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Graphs admitting transitive commutative group actions

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ABSTRACT

Let X be a compact metric space, and Homeo(X) be the group consisting of all homeomorphisms from X to X. A subgroup H of Homeo(X) is said to be transitive if there exists a point $x \in X$ such that $\{k(x)\colon k \in H\}$ is dense in X. In this paper we show that, if $X = G$ is a connected graph, then the following five conditions are equivalent: (1) Homeo(G) has a transitive commutative subgroup; (2) G admits a transitive $\mathbb{Z}^2$-action; (3) G admits an edge-transitive commutative group action; (4) G admits an edge-transitive $\mathbb{Z}^2$-action; (5) G is a circle, or a k-fold loop with $k \geqslant 2$, or a k-fold polygon with $k \geqslant 2$, or a k-fold complete bigraph with $k \geqslant 1$. As a corollary of this result, we show that a finite connected simple graph whose automorphism group contains an edge-transitive commutative subgroup is either a cycle or a complete bigraph.

1. Introduction

Let (X, d) be a metric space, and let Homeo(X) be the group consisting of all homeomorphisms from X to X. A homomorphism $\varphi : P \to \text{Homeo}(X)$ from any group P to Homeo(X) is called a ***group action*** (or, more precisely, a P-***action***) on X, and the homomorphism φ is called a ***commutative group action*** if the group P is commutative. Let $\varphi : P \to \text{Homeo}(X)$ be a given group homomorphism, and let $H = \varphi(P)$. For any $x \in X$, the set $H(x) \equiv \{h(x)\colon h \in H\}$ is called the ***orbit*** of x under the group action φ (or, under the subgroup H of Homeo(X)). φ and H are said to be (topologically) ***transitive*** if there is a point $x \in X$ such that the orbit $H(x)$ is dense in X. For any $W \subset X$, write $H(W) = \bigcup_{x \in W} H(x)$. If $H(W) = W$, then W is called an ***invariant set*** of φ (or of H). For any $\{h_1, \ldots, h_n\} \subset \text{Homeo}(X)$ ($h_1, \ldots, h_n$ need not be mutually different), let $\langle h_1, \ldots, h_n \rangle$ denote the subgroup of Homeo(X) generated by $\{h_1, \ldots, h_n\}$. For $i = 1, \ldots, n$, write $v_{ni} = (\delta_{1i}, \delta_{2i}, \ldots, \delta_{ni})$, where $\delta_{ki} = 0$ if $k \neq i$ and $\delta_{ki} = 1$ if $k = i$. It is easy to see that $h_1, \ldots, h_n$ are mutually commutative homeomorphisms if and only if there exists a unique $\mathbb{Z}^n$-action $\varphi : \mathbb{Z}^n \to \text{Homeo}(X)$ such that $\varphi(\mathbb{Z}^n) = \langle h_1, \ldots, h_n \rangle$ and $\varphi(v_{ni}) = h_i$ for $i = 1, \ldots, n$. An $h \in \text{Homeo}(X)$ is called a ***transitive homeomorphism*** if the group $\langle h \rangle = \{h^n\colon n \in \mathbb{Z}\}$ is transitive. Thus, a metric space X admits a transitive homeomorphism if and only if X admits a transitive $\mathbb{Z}$-action.

In the study of dynamical systems, transitivity of group actions is an important notion, and is discussed by many authors. For example, Cairns et al. [3] introduced the notion of chaotic group action, which is similar to the notion of chaotic maps raised by Devaney [7]. In the definition of chaotic group action (and of chaotic maps), transitivity is a necessary condition. In

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[3] and [4], some chaotic group actions on manifolds were constructed. Cairns et al. [6] discussed the transitivity of solvable group actions on the line, and they proved that each non-cyclic poly-cyclic infinite group has a faithful transitive action on the line.

Some notions stronger than transitivity (such as, minimality, weak mixing, k-transitivity, etc.) are also deeply discussed in the framework of group actions. For example, a famous theorem due to Furstenberg says that, for commutative group actions, weak mixing implies k-transitivity for any integer $k \geqslant 1$, see [8] or [10]. Also a structure theorem of minimal group actions on compact metric spaces was established by Furstenberg, Ellis, etc., see [1]. One may consult [5] for the detailed arguments of relations between all types of transitivity for group actions.

It is well known that there are many metric spaces admitting transitive homeomorphisms, such as the Cantor set, the circle, the plane, the disk, the n-dimensional torus, etc. A famous transitive homeomorphism of the plane was constructed by Besicovitch [2]. There are also many metric spaces admitting no transitive homeomorphism, even admitting no transitive group action. For example, it is easy to show that no connected graph except the circle admits a transitive homeomorphism, and that the space $X = ([0,1] \times [0,1]) \cup ([1,2] \times \{0\}) \subset \mathbb{R}^2$ admits no transitive group action.

Therefore, there is an interesting question: Given a metric space X which admits no transitive homeomorphism, does X admit a transitive $\mathbb{Z}^n$-action, or admit a transitive commutative group action? In this paper we will study this question for X being a graph. All graphs in this paper are compact as topological spaces, or equivalently, finite as combinatorial graphs. Our main result is the following theorem.

Theorem 4.2. *For any connected graph G, the following conditions are equivalent.*

(1) *G admits a transitive commutative group action.*
(2) *G admits a transitive $\mathbb{Z}^2$-action.*
(3) *G admits an edge-transitive commutative group action.*
(4) *G admits an edge-transitive $\mathbb{Z}^2$-action.*
(5) *G is a circle, or a k-fold loop with some $k \geqslant 2$, or a k-fold polygon with some $k \geqslant 2$, or a k-fold complete bigraph with some $k \geqslant 1$.*

2. Edge-transitive group actions on graphs

A metric space X is called an ***arc*** (resp. an ***open arc***, resp. a ***circle***) if it is homeomorphic to the interval $[0,1]$ (resp. the open interval $(0,1)$, resp. the unit circle S^1 in $\mathbb{R}^2$). For any arc A with a homeomorphism $h : [0,1] \to A$, $h(0)$ and $h(1)$ are called the ***endpoints*** of A, and we write $\partial A = \{h(0), h(1)\}$. A metric space G is called a ***graph*** if there exist finitely many arcs $A_1, \dots, A_n$ with $n \geqslant 1$ such that $G = \bigcup_{i=1}^n A_i$ and $A_i \cap A_j = \partial A_i \cap \partial A_j$ for $1 \leqslant i < j \leqslant n$. Let G be a connected graph with a metric d. For any $x \in G$ and any $r > 0$, write $B(x,r) = \{y \in G\colon\ d(y,x) < r\}$. Without loss of generality, we may assume that the metric d has such a property that every $B(x,r)$ is a connected subset of G. For sufficiently small $\varepsilon > 0$, the number of connected components of $B(x,\varepsilon) - \{x\}$, denoted by $val(x)$ or $val_G(x)$, is called the ***valence*** (or ***degree***) of x in G. A point $v \in G$ is called an ***endpoint*** (resp. ***branching point***) of G if $val(v) = 1$ (resp. $val(v) \geqslant 3$). Denote by $End(G)$ (resp. $Br(G)$) the set of all endpoints (resp. branching points) of G. Write $V(G) = End(G) \cup Br(G)$. Any point $v \in V(G)$ is called a (natural) ***vertex*** of G. Note that in this paper no point $x \in G$ with $val(x) = 2$ is a vertex. Obviously, a connected graph G is a circle if and only if $V(G) = \emptyset$.

Let G be a connected graph. Every connected component of $G - V(G)$ is called an ***edge*** of G. Denote by $\mathbf{E}(G)$ the set of all edges of G. If $V(G) = \emptyset$ (that is, G is a circle), then G has only one edge, which is the circle itself. If $V(G) \neq \emptyset$ (that is, G is not a circle), then every edge of G is an open arc, and an edge E of G is called a ***line*** (resp. ***loop***) if the closure $\overline{E}$ of E in G is an arc (resp. a circle). Write $\partial E = \overline{E} - E$. Then ∂E consist of two vertexes (resp. one vertex) of G if E is a line (resp. loop), and $\partial E = \emptyset$ if $E = G$ is a circle. Let u and v be vertexes of G (u and v may be equal). An edge E of G is said to ***join*** u and v if $\partial E = \{u, v\}$. Edges $E_1, \dots, E_k$ of G with $k \geqslant 2$ are called ***multiple edges*** if $\partial E_1 = \cdots = \partial E_k$.

The following lemma is trivial.

Lemma 2.1. *Let G be a connected graph, and let $h : G \to G$ be a homeomorphism. Then*

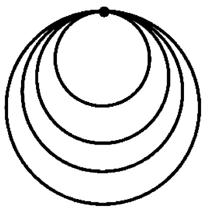
(1) *$h(V(G)) = V(G)$, and $val(h(v)) = val(v)$ for any $v \in V(G)$.*
(2) *$\{h(E)\colon\ E \in \mathbf{E}(G)\} = \mathbf{E}(G)$, and, for any $E \in \mathbf{E}(G)$, $\partial(h(E)) = h(\partial E)$, and $h(E)$ is a line (resp. loop) if E is a line (resp. loop).*
(3) *If $E_1, \dots, E_n$ are multiple lines (resp. loops), then $h(E_1), \dots, h(E_n)$ are also multiple lines (resp. loops).*

Definition 2.2. Let G be a connected graph, $\varphi : P \to \mathrm{Homeo}(G)$ be a group action, and $H = \varphi(P)$. φ and H are said to be ***edge-transitive*** if there exists an edge $E \in \mathbf{E}(G)$ such that $\{h(E)\colon\ h \in H\} = \mathbf{E}(G)$.

It is easy to see that a group action $\varphi : P \to \mathrm{Homeo}(G)$ and the group $H = \varphi(P)$ are edge-transitive if and only if $\{h(E)\colon\ h \in H\} = \mathbf{E}(G)$ for any $E \in \mathbf{E}(G)$.

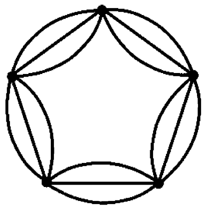
An $h \in \mathrm{Homeo}(G)$ is called an ***edge-transitive homeomorphism*** if the group $\langle h \rangle = \{h^n\colon\ n \in \mathbb{Z}\}$ is edge-transitive.

Evidently, a graph G admits an edge-transitive homeomorphism if and only if it admits an edge-transitive $\mathbb{Z}$-action. Note that if G is a circle then the circle itself is the unique edge of G. Thus any homeomorphism of a circle is edge-transitive.



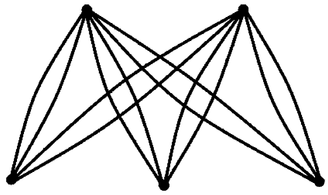
4-fold loop

Fig. 1.



3-fold polygon

Fig. 2.



2-fold complete bigraph

Fig. 3.

Definition 2.3. Let k be a positive integer. A connected graph G is called a ***k*-fold graph** if for any $(u, v) \in V(G) \times V(G)$, when G has an edge joining u and v then G has exactly k edges joining u and v.

By Lemma 2.1, the following lemma is clear.

Lemma 2.4. *Let G be a connected graph. If G has at least two vertexes, and G admits an edge-transitive group action, then G has no loop, and G is a k-fold graph, for some $k \geqslant 1$.*

Let $\varphi : P \to \mathrm{Homeo}(X)$ be a group action, $H = \varphi(P)$, and let U be a nonempty open subset of X. φ and H are said to be (topologically) ***transitive on*** U if there exists a point $x \in X$ such that the orbit $H(x)$ is dense in U (that is, the closure $\overline{H(x)} \supset U$). The following lemma is also clear.

Lemma 2.5. *Let G be a connected graph. Then a subgroup H of* $\mathrm{Homeo}(G)$ *is transitive if and only if H is edge-transitive and there exists an edge E of G such that H is transitive on E.*

3. Graphs admitting transitive $\mathbb{Z}^2$-actions

Since the real line $\mathbb{R}$ and the closed interval $[0, 1]$ admit no transitive homeomorphism, by Lemma 2.1, no connected graph G with $V(G) \neq \emptyset$ admits a transitive homeomorphism. However, in this section we will show, there exist graphs which admit transitive $\mathbb{Z}^2$-actions. First we have the following well-known lemma.

Lemma 3.1. *Let r be a positive irrational number. Define homeomorphisms ξ and ζ of $\mathbb{R}$ by, for any $t \in \mathbb{R}$,*

$$\xi(t) = t + 1, \qquad \zeta(t) = t + r.$$

Then ξ and ζ are commutative, and the group $\langle \xi, \zeta \rangle$ generated by ξ and ζ is transitive on $\mathbb{R}$.

Definition 3.2. Let G be a connected graph. G is called a ***k*-fold loop** for some $k \geqslant 2$ if $V(G)$ contains only one vertex and $\mathbf{E}(G)$ contains exactly k multiple loops (see Fig. 1).

G is called a ***k*-fold polygon** for some $k \geqslant 2$ if $V(G)$ contains n vertexes with $n \geqslant 3$, $\mathbf{E}(G)$ contains nk edges, and $V(G)$ can be written to be $V(G) = \{v_1, \dots, v_n\}$ with $v_0 = v_n$ such that, for any $i \in \{1, \dots, n\}$, there are k lines joining v_{i-1} and v_i (see Fig. 2).

G is called a ***k*-fold complete bigraph** for some $k \geqslant 1$ if $V(G)$ can be partitioned into two subsets V_1 and V_2 such that every edge of G is a line joining a vertex in V_1 and a vertex in V_2, and for any $v \in V_1$ and any $w \in V_2$ there exist exactly k lines joining v and w (see Fig. 3).

Proposition 3.3. *Let G be a k-fold loop with $k \geqslant 2$, or a k-fold polygon with $k \geqslant 2$, or a k-fold complete bigraph with $k \geqslant 1$. Then G admits a transitive $\mathbb{Z}^2$-action.*

Proof. Let r be a given positive irrational number.

(1) Let G be a k-fold loop. Assume that v is the unique vertex, and $E_1, \dots, E_k$ are the k loops of G. For $i = 1, \dots, k$, take a homeomorphism $\psi_i : \mathbb{R} \to E_i$. Write $E_0 = E_k$ and $\psi_0 = \psi_k$. Define homeomorphisms g and h of G by $g(v) = h(v) = v$, and for any $i \in \{1, \dots, k\}$ and any $t \in \mathbb{R}$,

$$g\psi_{i-1}(t) = \psi_i(t + 1/k), \qquad h\psi_{i-1}(t) = \psi_i(t + r/k).$$

Then g and h are commutative, and both g and h are edge-transitive homeomorphisms on G. By Lemma 3.1, the group $\langle g^k, h^k \rangle$ is transitive on E_0. Hence, by Lemma 2.5, the group $\langle g, h \rangle$ is transitive on G. This means that G admits a transitive $\mathbb{Z}^2$-action.

(2) Let G be a k-fold polygon. Assume that G has $n(\geqslant 3)$ vertexes $v_1, \dots, v_n$ with $v_0 = v_n$, and that the nk lines of G are $\{E_{i,j}\colon i = 1, \dots, n;\ j = 1, \dots, k\}$ with $\partial E_{i,j} = \{v_{i-1}, v_i\}$. For any $i \in \{1, \dots, n\}$ and any $j \in \{1, \dots, k\}$, take a homeomorphism $\psi_{i,j} : \mathbb{R} \to E_{i,j}$ such that $\lim_{t\to\infty} \psi_{i,j}(t) = v_i$ and $\lim_{t\to-\infty} \psi_{i,j}(t) = v_{i-1}$, and write $E_{(j-1)n+i} = E_{i,j}$ and $\psi_{(j-1)n+i} = \psi_{i,j}$ with $E_0 = E_{kn} (= E_{n,k})$ and $\psi_0 = \psi_{kn} (= \psi_{n,k})$. Define homeomorphisms g and h of G by $g(v_{i-1}) = h(v_{i-1}) = v_i$ for $i = 1, \dots, n$, and, for any $\mu \in \{1, 2, \dots, kn\}$ and any $t \in \mathbb{R}$,

$$g\psi_{\mu-1}(t) = \psi_\mu\big(t + 1/(kn)\big), \qquad h\psi_{\mu-1}(t) = \psi_\mu\big(t + r/(kn)\big).$$

Then g and h are commutative, and both g and h are edge-transitive homeomorphisms on G. By Lemma 3.1, the group $\langle g^{kn}, h^{kn}\rangle$ is transitive on E_0. Hence, by Lemma 2.5, the group $\langle g, h\rangle$ is transitive on G. This means that G admits a transitive $\mathbb{Z}^2$-action.

(3) Let G be a k-fold complete bigraph. We may assume that G has $m + n$ vertexes $v_1, \dots, v_m, w_1, \dots, w_n$ with $m \geqslant 1$ and $n \geqslant 1$, and that G has mnk edges

$$\big\{E_{i,j,p}\colon (i, j, p) \in \{1, \dots, m\} \times \{1, \dots, n\} \times \{1, \dots, k\}\big\}$$

with $\partial E_{i,j,p} = \{v_i, w_j\}$. Write $v_0 = v_m$ and $w_0 = w_n$. For every $(i, j, p) \in \{1, \dots, m\} \times \{1, \dots, n\} \times \{1, \dots, k\}$, take a homeomorphism $\psi_{i,j,p} : \mathbb{R} \to E_{i,j,p}$ such that

$$\lim_{t\to\infty} \psi_{i,j,p}(t) = v_i \quad \text{and} \quad \lim_{t\to-\infty} \psi_{i,j,p}(t) = w_j,$$

and write

$$E_{i,(p-1)n+j} = E'_{j,(p-1)m+i} = E_{i,j,p} \quad \text{and} \quad \psi_{i,(p-1)n+j} = \psi'_{j,(p-1)m+i} = \psi_{i,j,p}$$

with

$$E_{i,0} = E_{i,nk} (= E_{i,n,k}), \qquad E'_{j,0} = E'_{j,mk} (= E_{m,j,k})$$

and

$$\psi_{i,0} = \psi_{i,nk} (= \psi_{i,n,k}), \qquad \psi'_{j,0} = \psi'_{j,mk} (= \psi_{m,j,k}).$$

Define homeomorphisms g and h of G by, for any $(i, j) \in \{1, \dots, m\} \times \{1, \dots, n\}$,

$$g(v_i) = v_i, \qquad g(w_{j-1}) = w_j, \qquad h(w_j) = w_j, \qquad h(v_{i-1}) = v_i,$$

and for any $(i, \lambda) \in \{1, \dots, m\} \times \{1, \dots, nk\}$, any $(j, \mu) \in \{1, \dots, n\} \times \{1, \dots, mk\}$, and any $t \in \mathbb{R}$

$$g\psi_{i,\lambda-1}(t) = \psi_{i,\lambda}\big(t + 1/(kn)\big), \qquad h\psi'_{j,\mu-1}(t) = \psi'_{j,\mu}\big(t + r/(km)\big).$$

It is easy to see that g and h are commutative, and the group $\langle g, h\rangle$ is edge-transitive on G. By Lemma 3.1, $\langle g^{nk}, h^{mk}\rangle$ is transitive on the edge $E_{1,1,1}$, and hence, by Lemma 2.5, $\langle g, h\rangle$ is transitive on G. Proposition 3.3 is proven. □

4. Graphs admitting transitive commutative group actions

To complete the proof of the main result of this paper, we also need the following

Proposition 4.1. *Let G be a connected graph with $V(G) \neq \emptyset$. If G admits an edge-transitive commutative group action, then G is a k-fold loop with some $k \geqslant 2$, or a k-fold polygon with some $k \geqslant 2$, or a k-fold complete bigraph with some $k \geqslant 1$.*

Proof. If G has only one vertex, then G must be a k-fold loop, for some $k \geqslant 2$, and the proposition holds. If G has exactly two vertexes, then, by Lemma 2.4, G must be a k-fold complete bigraph, for some $k \geqslant 1$ with $k \neq 2$, and the proposition also holds. We now assume that G has at least three vertexes. By Lemma 2.4, all edges of G are lines. Let E_0 be a line of G with endpoints v_0 and w_0. Since G admits an edge-transitive commutative group action, there exists a commutative subgroup H of Homeo(G) such that $\{h(E_0)\colon h \in H\} = \mathbf{E}(G)$, and hence we have $V(G) = \bigcup\{\partial(h(E_0))\colon h \in H\} = \bigcup\{h(\partial E_0)\colon h \in H\} = H(v_0) \cup H(w_0)$. Write $V = H(v_0)$ and $W = H(w_0)$.

If the orbits V and W are the same, then there exist $h \in H$ and $n \in \mathbb{N}$ such that $w_0 = h(v_0)$, $h^n(v_0) = v_0$, and $h^i(v_0) \neq v_0$ for $0 < i < n$. For each $i \in \mathbb{Z}$, write $v_i = h^i(v_0)$. Then $v_{i+n} = v_i$ and $\partial(h^i(E_0)) = \{v_i, v_{i+1}\}$. For any $g \in H$, if there is $i \in \{1, \dots, n\}$ such that $v_i \in \partial(g(E_0))$, then $v_i = g(v_0)$ or $v_i = g(w_0)$, which implies $g(w_0) = gh(v_0) = hg(v_0) = h(v_i) = v_{i+1}$ or $g(v_0) = gh^{-1}(w_0) = h^{-1}g(w_0) = h^{-1}(v_i) = v_{i-1}$. Therefore for any $E \in \mathbf{E}(G)$, since G is connected and H is edge-transitive, there exists $j = j_E \in \{1, \dots, n\}$ such that $\partial E = \{v_{j-1}, v_j\}$, and hence, by Lemma 2.4, G is a k-fold polygon, for some $k \geqslant 2$.

If the orbits V and W are different, then $V \cap W = \emptyset$. Since H is edge-transitive, any line of G has an endpoint in V and has the other in W. Let $H_0 = \{h \in H\colon h(v_0) = v_0\}$. Then H_0 is a subgroup of H. Let $W_0 = H_0(w_0)$. If $W - W_0 \neq \emptyset$, then, since G is connected, there exist $w_1 \in W_0$, $v_1 \in V$, $w_2 \in W - W_0$ and $\{E_1, E_2\} \subset \mathbf{E}(G)$ such that $\partial E_1 = \{w_1, v_1\}$ and $\partial E_2 = \{v_1, w_2\}$. Since H is edge-transitive, there exist $\{h_1, h_2\} \subset H$ such that $h_1(E_0) = E_1$ and $h_2(E_0) = E_2$, which imply $h_1(v_0) = h_2(v_0) = v_1$, $h_1(w_0) = w_1$ and $h_2(w_0) = w_2$. Take an $h_0 \in H_0$ such that $h_0(w_0) = w_1$. Then $h_1^{-1}h_2h_0(v_0) = h_1^{-1}h_2(v_0) = h_1^{-1}(v_1) = v_0$, which implies $h_1^{-1}h_2h_0 \in H_0$, and hence $w_2 = h_2(w_0) = h_2h_1^{-1}(w_1) = h_2h_1^{-1}h_0(w_0) \in H_0(w_0) = W_0$. But this will lead to a contradiction. Thus it must hold that $W - W_0 = \emptyset$. Therefore, for any $(v, w) \in V \times W$, we can take a $g \in H$ and an $h \in H_0$ such that $g(v_0) = v$ and $h(w_0) = g^{-1}(w) \in W = W_0$, which lead to $gh(v_0) = g(v_0) = v$ and $gh(w_0) = gg^{-1}(w) = w$. Thus G has at least an edge $gh(E_0)$ joining v and w, for any $(v, w) \in V \times W$. Hence, by Lemma 2.4, G is a k-fold complete bigraph, for some $k \geqslant 1$. Proposition 4.1 is proven. □

We now complete the proof of the main result of this paper.

Theorem 4.2. *For any connected graph G, the following conditions are equivalent.*

(1) *G admits a transitive commutative group action.*
(2) *G admits a transitive $\mathbb{Z}^2$-action.*
(3) *G admits an edge-transitive commutative group action.*
(4) *G admits an edge-transitive $\mathbb{Z}^2$-action.*
(5) *G is a circle, or a k-fold loop with some $k \geqslant 2$, or a k-fold polygon with some $k \geqslant 2$, or a k-fold complete bigraph with some $k \geqslant 1$.*

Proof. $(2) \Rightarrow (1) \Rightarrow (3)$ and $(2) \Rightarrow (4) \Rightarrow (3)$ are trivial. If $V(G) = \emptyset$, then G is a circle, and all of the five conditions hold. If $V(G) \neq \emptyset$, then we have $(3) \Rightarrow (5)$ by Proposition 4.1 and $(5) \Rightarrow (2)$ by Proposition 3.3. Thus the five conditions in this theorem are equivalent. □

Remark 4.3. A graph G has at most finitely many connected components. If G admits an edge-transitive commutative group action, then all the connected components of G are homeomorphic to each other. It is easy to give a theorem similar to Theorem 4.2 for non-connected graphs. The details may be omitted.

Remark 4.4. Let G be a (topological) graph defined as in Section 2, and let $V(G)$ be the set of natural vertexes of G. Suppose that $V_1(G)$ is a finite subset of G which contains $V(G)$, and $\mathbf{E}_1(G)$ is the set of all connected components of $G - V_1(G)$. Then $(G, V_1(G), \mathbf{E}_1(G))$ is called a ***combinatorial graph***. Let $\Gamma = (G, V_1(G), \mathbf{E}_1(G))$. A bijection $\eta : V_1(G) \cup \mathbf{E}_1(G) \to V_1(G) \cup \mathbf{E}_1(G)$ is called an ***automorphism*** of Γ if $\eta(V_1(G)) = V_1(G)$, $\eta(\mathbf{E}_1(G)) = \mathbf{E}_1(G)$, and $\partial(\eta(E)) = \eta(\partial E)$ for any $E \in \mathbf{E}_1(G)$. $\Gamma = (G, V_1(G), \mathbf{E}_1(G))$ is called a ***simple graph*** if any circle in G contains at least three points of $V_1(G)$. A simple graph $\Gamma = (G, V_1(G), \mathbf{E}_1(G))$ is called a ***cycle*** if G itself is a circle. Note that Proposition 4.1 and its proof are still true for combinatorial graphs. Thus, as a corollary of Proposition 4.1, we have the following purely graph theoretic result, which is stronger than [9, Lemma 6.1].

Proposition 4.5. *Let Γ be a finite connected simple graph and suppose that the automorphism group of Γ contains a commutative subgroup that acts edge-transitively. Then Γ is either a cycle or a complete bigraph.*

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