CAN POINTS OF BOUNDED ORBITS SURROUND POINTS OF UNBOUNDED ORBITS ?

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ABSTRACT. We show a somewhat surprising result: if E is a disk in the plane $\mathbb{R}^2$, then there is a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that, for every $x \in \partial E$, the orbit $O(x,h)$ is bounded, but for every $y \in \text{Int}(E)$, the orbit $O(y,h)$ is doubly divergent. To prove this, we define a class of homeomorphisms on the square $[-1,1]^2$, called normally rising homeomorphisms, and show that a normally rising homeomorphism can have very complex ω-limit sets and α-limt sets, though the homeomorphism itself looks very simple.

1. INTRODUCTION

By a *dynamical system*, we mean a pair (X, f), where the *phase space* X is a metric space and $f : X \to X$ is a continuous map. For $x, y \in X$, if there is a sequence of positive integers $n_1 < n_2 < \cdots$ such that $f^{n_i}(x) \to y$ then we call y an ω*-limit point* of x. We denote by $\omega_f(x)$ or $\omega(x, f)$ the set of all ω-limit points of x and call it the ω*-limit set* of x. The ω-limit sets are important in understanding the long term behavior of a dynamical system and the properties of which have been intensively studied. It is well known that, if X is a compact metric space, then $\omega_f(x)$ is nonempty, closed and strongly f-invariant for any $x \in X$. Bowen [9] gave an intrinsic characterization of abstract ω-limit sets as those having no non-trivial filtrations and used shadowing to study the ω-limit sets of Axiom A diffeomorphisms. Hirsch, Smith, and Zhao [14] showed that the ω-limit set of any precompact orbit is internally chain transitive; the opposite direction is shown for tent maps with periodic critical points by Barwell-Davies-Good [5], and for subshifts of finite type by Barwell-Good-Knight-Raines [6], respectively. Barwell, Good, Oprocha, and Raines [7] showed the equivalence between internal chain transitivity, weak incompressibility, and being an ω-limit set for topologically hyperbolic systems. Good and Meddaugh [11] studied the relations between the collections of all ω-limit sets and those of all internally

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chain transitive sets under various shadowing properties. The notion of ω-limit set is also key in several definitions of attractors and chaos (see e.g. [20, 17]).

For one-dimensional systems, the structures of ω-limit sets have been well characterized. For interval maps $f : I \to I$ and $x \in I$, Blokh, Bruckner, Humke, and Smítal [8] showed that the family of all ω-limit sets of f forms a closed subset of the hyperspace of I endowed with the Hausdorff metric, and Agronsky, Bruckner, Ceder, and Pearson [2] showed that $\omega_f(x)$ is either a finite periodic orbit, or an infinite nowhere-dense set, or the union of periodic nondegenerate subintervals of I. These results were extended to the cases when the phase space is either a circle [22], or a graph [18, 13, 10], or a dendrite [1], or a hereditarily locally connected continua [23], or a quasi-graph [19]. However, when the phase space X has dimension ≥ 2, only partial results are known. For instance, Agronsky and Ceder [3] showed that every finite union of the nondegenerate Peano continua of the square I^k is an ω-limit set of some continuous map on the square I^k, and Jiménez López and Smítal [15] found the necessary and sufficient conditions for a finite union of Peano continua to be an ω-limit set of a triangular map. The family of all ω-limit sets of the Stein-Ulam spiral map was identified by Barański and Misiurewicz [4, 16].

Let $\mathbb{Z}$, $\mathbb{Z}_+$, and $\mathbb{Z}_-$ be the sets of integers, nonnegative integers, and nonpositive integers, respectively. Let X be a topological space and $f : X \to X$ be a homeomorphism. Then the sets $O(x, f) \equiv \{f^n(x) : n \in \mathbb{Z}\}$, $O_+(x, f) \equiv \{f^n(x) : n \in \mathbb{Z}_+\}$, and $O_-(x, f) \equiv \{f^{-n}(x) : n \in \mathbb{Z}_-\}$ are called the *orbit*, *positive orbit*, and *negative orbit* of x, respectively. For $x, y \in X$, if $y \in \omega_{f^{-1}}(x)$, then we call y an *α-limit point* of x. We denote by $\alpha_f(x)$ or $\alpha(x, f)$ the set of all α-limit points of x and call it the *α-limit set* of x.

The aim of the paper is to study the ω-limit sets and α-limit sets of homeomorphisms on the plane $\mathbb{R}^2$. This topic has also been discussed by some ahtours. For instance, by means of ω-limit sets and α-limit sets, Handel [12] obtained a fixed point theorem for homeomorphisms on the plane. In Section 2, we introduce a class of homeomorphisms on the square J^2 where $J = [-1, 1]$, called *normally rising homeomorphisms*. Every normally rising homeomorphism fixes point-wise the top edge and bottom edge of J^2, and moves up other horizontal line segments. So, a normally rising homeomorphism looks very simple. However, the first main result we obtained shows that the ω-limit set and the α-limit set of a normally rising homeomorphism can be very complex. Actually, any family of predescribed reasonable sets can always be realised as the limit sets of a normally rising homeomorphism.

For $s \in J$, let $J_s = J \times \{s\}$ and $\mathscr{C}_s$ be the collection of all nonempty connected closed subsets of J_s. A map $\phi : J \to \mathscr{C}_s$ is *increasing* if the abscissae and ordinates of the endpoints of $\phi(r)$ are increasing functions of r and is *endpoint preserving* if $\phi(-1) = (-1, s)$ and $\phi(1) = (1, s)$. If f is a normally rising homeomorphism on J^2 and $s \in J$, then $\omega_{sf}(r) \equiv \omega_f(r, s)$ defines an increasing function ω_{sf} from J to $\mathscr{C}_1$. Similarly, we also have an increasing function α_{sf} from J to $\mathscr{C}_{-1}$.

Let $\mathscr{A} = \mathscr{C}_1$ and $\mathscr{A}' = \mathscr{C}_{-1}$. Then we have

Theorem 1.1. *Let $\mathbb{N}'$ and $\mathbb{N}''$ be two nonempty subsets of $\mathbb{N}$, and let $\mathscr{V} = \{V_n : n \in \mathbb{N}'\}$ and $\mathscr{W} = \{W_j : j \in \mathbb{N}''\}$ be two families of pairwise disjoint nonempty connected subsets of the semi-open interval $(0, 1/2]$. For each $n \in \mathbb{N}'$ and each $j \in \mathbb{N}''$, let $\omega_n : J \to \mathscr{A}$ and $\alpha_j : J \to \mathscr{A}'$ be given increasing and endpoint preserving maps. Then there exists a normally rising homeomorphism $f : J^2 \to J^2$ such that $\omega_{sf} = \omega_n$ for any $n \in \mathbb{N}'$ and any $s \in V_n$, and $\alpha_{tf} = \alpha_j$ for any $j \in \mathbb{N}''$ and any $t \in W_j$.*

Let d be the Euclidean metric on $\mathbb{R}^2$. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism and $x \in \mathbb{R}^2$. The orbit $O(x, f)$ is said to be *positively divergent* (resp. *negatively divergent*) if $d(f^n(x), (0,0)) \to \infty$ (resp. $d(f^{-n}(x), (0,0)) \to \infty$) as $n \to \infty$. If $O(x, f)$ is both positively divergent and negatively divergent, then we say $O(x, f)$ is *doubly divergent*.

A subset E of $\mathbb{R}^2$ is a *disk* if it is homeomorphic to the unit closed ball of $\mathbb{R}^2$. We use ∂E and $\overset{\circ}{E}$ (or, $\mathrm{Int}(E)$) to denote the boundary and interior of E, respectively. By means of Theorem 1.1, we get in Section 3 the following somewhat surprising result.

Theorem 1.2. *Let E be a disk in $\mathbb{R}^2$. Then there exists a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that, for any $x \in \partial E$, the orbit $O(x, h)$ is bounded, but for any $y \in \overset{\circ}{E}$, the orbit $O(y, h)$ is doubly-divergent.*

2. Limit sets of normally rising homeomorphisms on J^2

In this paper, for $r, s \in \mathbb{R}$, we use (r, s) to denote a point in $\mathbb{R}^2$. If $r < s$, we also use (r, s) to denote an open interval in $\mathbb{R}$. These will not lead to confusion. For example, if we write $(r,s) \in X$, then (r,s) will be a point; if we write $t \in (r,s)$, then (r,s) will be a set, and hence is an open interval.

We always write $J = [-1, 1]$. Define the homeomorphism $f_{01} : J \to J$ by, for any $s \in J$,

$$f_{01}(s) = \begin{cases} (s+1)/2, & \text{if } 0 \leq s \leq 1; \\ s+1/2, & \text{if } -1/2 \leq s \leq 0; \\ 2s+1, & \text{if } -1 \leq s \leq -1/2. \end{cases}$$

Define the homeomorphism $f_{02} : J^2 \to J^2$ by, for any $(r,s) \in J^2$,

$$f_{02}(r,s) = \big(r, f_{01}(s)\big).$$

For any compact connected manifold M, denote by ∂M the boundary, and by $\overset{\circ}{M}$ the interior of M. Specially, we have $\partial J = \{-1, 1\}$, and $\overset{\circ}{J} = (-1,1)$. Note that ∂J^2 is $\partial(J^2)$, not $(\partial J)^2$.

Definition 2.1. For any $s \in J$, write $J_s = J \times \{s\}$. A homeomorphism $f : J^2 \to J^2$ is said to be *normally rising* if

$$f\,|\,\partial J^2 = f_{02}\,|\,\partial J^2, \qquad \text{and} \qquad f(J_s) = f_{02}(J_s) \quad \text{for any } s \in J.$$

Define the maps $p, q : \mathbb{R}^2 \to \mathbb{R}$ by $p(x) = r$ and $q(x) = s$ for any $x = (r,s) \in \mathbb{R}^2$, that is, we denote by $p(x)$ and $q(x)$ the abscissa and the ordinate of x, respectively. By the definition, the following lemma is obvious.

Lemma 2.2. *Let $f : J^2 \to J^2$ be a normally rising homeomorphism. Then*

(1) $f\,|\,J_{-1} \cup J_1$ is the identity map;

(2) $pf(r_1, s) < pf(r_2, s)$, for any $\{r_1, r_2, s\} \subset J$ with $r_1 < r_2$;

(3) $\omega_f(x) = \alpha_f(x) = \{x\}$, for any $x \in J \times \partial J$;

(4) $\omega_f(r,s) = (r,1)$, and $\alpha_f(r,s) = (r,-1)$, for any $(r,s) \in \partial J \times \overset{\circ}{J}$;

(5) for any $(r,s) \in \overset{\circ}{J}{}^2 = (-1,1)^2$, $\omega_f(r,s)$ is a nonempty connected closed subset of J_1, and $\alpha_f(r,s)$ is a nonempty connected closed subset of J_{-1}.

Denote by $\mathscr{A}$ the family of all nonempty connected closed subsets of J_1, and by $\mathscr{A}'$ the family of all nonempty connected closed subsets of J_{-1}. From Lemma 2.2 we see that

$$\omega_f(x) \in \mathscr{A}, \qquad \text{and} \qquad \alpha_f(x) \in \mathscr{A}', \quad \text{for any } x \in J \times \overset{\circ}{J}.$$

Definition 2.3. Let $v_1 = (-1, 1)$, $v_2 = (1, 1)$, $v_3 = (-1, -1)$ and $v_4 = (1, -1)$ be the four vertices of the square J^2. A map $\omega : J \to \mathscr{A}$ is said to be *increasing* if

$$\min\big(p\,\omega(r_1)\big) \le \min\big(p\,\omega(r_2)\big) \quad \text{and} \quad \max\big(p\,\omega(r_1)\big) \le \max\big(p\,\omega(r_2)\big)$$

for any $\{r_1, r_2\} \in J$ with $r_1 < r_2$. The map ω is said to be *endpoints preserving* if $\omega(-1) = \{v_1\}$ and $\omega(1) = \{v_2\}$.

Similarly, a map $\alpha : J \to \mathscr{A}'$ is said to be *increasing* if

$$\min\big(p\,\alpha(r_1)\big) \le \min\big(p\,\alpha(r_2)\big) \quad \text{and} \quad \max\big(p\,\alpha(r_1)\big) \le \max\big(p\,\alpha(r_2)\big)$$

for any $\{r_1, r_2\} \in J$ with $r_1 < r_2$. The map α is said to be *endpoints preserving* if $\alpha(-1) = \{v_3\}$ and $\alpha(1) = \{v_4\}$.

From Lemma 2.2 we get the following lemma at once.

Lemma 2.4. *Let $f : J^2 \to J^2$ be a normally rising homeomorphism. For any given $s \in \mathring{J}$, define maps $\omega_{sf} : J \to \mathscr{A}$ and $\alpha_{sf} : J \to \mathscr{A}'$ by*

$$\omega_{sf}(r) = \omega_f(r,s) \quad \textit{and} \quad \alpha_{sf}(r) = \alpha_f(r,s), \quad \textit{for any } r \in J. \tag{2.1}$$

Then both ω_{sf} and α_{sf} are increasing and endpoints preserving.

Write $t_0 = 0$, $I_1 = \big(0, 1/2\big]$, $D_1 = J \times \big[0, 1/2\big]$, and

$$t_n = f_{01}^n(t_0), \quad D_n = f_{02}^{n-1}(D_1), \quad \text{for any } n \in \mathbb{Z}.$$

Then $D_n = J \times [t_{n-1}, t_n]$, and $D_n \cap D_{n+1} = J_{t_n}$. A main result of this paper is the following theorem, which shows that the ω-limit sets and α-limit sets of homeomorphisms of J^2 may have very complex structure, even if these homeomorphisms are normally rising.

Theorem 2.5. *Let $\mathbb{N}'$ and $\mathbb{N}''$ be two nonempty subsets of $\mathbb{N}$, and let $\mathscr{V} = \{V_n : n \in \mathbb{N}'\}$ and $\mathscr{W} = \{W_j : j \in \mathbb{N}''\}$ be two families of pairwise disjoint nonempty connected subsets of the semi-open interval I_1. For each $n \in \mathbb{N}'$ and each $j \in \mathbb{N}''$, let $\omega_n : J \to \mathscr{A}$ and $\alpha_j : J \to \mathscr{A}'$ be arbitarily given increasing and endpoint preserving maps. Then there exists a normally rising homeomorphism $f : J^2 \to J^2$ such that $\omega_{sf} = \omega_n$ for any $n \in \mathbb{N}'$ and any $s \in V_n$, and $\alpha_{tf} = \alpha_j$ for any $j \in \mathbb{N}''$ and any $t \in W_j$, where the definitions of ω_{sf} and α_{tf} are given by (2.1).*

Remark 2.6. Before the proof of Theorem 2.5, we make a survey of the differences between ω, ω_f, ω_{sf} and ω_n. Let $\mathscr{F}$ be the family of all normally rising homeomorphisms from J^2 to J^2. If we consider only the space J^2 and the family $\mathscr{F}$, then

(1) ω is a map from $J^2 \times \mathscr{F}$ to the family $\mathscr{A}$ of some subsets of J_1;

(2) ω_f is a map from J^2 to $\mathscr{A}$, with a given $f \in \mathscr{F}$ as a parameter;

(3) ω_{sf} is a map from J to $\mathscr{A}$, with a given $s \in J$ and an $f \in \mathscr{F}$ as parameters;

(4) ω_n is a map from J to $\mathscr{A}$, with a given $n \in \mathbb{N}$ as a parameter;

(5) The definition of ω depends on the definition of ω-limit sets of orbits of maps, and ω_f can be regard as the restriction of ω to $J^2 \times \{f\}$ (really, it should be $\omega_f = \omega \mathbf{i}_f$, that is the composition of an imbedding $\mathbf{i}_f : J^2 \to J^2 \times \mathscr{F}$ and ω). ω_{sf} can be again regard as the restriction of ω_f to J_s (really, it should be $\omega_{sf} = \omega_f \mathbf{i}_s$, that is the composition of an imbedding $\mathbf{i}_s : J \to J^2$ and ω_f). Hence, the definitions of ω, ω_f and ω_{sf} depend on orbits of maps. However, ω_n is only a pure map, of which the definition is directly assigning a set $\omega_n(r) \in \mathscr{A}$ for each $r \in J$, which does not depend on any $f \in \mathscr{F}$.

The differences between α, α_f, α_{sf} and α_n are analogous.

Now we begin the proof. For $k \in \mathbb{N}$, write $\mathbb{N}_k = \{1, 2, \cdots, k\}$.

Proof. We may consider only the case that $\mathbb{N}' = \mathbb{N}$ since the case that $\mathbb{N}'$ is a finite subset of $\mathbb{N}$ is similar and is simpler.

If $1/2 \notin \bigcup_{n=1}^{\infty} V_n$, then we can add the one-point-set $\{1/2\}$ to the family $\mathscr{V}$. If there exist $n \in \mathbb{N}$ and $s \in (0, 1/2)$ such that $[s, 1/2] \subset V_n$, then we can divide V_n into two connected sets $\{1/2\}$ and $V_n - \{1/2\}$. Therefore, we may assume that $\{1/2\} \in \mathscr{V}$, and $V_1 = \{1/2\}$.

For any $n, k \in \mathbb{N}$, write

$$V_{nk} = \begin{cases} V_n, & \text{if } V_n \text{ is a one-point-set or a closed interval;} \\ [a, (a+kb)/(k+1)], & \text{if } V_n = [a, b) \text{ for some } a < b; \\ [(ka+b)/(k+1), b], & \text{if } V_n = (a, b] \text{ for some } a < b; \\ [(ka+b)/(k+1), (a+kb)/(k+1)], & \text{if } V_n = (a, b) \text{ for some } a < b. \end{cases}$$

Then $V_{n1} \subset V_{n2} \subset V_{n3} \subset \cdots \subset V_n$, $\bigcup_{k=1}^{\infty} V_{nk} = V_n$, and for every fixed $k \in \mathbb{N}$, $\{V_{nk} : n \in \mathbb{N}\}$ is a family of pairwise disjoint nonempty connected closed subsets of I_1 .

For each $n \in \mathbb{N}$, since $\omega_n : J \to \mathscr{A}$ is increasing and endpoint preserving , there exist increasing functions $\xi_{n1} : J \to J$ and $\xi_{n2} : J \to J$ such that , for any $r \in J$, it holds that

$$\xi_{n1}(r) \leq \xi_{n2}(r) , \qquad \omega_n(r) = \left[\, \xi_{n1}(r) \, , \, \xi_{n2}(r) \, \right] \times \{1\} , \tag{2.2}$$

and $\xi_{n1}(-1) = \xi_{n2}(-1) = -1$, $\xi_{n1}(1) = \xi_{n2}(1) = 1$. For $j = 1, 2$, let Y_{nj} be the set of all discontinuous points of ξ_{nj} . Since the set of discontinuous points of an increasing function is countable , there exists a countable dense subset $R = \{r_{-1}, r_0, r_1, r_2, r_3, \cdots\}$ of J such that $r_{-1} = -1$, $r_0 = 1$, and $\bigcup_{n=1}^{\infty} (Y_{n1} \cup Y_{n2}) \subset R$.

For each $k \in \mathbb{N}$, write

$$R_k = \{r_{-1}, r_0, r_1, r_2, \cdots, r_k\} , \qquad \text{and} \qquad V = \bigcup_{n=1}^{\infty} V_n .$$

Then we have

$$\bigcup_{k=1}^{\infty} \left(R_k \times (V_{1k} \cup V_{2k} \cup \cdots \cup V_{kk}) \right) = R \times V \subset D_1 - J_0 . \tag{2.3}$$

For any $j, m \in \mathbb{Z}$ with $j \leq m$, write

$$D_j^m = \bigcup_{i=j}^{m} D_i , \quad D_j^{\infty} = \bigcup_{i=j}^{\infty} D_i , \quad \text{and} \quad D_{-\infty}^m = \bigcup_{i=-\infty}^{m} D_i .$$

Then $D_j^m = J \times [t_{j-1}, t_m]$, $D_j^{\infty} = J \times [t_{j-1}, 1)$, $D_{-\infty}^m = J \times (-1, t_m]$, and the closure $\overline{D_j^{\infty}} = D_j^{\infty} \cup J_1$, $\overline{D_{-\infty}^m} = D_{-\infty}^m \cup J_{-1}$. (See Figure 2.1).

In order to construct the homeomorphism $f : J^2 \to J^2$ mentioned in Theorem 2.5 , we

First, put $f_0 = f \,|\, D_0 = f_{02} \,|\, D_0 : D_0 \to D_1$.

Secondly , assume that , for some $k \in \mathbb{N}$, we have defined a homeomorphism

$$f_{2.4} = f \,|\, D_0^{k2-1} : D_0^{k2-1} \to D_1^{k2} , \tag{2.4}$$

which satisfy the following conditions : (For avoiding that there are overmany subscripts , in the already explicit domain D_0^{k2-1} , we will use f to replace $f_{2.4}$, although the entire $f : J_2 \to J_2$ has not yet been defined . Similarly here in after.)

(C.1.k) $f(J_s) = f_{02}(J_s)$, for any $J_s \subset D_0^{k2-1}$;

(C.2.k) $|pf(r,s)-r| < 2/(2m+3)$, for any $m \in \mathbb{N}_k$ and any $(r,s) \in D^{m^2-1}_{(m-1)^2}$;

(C.3.k) $|pf(r,s)-r| < 2/(2k+5)$, for any $(r,s) \in D_{k^2-1} \cap D_{k^2} = J_{t_{k^2-1}}$.

We will extend the homeomorphism $f_{2.4}$ to a homeomorphism

$$f_{2.5} = f \,|\, D_0^{(k+1)^2-1} \;:\; D_0^{(k+1)^2-1} \;\to\; D_1^{(k+1)^2} \tag{2.5}$$

as follows :

Step 1. For any $n \in \mathbb{N}_k$ and any $(r,s) \in R_k \times V_{nk}$, we define $f\big(f^{i-1}f^{k^2-1}(r,s)\big) = f^i\big(f^{k^2-1}(r,s)\big)$ for $i = 1, \cdots, 2k+1$ in the natural order by putting the ordinate

$$q\big(f\big(f^{i-1}f^{k^2-1}(r,s)\big)\big) = q\big(f^i\big(f^{k^2-1}(r,s)\big)\big) = q\big(f_{02}^{k^2+i-1}(r,s)\big), \tag{2.6}$$

and putting the abscissa

$$\begin{aligned} p\big(f\big(f^{i-1}f^{k^2-1}(r,s)\big)\big) &= p\big(f^i\big(f^{k^2-1}(r,s)\big)\big) \\ &= p\big(f^{k^2-1}(r,s)\big) + i\big(\xi_{n\lambda_k}(r) - p\big(f^{k^2-1}(r,s)\big)\big)/(2k+7), \end{aligned} \tag{2.7}$$

where $\lambda_k = 1$ if k is odd , and $\lambda_k = 2$ if k is even . Note that , for any $n \in \mathbb{N}_k$ and any $(r,s), (r',s) \in R_k \times V_{nk}$ with $r < r'$, since ω_n is increasing, from (2.7) we get

$$p\big(f^i\big(f^{k^2-1}(r,s)\big)\big) < p\big(f^i\big(f^{k^2-1}(r',s)\big)\big) \qquad \text{for } i = 1, \cdots, 2k+1 .$$

Step 2. In Step 1 , for $i = 1, \cdots, 2k+1$, the set $f^{i-1}\big(f^{k^2-1}\big(R_k \times \bigcup_{n=1}^k V_{nk}\big)\big) \subset D_{k^2+i-1} - J_{t_{k^2+i-2}}$ has been really defined . Thus we can define

$$X_k = \bigcup_{i=1}^{2k+1} f^{i-1}\big(f^{k^2-1}\big(R_k \times \bigcup_{n=1}^{k} V_{nk}\big)\big). \tag{2.8}$$

Note that $X_k \subset \bigcup_{i=1}^{2k+1}\big(D_{k^2+i-1} - J_{t_{k^2+i-2}}\big) = D_{k^2}^{k^2+2k} - J_{t_{k^2-1}}$. Write $S_k = q(X_k)$. Then $S_k = \{s \in J : J_s \cap X_k \neq \emptyset\}$. Noting that R_k contains just $k+2$ points, $R_k \times \bigcup_{n=1}^k V_{nk}$ contains just $k\,(k+2)$ connected components , and every connected component of $R_k \times \bigcup_{n=1}^k V_{nk}$ is a point or a vertical line segment in $D_1 - J_0$, from (2.6) , (2.7) , (2.8) and the conditions (C.1.k) we see that X_k with S_k has the following properties :

(A) X_k contains just $k(k+2)(2k+1)$ connected components , S_k contains just $k(2k+1)$ connected components , and every connected component of X_k is a point or an arc , every connected component of S_k is a point or a closed interval ;

(B) For any $s \in S_k$, $J_s \cap X_k$ contains just $k+2$ points. Specially, since $\{r_{-1}, r_0\} = \{-1, 1\} \subset R_k$, $(-1,s)$ and $(1,s)$ are two points in $J_s \cap X_k$;

(C) If L is a connected component of S_k and is a closed interval, then $(J \times L) \cap X_k$ is the union of just $k+2$ arcs, of which each arc is a connected component of X_k . Specially, $\{-1\} \times L$ and $\{1\} \times L$ are two connected components of X_k . Moreover, for any $s \in L$ and any connected component A of X_k in $(J \times L) \cap X_k$, $A \cap (J \times \{s\})$ contains just one point.

In Step 1, we actually have given the definition of $f \,|\, X_k$, hence we actually have obtained the map

(2.9) $$f_{2.9} = f \,|\, D_0^{k^2-1} \cup X_k \ : \ D_0^{k^2-1} \cup X_k \ \to \ D_1^{(k+1)^2} .$$

From (2.6) and (2.7) we see that the map $f_{2.9}$ is a continuous injection. From the properties of X mentioned above we see that the map $f_{2.9}$ can be uniquely extended to a continuous injection

(2.10) $$f_{2.10} = f \,|\, D_0^{k^2-1} \cup (J \times S_k) \ : \ D_0^{k^2-1} \cup (J \times S_k) \ \to \ D_1^{(k+1)^2}$$

such that, for any $s \in S_k$ and any connected component L of $J_s - X_k$, $f \,|\, L$ is linear. Such an injection $f_{2.10}$ will be called the *level linear extension* of $f_{2.9}$. Obviously, $f_{2.10}$ can also be uniquely extended to a homeomorphism

$$f_{2.5} = f \,|\, D_0^{(k+1)^2-1} \ : \ D_0^{(k+1)^2-1} \ \to \ D_1^{(k+1)^2}$$

such that, for any $r \in J$ and any connected component L of $\big(\{r\} \times J\big) \cap D_{k^2}^{(k+1)^2-1} - (J \times S_k)$, $f \,|\, L$ is linear. Such a homeomorphism $f_{2.5}$ will be called the *vertical linear extension* of the injection $f_{2.10}$.

Since the level linear extension from $f_{2.9}$ to $f_{2.10}$ is before the vertical linear extension from $f_{2.10}$ to $f_{2.5}$, by (2.6) and (C.1.k) we see that, for the homeomorphism $f_{2.5}$, the condition (C.1.$k+1$) holds.

Since ω_n is endpoint preserving, in (2.7), if $r \in \partial J$ then $p\big(f^{k^2-1}(r,s)\big) = \xi_{n\lambda_k}(r) = r$. If $r \in \overset{\circ}{J}$, then $p\big(f^{k^2-1}(r,s)\big) \in \overset{\circ}{J}$, which with $\xi_{n\lambda_k}(r) \in J$ implies $\big|\xi_{n\lambda_k}(r) - p\big(f^{k^2-1}(r,s)\big)\big| < 2$. Thus, no matter whether $r \in \partial J$ or $r \in \overset{\circ}{J}$, we have

$$\big|\xi_{n\lambda_k}(r) - p\big(f^{k^2-1}(r,s)\big)\big| \big/ (2k+7) < 2/(2k+7) = 2\big/\big(2(k+1)+5\big),$$

which with (2.7) and (2.8) implies

$$(2.11)\qquad |\,pf(x)-p(x)\,| \;<\; 2\big/\big(2(k+1)+5\big), \qquad \text{for any } x\in X_k\,.$$

Clearly, after the level linear extension, for any $x\in J\times S_k$, (2.11) still holds. Specially, noting $t_{(k+1)^2-1}\in S_k$, we have $J_{t_{(k+1)^2-1}}\subset J\times S_k$. Thus, for the homeomorphism $f_{2.5}$, the condition (C.3.$k+1$) holds.

In addition, after the vertical linear extension, for any $x\in D_{k^2}^{(k+1)^2-1}$, by (C.3.k) and (2.11) (for all $x\in J\times S_k$) we obtain

$$|\,pf(x)-p(x)\,| \;<\; \max\left\{2/(2k+5)\,,\,2\big/\big(2(k+1)+5\big)\right\} \;=\; 2\big/\big(2(k+1)+3\big)\,.$$

This with (C.2.k) implies that, for the homeomorphism (2.5), the condition (C.2.$k+$ 1) also holds.

Therefore, by induction, we obtain a homeomorphism

$$(2.12)\qquad f_{2.12} \;=\; f\,|\,D_0^\infty \;:\; D_0^\infty \;\to\; D_1^\infty,$$

which satisfies the conditions (C.1.k), (C.2.k) and (C.3.k) for all $k\in\mathbb{N}$, and from these conditions we can directly extend the homeomorphism $f_{2.12}$ to a homeomorphism

$$(2.13)\qquad f_{2.13} \;=\; f\,|\,\overline{D_0^\infty} \;:\; \overline{D_0^\infty} \;\to\; \overline{D_1^\infty}$$

by putting $f(x)=x$ for any $x\in J_1$.

As mentioned above, in the domain $\overline{D_0^\infty}$, we will replace $f_{2.13}$ by f, even if the definition of the entire $f:J^2\to J^2$ has not yet been given. Specially, for any $x\in\overline{D_0^\infty}$, we can write $\omega(x,f)$ for $\omega(x,f_{2.13})$, since $\omega(x,f) = \omega(x,f_{2.13})$, no matter how $f\,\big|\,D_{-\infty}^{-1} \,:\, D_{-\infty}^{-1} \to D_{-\infty}^{0}$ is defined.

Claim 2.5.1. for any $n\in\mathbb{N}$ and any given $s\in V_n$, it holds that $\omega_{sf} = \omega_n$.

Proof of Claim 2.5.1. Consider any given $r\in R$. Take an integer $j\geq n$ such that $r\in R_j$ and $s\in V_{nj}$. Then $(r,s)\in R_k\times V_{nk}$ for any integer $k\geq j$. By (2.6) and (2.7) we can easily verify that $\omega_f(r,s) = \big[\xi_{n1}(r)\,,\xi_{n2}(r)\big]\times\{1\}$. This with (2.1) and (2.2) implies $\omega_{sf}(r) = \omega_n(r)$. Thus we have $\omega_{sf}\,|\,R = \omega_n\,|\,R$.

Further, consider any given $t\in J-R$. Since R contains all discontinuous points of ξ_{n1} and ξ_{n2}, it follows that t is a continuous point both of ξ_{n1} and of ξ_{n2}. Since R

is a dense subset of J, $\partial J \subset R$, and since ξ_{n1} and ξ_{n2} are increasing, there exist $t_i \in (t, t+1/i] \cap R$ and $\tau_i \in [t-1/i, t) \cap R$ for each $i \in \mathbb{N}$ such that

$$\xi_{n1}(t) - 1/i < \xi_{n1}(\tau_i) \leq \xi_{n1}(t) \leq \xi_{n1}(t_i) < \xi_{n1}(t) + 1/i \tag{2.14}$$

and

$$\xi_{n2}(t) - 1/i < \xi_{n2}(\tau_i) \leq \xi_{n2}(t) \leq \xi_{n2}(t_i) < \xi_{n2}(t) + 1/i\,. \tag{2.15}$$

On the other hand, from Lemma 2.4 we know that there exist increasing functions $\psi_{n1} : J \to J$ and $\psi_{n2} : J \to J$ such that, for any $r \in J$, it holds that

$$\psi_{n1}(r) \leq \psi_{n2}(r) \quad \text{and} \quad \omega_{sf}(r) = \big[\psi_{n1}(r), \psi_{n2}(r)\big] \times \{1\}\,. \tag{2.16}$$

Noting $\omega_{sf}|R = \omega_n|R$, from (2.16) and (2.2) we get, for any $i \in \mathbb{N}$,

$$\psi_{n1}(\tau_i) = \xi_{n1}(\tau_i)\,, \quad \psi_{n1}(t_i) = \xi_{n1}(t_i)\,, \quad \psi_{n2}(\tau_i) = \xi_{n2}(\tau_i)\,, \quad \psi_{n2}(t_i) = \xi_{n2}(t_i)\,,$$

which with $\psi_{n1}(\tau_i) \leq \psi_{n1}(t) \leq \psi_{n1}(t_i)$, $\psi_{n2}(\tau_i) \leq \psi_{n2}(t) \leq \psi_{n2}(t_i)$ and (2.14), (2.15) imply $\psi_{n1}(t) = \xi_{n1}(t)$ and $\psi_{n2}(t) = \xi_{n2}(t)$. Thus we have $\omega_{sf}(t) = \omega_n(t)$ and hence $\omega_{sf} = \omega_n$. Claim 2.5.1 is proved.

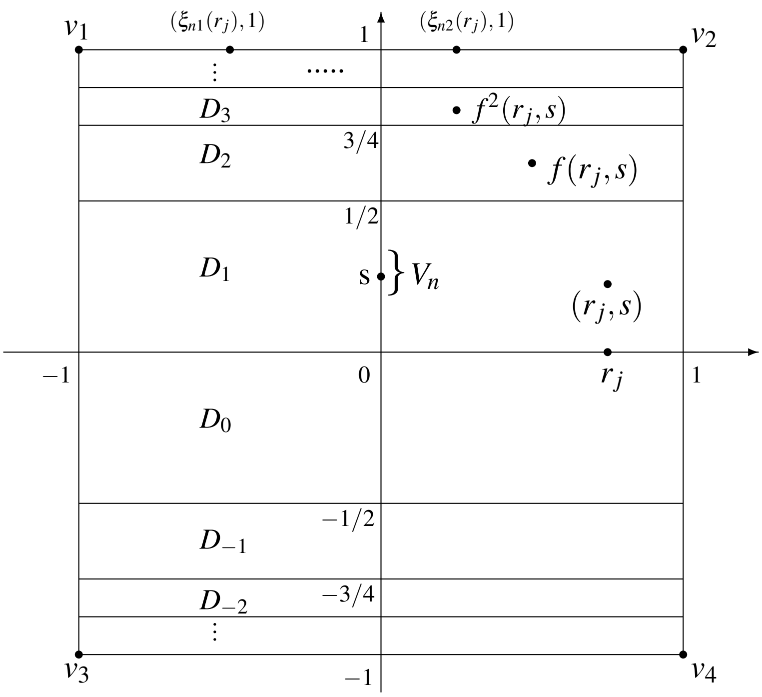

Figure 2.1

Similarly , we can construct a homeomorphism

$$\varphi \,:\, \overline{D^1_{-\infty}} \;\to\; \overline{D^0_{-\infty}} \tag{2.17}$$

such that

(1) $\varphi \mid D_1 \cup J_{-1} \;=\; f_{02}^{-1} \mid D_1 \cup J_{-1} \,:\, D_1 \cup J_{-1} \;\to\; D_0 \cup J_{-1}$;

(2) $\varphi\,(J_s) \;=\; f_{02}^{-1}(J_s)$, for any $s \in [\,-1\,,\,1/2\,]$;

(3) $\omega_\varphi(r,s) \;=\; \alpha_j\,(r)$, for any $r \in J$, any $j \in \mathbb{N}''$, and any $s \in W_j$.

Define $f : J^2 \to J^2$ by

$$f \mid \overline{D_0^\infty} \;=\; f_{2.13}\ , \qquad \text{and} \qquad f \mid \overline{D^{-1}_{-\infty}} \;=\; \varphi^{-1} \mid \overline{D^{-1}_{-\infty}}\ .$$

Then f is a homeomorphism which satisfies the conditions mentioned in Theorem 2.5 , and the proof is complete. □

3. Bounded and unbounded orbits of homeomorphisms on $\mathbb{R}^2$

In this section, we will use Theorem 2.5 to construct a homeomorphism on the plane which illustrates an interesting phenomenon: points of bounded orbits can surround points of divergent orbits.

Let $\Psi : J^2 \to J^2$ be the *level reflect* and let $\Psi_v : J^2 \to J^2$ be the *vertical reflect* defined by

$$\Psi(r,s) = (-r,s) \quad \text{and} \quad \Psi_v(r,s) = (r,-s) \quad \text{for any } (r,s) \in J^2\,. \tag{3.1}$$

Lemma 3.1. *Let $K = [1/3\,,\,1/2]$, and let the rectangle $F = [-1/2\,,\,1/2] \times K$. Write $L_1 = \{-1/2\} \times \overset{\circ}{K}$, and $L_2 = \partial F - L_1$. Let $u_1\,,\,\cdots\,,\,u_6$ be six points in $J \times \partial J$ with*

$$u_1 = (\,-1/2,\,1\,)\ ,\ \ u_2 = (\,0\,,\,1\,)\ ,\ \ u_3 = (\,1/2,\,1\,)\ ,\ \ \text{and}\ \ u_{i+3} = \Psi_v(u_i)\ \ \text{for}\ \ i \in \mathbb{N}_3$$

(see Fig. 3.1 below) . Then there exists a normally rising homeomorphism $f : J^2 \to J^2$ such that

(1) $\omega(x,f) = \{u_1\}$ and $\alpha(x,f) = \{u_4\}$ for any $x \in L_1$,
(2) $\omega(y,f) = \{u_2\}$ and $\alpha(y,f) = \{u_5\}$ for any $y \in \overset{\circ}{F}$, and
(3) $\omega(z,f) = \{u_3\}$ and $\alpha(z,f) = \{u_6\}$ for any $z \in L_2$.

Proof. Let $\mathscr{V} = \{V_1, V_2, V_3\}$ with $V_1 = \{1/3\}$, $V_2 = \overset{\circ}{K}$, $V_3 = \{1/2\}$. Let $\{\omega_i : J \to \mathscr{A}\}_{i=1}^3$ and $\{\alpha_i : J \to \mathscr{A}'\}_{i=1}^3$ be endpoint preserving and increasing maps, which satisfy

(a) $\omega_1(r) = \omega_3(r) = \{u_3\}$, for any $r \in [-1/2, 1/2]$;

(b) $\omega_2(-1/2) = \{u_1\}$, $\omega_2(1/2) = \{u_3\}$, and $\omega_2(r) = \{u_2\}$ for any $r \in (-1/2, 1/2)$;

(c) $\alpha_1(r) = \alpha_3(r) = \{u_6\}$, for any $r \in [-1/2, 1/2]$;

(d) $\alpha_2(-1/2) = \{u_4\}$, $\alpha_2(1/2) = \{u_6\}$, and $\alpha_2(r) = \{u_5\}$ for any $r \in (-1/2, 1/2)$.

Then by Theorem 2.5, there exists a normally rising homeomorphism $f : J^2 \to J^2$ such that $\omega_{sf} = \omega_i$ and $\alpha_{sf} = \alpha_i$ for any $i \in \mathbb{N}_3$ and any $s \in V_i$. Such an f will satisfies the requirements. The proof is complete. □

Let X and Y be topological spaces, and $\beta : X \to X$ and $\gamma : Y \to Y$ be continuos maps. If there exists a continuos surjection (resp. a homeomorphism) $\eta : X \to Y$ such that $\eta\beta = \gamma\eta$, then β and γ are said to be *topologically semi-conjugate* (resp. *topologically conjugate*), and η is called a *topological semi-conjugacy* (resp. a *topological conjugacy*) from β to γ. The following lemma is well known, however, for convenience, we still give a short proof.

Lemma 3.2. *Let $\beta : X \to X$ and $\gamma : Y \to Y$ with a topological semi-conjugacy $\eta : X \to Y$ be as above. If both X and Y are compact metric spaces, then*

(1) *For any $x \in X$, the ω-limit set $\omega(\eta(x), \gamma) = \eta(\omega(x, \beta))$;*

(2) *If both β and γ are homeomorphisms, then, for any $x \in X$, the α-limit set $\alpha(\eta(x), \gamma) = \eta(\alpha(x, \beta))$.*

Proof. (1) For any given $x \in X$, let $y = \eta(x)$. For any $n \in \mathbb{N}$, write $x_n = \beta^n(x)$, and $y_n = \gamma^n(y)$. If some point $w \in \omega(x, \beta)$, then there is a sequence $n_1 < n_2 < \cdots$ in $\mathbb{N}$ such that $\lim_{i\to\infty} x_{n_i} = w$. By the continuity of η, we have $\lim_{i\to\infty} y_{n_i} = \eta(w)$, which means $\eta(w) \in \omega(y, \gamma)$, and hence $\eta(\omega(x, \beta)) \subset \omega(\eta(x), \gamma)$. Conversely, if some point $u \in \omega(y, \gamma)$, then there is a sequence $n_1 < n_2 < \cdots$ in $\mathbb{N}$ such that $\lim_{i\to\infty} y_{n_i} = u$. Since X is compact, there is a point $w \in X$ and a subsequence $m_1 < m_2 < \cdots$ of the sequence $n_1 < n_2 < \cdots$ such that $\lim_{i\to\infty} x_{m_i} = w$, which means

$w \in \omega(x, \beta)$ and leads to $\lim_{i\to\infty} y_{m_i} = \eta(w) = u$. Thus we have $\omega\big(\eta(x), \gamma\big) \subset \eta\big(\omega(x, \beta)\big)$.

(2) If both β and γ are homeomorphisms, then η is also a topological semi-conjugacy from β^{-1} to γ^{-1}, and from the conclusion (1) we get

$$\alpha\big(\eta(x), \gamma\big) = \omega\big(\eta(x), \gamma^{-1}\big) = \eta\big(\omega(x, \beta^{-1})\big) = \eta\big(\alpha(x, \beta)\big). \qquad \square$$

The following theorem is well known, which is an equivalent form of the Schönflies theorem (see e.g. [21, p.72]).

Theorem 3.3. *For any disks E and G in $\mathbb{R}^2$, there exists a homeomorphism $\zeta : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\zeta(G) = E$.*

Let d be the Euclidean metric on $\mathbb{R}^2$. For any homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ and any $x \in \mathbb{R}^2$, the orbit $O(x,h)$ is said to be *positively bounded* (resp. *negatively bounded*) if $O_+(x,h)$ (resp. $O_-(x,h)$) is bounded. If $O(x,h)$ is not positively bounded (resp. not negatively bounded), then $O(x,h)$ is said to be *positively unbounded* (resp. *negatively unbounded*).

The following lemma is clear.

Lemma 3.4. *Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism, and $x \in \mathbb{R}^2$. Then*

(1) *$O(x,h)$ is positively divergent if and only if $\omega(x,h) = \emptyset$;*

(2) *If $O(x,h)$ is positively unbounded, then $\omega(x,h) \neq \emptyset$ if and only if $\omega(x,h)$ is an unbounded set;*

(3) *If $\omega(x,h)$ is a nonempty bounded set, then $O(x,h)$ is positively bounded.*

In the negative direction of the orbit $O(x,h)$, we also have similar conclusions.

Theorem 3.5. *Let E be a disk in $\mathbb{R}^2$. Then there exists a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that, for any $x \in \partial E$, the orbit $O(x,h)$ is bounded, but for any $y \in \mathring{E}$, the orbit $O(y,h)$ is doubly-divergent.*

Proof. Continue to use the all notations in Lemma 3.1. Let $v_1, \cdots, v_4$ and $w_1, \cdots, w_8$ be points in $J \times \partial J$ with (see Fig. 3.1)

$$v_1 = (-1, 1), \quad v_2 = (1, 1), \quad v_3 = (-1, -1), \quad v_4 = (1, -1),$$

$$w_1 = (-3/4, 1), \quad w_2 = (-1/4, 1), \quad w_3 = (1/4, 1), \quad w_4 = (3/4, 1),$$

and $w_{i+4} = \Psi_v(w_i)$ for $i \in \mathbb{N}_4$. Let $x_1, \cdots, x_6$ be points in $\overset{\circ}{J}{}^2$ with

$$x_1 = (-1/2, 3/4), \qquad x_2 = (0, 3/4), \qquad x_3 = (1/2, 3/4)$$

and $x_{i+3} = \Psi_v(x_i)$ for $i \in \mathbb{N}_3$. For any points $y_1, y_2, \cdots, y_n$ in $\mathbb{R}^2$ with $n \geq 2$, denote by $[y_1, y_2, \cdots, y_n]$ or by $[y_1 y_2 \cdots y_n]$ the smallest convex set containing $y_1, y_2, \cdots, y_n$. Clearly, there is a continuous map $\xi : J^2 \to J^2$ satisfying the following conditions :

(a) $\xi \,|\, \big(J \times [-1/2, 1/2]\big) \cup \big(\{-1, 0, 1\} \times J\big)$ is the identity map ;

(b) $\xi(u_3) = x_3$, $\xi(w_3) = \xi(w_4) = u_3$, and $\xi\,|\,[u_2 w_3]$, $\xi\,|\,[w_3 u_3]$, $\xi\,|\,[u_3 w_4]$ and $\xi\,|\,[w_4 v_2]$ are linear ;

(c) $\xi\Psi = \Psi\xi$, and $\xi\Psi_v = \Psi_v\xi$, where $\Psi : J^2 \to J^2$ is the level reflect, and $\Psi_v : J^2 \to J^2$ is the vertical reflect, defined as in (3.1).

(d) $\xi \,|\, \overset{\circ}{J}{}^2$ is an injection, and $\xi(\overset{\circ}{J}{}^2) = \overset{\circ}{J}{}^2 - [u_1 x_1] - [u_3 x_3] - [u_4 x_4] - [u_6 x_6]$.

Let $f : J^2 \to J^2$ be the homeomorphism given in Lemma 3.1 . Define a map $g : J^2 \to J^2$ by $g = \xi f \xi^{-1}$. Note that, if $x \in [u_1 x_1] \cup [u_3 x_3] \cup [u_4 x_4] \cup [u_6 x_6] - \{x_1, x_3, x_4, x_6\}$, then $\xi^{-1}(x)$ contains two points, but $\xi f \xi^{-1}(x)$ still contains only one point. Thus g is well defined. It is easy to see that g is a bijection, and g is continuous. Thus $g : J^2 \to J^2$ is an orientation preserving homeomorphism. Moreover, from $g = \xi f \xi^{-1}$ we obtain $g\xi = \xi f$, this means that f and g are topologically semi-conjugate, and ξ is a topological semi-conjugacy from f to g. By Lemmas 3.1 and 3.2 we get

Claim 3.4.1. (1) $\omega(x, g) = \{x_1\}$ and $\alpha(x, g) = \{x_4\}$ for any $x \in L_1$,
(2) $\omega(y, g) = \{u_2\}$ and $\alpha(y, g) = \{u_5\}$ for any $y \in \overset{\circ}{F}$, and
(3) $\omega(z, g) = \{x_3\}$ and $\alpha(z, g) = \{x_6\}$ for any $z \in L_2$.

Define a homeomorphism $\psi : \overset{\circ}{J}{}^2 \to \mathbb{R}^2$ by $\psi(r, s) = \big(\mathrm{tg}(\pi r/2), \mathrm{tg}(\pi s/2)\big)$, for any $(r, s) \in \overset{\circ}{J}{}^2$. Write $G = \psi(F)$. Then $G = [-1, 1] \times [\sqrt{3}/3, 1]$ is also a rectangle, and $\psi(L_1) = \{-1\} \times (\sqrt{3}/3, 1)$, $\psi(L_2) = \partial G - \psi(L_1)$. By Theorem 3.3, there exists a homeomorphism $\zeta : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\zeta(G) = E$. Let $h = \zeta\psi g \psi^{-1}\zeta^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$. Then h is also an orientation preserving homeomorphism, which is topologically conjugate to g, and $\zeta\psi$ is a topological conjugacy from g to h. By Claim 3.4.1 and Lemma 3.2, we have

Claim 3.4.2. (1) $\omega(x,h) = \{\zeta\psi(x_1)\}$, and $\alpha(x,h) = \{\zeta\psi(x_4)\}$, for any $x \in \zeta\psi(L_1)$;

(2) $\omega(y,h) = \alpha(y,h) = \emptyset$, for any $y \in \zeta\psi(\mathring{F})$;

(3) $\omega(z,h) = \{\zeta\psi(x_3)\}$, and $\alpha(z,h) = \{\zeta\psi(x_6)\}$, for any $z \in \zeta\psi(L_2)$.

Noting that $\mathring{E} = \zeta\psi(\mathring{F})$ and $\partial E = \zeta\psi(\partial F) = \zeta\psi(L_1) \cup \zeta\psi(L_2)$, from Claim 3.4.2 and Lemma 3.4 we see that the homeomorphism h satisfies the requirement. The proof is complete. $\square$

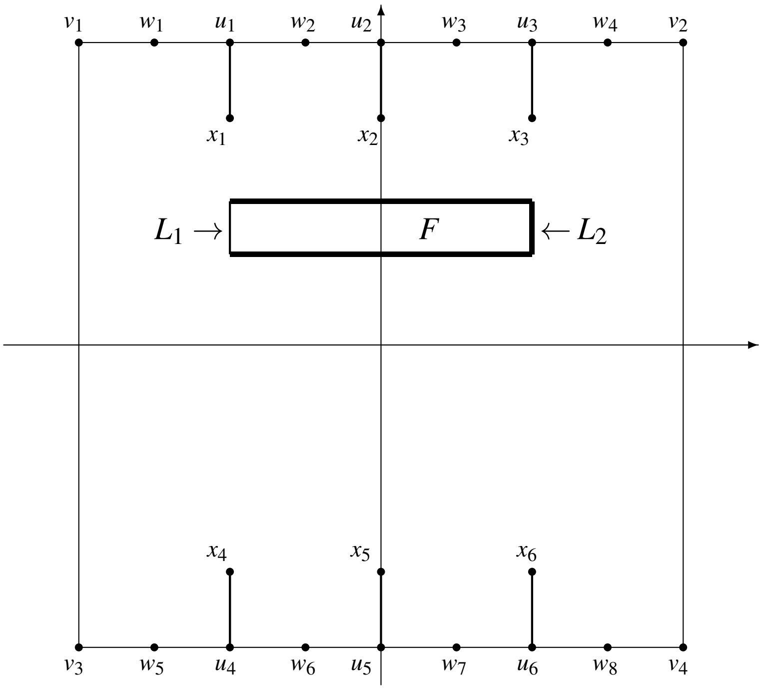

Figure 3.1

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