{n_1}\in[a_n,b_n]$ satisfy $d(a_{n_1},a'_{n_1})=d(b_{n_1},b'_{n_1})=\varepsilon/3$. This together with Lemma 2.4(b) implies that $X([a'_{n_1},b'_{n_1}])\subset X([a_{n_i},b_{n_i}])(=X_{n_i})$ for all i. Then $\operatorname{diam}(g_{n_i}^{-1}X([a'_{n_1},b'_{n_1}]))<1/n_i$ from (3.1). Thus the subdendrite $X([a'_{n_1},b'_{n_1}])$ is contractible, which together with the minimality of G-action implies that every point of X has a contractible neighborhood. □

4. Proximality

Let a group G act on a compact metric space (X,d). Two points $x,y\in X$ are said to be ***proximal*** if there is a sequence $\{g_i\}$ such that $\lim_{i\to\infty} d(g_ix,g_iy)=0$. If x and y are not proximal, then we say x and y are ***distal***, that is, there is some $c>0$ such that $d(gx,gy)>c$ for all $g\in G$. We say the action of G on X is ***proximal*** if any two points $x,y\in X$ are proximal. An nonempty open subset U of X is said to be ***strongly proximal*** if every compact subset K of U is contractible.

Lemma 4.1. *Assume that a group G acts on a nondegenerate dendrite X minimally. Then the set $\{(x,y)\in X\times X\colon x,y \text{ are proximal}\}$ is open in $X\times X$.*

Proof. Let $x,y\in X$ be proximal. By Proposition 3.4, we can select a nonempty contractible open subset U of X. Since x, y are proximal and the G-action is minimal, there is some $g\in G$ such that $gx\in U$ and $gy\in U$. By the continuity of G-action, there are open neighborhoods U_x of x and U_y of y such that $gU_x\subset U$ and $gU_y\subset U$. Since U is contractible, we see that u and v are proximal for any $u\in U_x$ and $v\in U_y$. The proof is complete. □

Let y be a cut point of a nondegenerate dendrite X and $x\neq y\in X$. We use the symbol $U_y(x)$ to denote the connected component of $X\setminus\{y\}$ containing x. If the number of connected components of $X\setminus\{y\}$ is two, then y is called a 2-***cut point***. It is well known that for any nondegenerate arc $[a,b]\subset X$, the set of 2-cut points in $[a,b]$ is dense in $[a,b]$.

Lemma 4.2. *Let a group G act minimally on a nondegenerate space X. If $x,y\in X$ are distal and every point in the open arc $]x,y[$ ($\equiv[x,y]\setminus\{x,y\}$) is proximal to x, then the connected component $U_y(x)$ of $X\setminus\{y\}$ is strongly proximal.*

Proof. Since x, y are distal, there is some $c>0$ such that $d(hx,hy)>c$ for all $h\in G$. For any given $0<\varepsilon<c$, select a 2-cut point $w\in X$ such that one of the components, say W, of $X\setminus\{w\}$ has $\operatorname{diam}(W)<\varepsilon$ (it is not difficult to see that this can always be done). For any compact subset K of $U_y(x)$, select a 2-cut point $v\in]x,y[$ which is sufficiently close to y such that the connected component $U_v(x)$ contains K and x, v are proximal (see Lemma 4.1). Since the G-action is minimal, there is some $g\in G$ such that $g([x,v])\subset W$. In particular, $gx\in W$. This implies that $gy\notin W$, because $\operatorname{diam}(W)<\varepsilon$ and $d(gx,gy)>c>\varepsilon$. It follows that $g(U_v(x))\subset W$, which means that $g(K)\subset W$. So $\operatorname{diam}(gK)<\varepsilon$. By the arbitrariness of ε and K, we obtain that $U_y(x)$ is strongly proximal. Thus we complete the proof. □

Proposition 4.3. *Assume that a group G acts on a nondegenerate dendrite X minimally. Then this action is proximal.*

Proof. Assume to the contrary that there are two points $x, y \in X$ which are distal. According to Lemma 4.1 and Proposition 3.4, we may suppose that every point $v \in]x, y[$ is proximal to x. By the minimality of G-action, it is easy to see that the arc $[x, y]$ is nowhere dense in X, and so, there is some $g \in G$ such that $g(y) \in U_y(x) \setminus [x, y]$. (Recall that $U_y(x)$ is the connected component of $X \setminus \{y\}$ containing x.) Since $U_y(x)$ is strongly proximal by Lemma 4.2 and gx, gy are distal, we must have $g(x) \notin U_y(x)$. It follows that $y \in [gx, gy]$ and then $[x, y] \subset U_{gy}(gx)$ (noting that $g(y) \notin [x, y]$). But $U_{gy}(gx)$ is still strongly proximal, which contradicts the distality of x and y. □

5. Free subgroup

In this section, we shall prove the main result of the paper.

Lemma 5.1. *Assume that a group G acts on a nondegenerate dendrite X minimally and μ is the probability measure defined as in (2.2). Then there are an endpoint $x \in \mathrm{End}(X)$ and a sequence $g_i \in G$ such that $g_i\mu \to \delta_x$ as $i \to \infty$, where δ_x is the Dirac measure with support x.*

Proof. From Lemma 3.1 and Proposition 3.4, we see that the G-action on X is strongly ε-proximal for some $\varepsilon > 0$. Then from Lemma 3.2 we obtain that there is some measure $\nu \in \overline{G\mu} \subset \mathcal{M}(X)$ with finite support. Since the G-action is proximal by Proposition 4.3, we get further that there is a Dirac measure $\delta_x \in \overline{G\nu}$ and we can take x to be an endpoint of X by the minimality of G-action. Since $\delta_x \in \overline{G\nu} \subset \overline{G\mu}$, there exists a sequence $g_i \in G$ such that $g_i\mu \to \delta_x$ as $i \to \infty$. Thus we complete the proof. □

Proposition 5.2. *Assume that a group G acts on a nondegenerate dendrite X minimally. Then there are two endpoints $x \neq y \in \mathrm{End}(X)$ and a sequence $g_i \in G$ such that for any subdendrite K of $X \setminus \{x\}$ we have $g_i(K) \to y$ and for any subdendrite K of $X \setminus \{y\}$ we have $g_i^{-1}(K) \to x$ as $i \to \infty$.*

Proof. Let μ be the probability measure defined as in (2.2). From Lemma 5.1, there exists a sequence $h_i \in G$ such that $h_i\mu \to \delta_x$ $(i \to \infty)$ for some $x \in \mathrm{End}(X)$. Thus, for any subdendrite $K \subset X \setminus \{x\}$, we have $\mu(h_i^{-1}K) = h_i\mu(K) \to 0$ as $i \to \infty$. It follows from Proposition 2.3 that

$$\mathrm{diam}\big(h_i^{-1}K\big) \to 0 \quad \text{as } i \to \infty. \tag{5.1}$$

Now choose an endpoint $y \neq x$. Suppose x_n is a sequence in the arc $[x, y]$ such that $x_n \to x$ as $n \to \infty$. Let $K_n = X([y, x_n])$ be the subdendrite of X strung by $[y, x_n]$ (see the definition before Lemma 2.4). Passing to a subsequence if necessary, we may suppose that $K_1 \subset K_2 \subset \cdots$, and $h_i^{-1}K_1 \to z$ for some $z \in X$ by (5.1). Since $K_1 \subset K_n$ for all $n \geqslant 2$ and $\lim_{i\to\infty} \mathrm{diam}(h_i^{-1}K_n) = 0$ by (5.1), we obtain further that $h_i^{-1}K_n \to z$ $(i \to \infty)$ for all n. By the minimality of G-action, there is a sequence $r_n \in G$ such that $r_n z \to y$ as $n \to \infty$. Then for each K_n we may choose an i_n such that $h_{i_n}^{-1}K_n$ is sufficiently close to z that $d(r_n h_{i_n}^{-1}K_n, y) < 1/n$. Write $g_n = r_n h_{i_n}^{-1}$. Because $X \setminus \{x\} = \bigcup_{n=1}^{\infty} K_n$ and every subdendrite of $X \setminus \{x\}$ is contained in K_n for some n, we obtain that

$$g_n K \to y \ (n \to \infty) \quad \text{for all subdendrites } K \subset X \setminus \{x\}. \tag{5.2}$$

Choose a sequence $y_n \in [x, y]$ such that $y_n \to y$ and $X([x, y_n]) \subset X([x, y_{n+1}])$ for all n. Write $F_n = X([x, y_n])$ and $U_n = X \setminus F_n$. Then $X \setminus \{y\} = \bigcup_{n=1}^{\infty} F_n$ and the family $\{U_n\colon\ n = 1, 2, \ldots\}$ becomes a neighborhood base of y. It follows from (5.2) that for each n, there is an N such that $g_i K_n \subset U_n$ for all $i > N$. This implies that $g_i(K_n \cap g_i^{-1}F_n) = g_i K_n \cap F_n = \emptyset$ for all $i > N$. So $K_n \cap g_i^{-1}F_n = \emptyset$ for all $i > N$. Thus $g_i^{-1}F_n \to x$ as $i \to \infty$ for each n. Since $X \setminus \{y\} = \bigcup_{n=1}^{\infty} F_n$ and every subdendrite of $X \setminus \{y\}$ is contained in F_n for some n, we obtain that

$$g_i^{-1}K \to x \ (i \to \infty) \quad \text{for all subdendrites } K \subset X \setminus \{y\}. \tag{5.3}$$

Thus we complete the proof from (5.2) and (5.3). □

A group with two generators acting on a topological space X is called ***quasi-Schottky*** if there exist generators h_1, h_2 of H and nonempty disjoint open subsets U_1, U_2, V_1, V_2 and W of X such that for $i = 1, 2$,

$$h_i(U_i \cup U_{3-i} \cup V_{3-i} \cup W) \subset U_i \quad \text{and} \quad h_i^{-1}(V_i \cup V_{3-i} \cup U_{3-i} \cup W) \subset V_i.$$

The "ping-pong" argument of Tits shows that H is a free noncommutative group.

The following lemma is taken from [11].

Lemma 5.3. *A group H cannot be represented as a union of finite number of left cosets $h_i H_i$ of subgroups $H_i \subset H$ of infinite index.*

The following lemma can be easily deduced from Lemma 5.3 (see the proof of Theorem 1 in [7]).

Lemma 5.4. *Let a group G act on a nondegenerate dendrite minimally. Then for any two different points $x, y \in X$, there is a $g \in G$ such that x, y, gx, gy are pairwise different.*

Now we start to prove the main theorem of this paper.

Theorem 5.5. *If a group G acts on a nondegenerate dendrite X minimally, then G must contain a noncommutative free subgroup.*

Proof. From Proposition 5.2, there are a sequence $g_i \in G$ and two different endpoints $x, y \in X$ such that for any subdendrite K of $X \setminus \{x, y\}$ we have

$$g_i(K) \to x \quad \text{and} \quad g_i^{-1}(K) \to y \quad \text{as } i \to \infty. \tag{5.4}$$

From Lemma 5.4, there is a $g \in G$ such that x, y, gx, gy are pairwise different. Write $h_i = g g_i g^{-1}$ for all i. Then from (5.4) we have that for each subdendrite K of $X \setminus \{gx, gy\}$,

$$h_i(K) \to gx \quad \text{and} \quad h_i^{-1}(K) \to gy \quad \text{as } i \to \infty. \tag{5.5}$$

Let U_1, V_1, U_2, V_2 be some connected open neighborhoods of x, y, gx, gy respectively such that the closures $\overline{U}_1$, $\overline{V}_1$, $\overline{U}_2$, $\overline{V}_2$ are pairwise disjoint. Choose an open set $W \subset X$ such that $\overline{W} \cap \overline{U}_i = \emptyset$ and $\overline{W} \cap \overline{V}_i = \emptyset$ for $i = 1, 2$. Then from (5.4) and (5.5) we obtain that there is a sufficiently large i_0 such that

$$g_{i_0}(U_1 \cup U_2 \cup V_2 \cup W) \subset U_1, \qquad g_{i_0}^{-1}(U_2 \cup V_1 \cup V_2 \cup W) \subset V_1,$$

and

$$h_{i_0}(U_1 \cup U_2 \cup V_1 \cup W) \subset U_2, \qquad h_{i_0}^{-1}(U_1 \cup V_1 \cup V_2 \cup W) \subset V_2.$$

It follows that the subgroup $\langle g_{i_0}, h_{i_0} \rangle$ of G generated by g_{i_0} and h_{i_0} is quasi-Schottky. Thus $\langle g_{i_0}, h_{i_0} \rangle$ is a noncommutative free group and this completes the proof. □

Clearly, circle and dendrites are both locally connected one-dimensional continua. It is natural to ask whether Margulis's conclusion holds for such more general spaces. Namely, we have the following

Question A. Let X be a locally connected one-dimensional continuum and let a group G act on X minimally. Is it true that either G contains a free noncommutative subgroup or there is a G-invariant probability measure on X?

Acknowledgement

The author would like to thank the referee for his helpful suggestions, especially for pointing out Question A to us.

References

[1] J. Auslander, Minimal Flows and Their Extensions, North-Holl. Math. Stud., vol. 153, North-Holland, 1988.
[2] B.H. Bowditch, Hausdorff dimension and dendritic limit sets, Math. Ann. 332 (2005) 667–676.
[3] M. Brin, C. Squier, Groups of piecewise linear homeomorphsims of the real line, Invent. Math. 79 (1985) 485–498.
[4] B. Deroin, V. Kleptsyn, A. Navas, Sur la dynamique unidimensionnelle en régularité intermédiaire, Acta Math. 199 (2007) 199–262.
[5] H. Furstenberg, Boundary theory and stochastic processes on homogeneous spaces, Proc. Sympos. Pure Math. 26 (1973) 193–229.
[6] J.H. Mai, E.H. Shi, The nonexistence of expansive commutative group actions on Peano continua having free dendrites, Topology Appl. 155 (2007) 33–38.
[7] G. Margulis, Free subgroups of the homeomorphism group of the circle, C. R. Math. Acad. Sci. Paris 331 (2000) 669–674.
[8] Y.N. Minsky, Teichmüller geodesics and ends of hyperbolic 3-manifolds, Topology 32 (1993) 625–647.
[9] Y.N. Minsky, On rigidity, limit sets, and ends of hyperbolic 3-manifolds, J. Amer. Math. Soc. 7 (1994) 539–588.
[10] A. Navas, Groups of Circle Diffeomorphisms, Sociedade Brasileira de Matemática, 2007, English translation of the original Grupos de difeomorfismos del círculo.
[11] B. Neumann, Groups covered by permutable subsets, J. Lond. Math. Soc. 29 (1954) 236–248.
[12] É. Ghys, V. Sergiescu, Sur un groupe remarquable de difféomorhismes du cercle, Comment. Math. Helv. 62 (1987) 185–239.
[13] É. Ghys, Groups acting on the circle, Enseign. Math. 47 (2001) 329–407.
[14] J.G. Ratcliffe, Foundations of Hyperbolic Manifolds, Grad. Texts in Math., vol. 149, Springer, 2006.

[15] S.B. Nadler Jr., Continuum Theory, Marcel Dekker, Inc., New York, 1992.
[16] E.H. Shi, S.H. Wang, The ping-pong game, geometric entropy and expansiveness for group actions on Peano continua having free dendrites, Fund. Math. 203 (2009) 21–37.
[17] E.H. Shi, S.H. Wang, L.Z. Zhou, Minimal group actions on dendrites, Proc. Amer. Math. Soc. 138 (2010) 217–223.
[18] J. Tits, Free subgroups in linear groups, J. Algebra 20 (1972) 250–270.