OBSTRUCTIONS FOR MINIMAL DISTAL ACTIONS

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ABSTRACT. In this paper, we mainly consider the nonexistences of minimal distal actions by some groups on compact manifolds, particularly on surfaces. Suppose that X is a compact manifold and Γ is a finitely generated group acting on X. We show in the following cases that Γ cannot act on X minimally and distally. (1) X is connected and the first Čech cohomology group $\check{H}^1(X)$ with integer coefficients is nontrivial and Γ is amenable; (2) X is the 2-sphere $\mathbb{S}^2$ or the real projective plane $\mathbb{RP}^2$ and Γ contains no nonabelian free subgroup; (3) X is a closed surface and Γ is a lattice of $\mathrm{SL}(n,\mathbb{R})(n \geq 3)$.

1. INTRODUCTION

The notion of distality was introduced by Hilbert for better understanding equicontinuity ([8]). The study of minimal distal systems culminates in the beautiful structure theorem of H. Furstenberg ([10]), which describes completely the relations between distality and equicontinuity for minimal systems. Considering minimal distal actions on compact manifolds, Rees proved a sharpening of Furstenberg's structure theorem ([21]). An interesting question is that given a discrete group G and a compact metric space X, can G act on X distally and minimally? Clearly, the answer to this question depends on the topology of the phase space and the algebraic structure of the acting group.

Due to the fact that every nontrivial minimal distal system admits a nontrivial equicontinuous factor, there are groups that have no nontrivial minimal equicontinuous actions and hence have no nontrivial minimal distal actions, e.g. minimally almost periodic groups that are groups having trivial Bohr compactifications. In [1], the author shows that an almost connected locally compact second countable group admits a faithfully distal action if and only if it is polynomially growing (some partial results are also obtained in [18]) and also gives some necessary conditions for a countable discrete group admitting faithfully distal actions. These indicate that there are obstructions from groups on distal actions.

On the other hand, the topology of phase spaces also can prevent groups acting distally and minimally. A remarkable result due to Furstenberg says that if a nontrivial space X admits a distal minimal action by a locally compact abelian group, then X cannot be simply connected (see [10, Theorem 11.1] or [2, Chapter 7-Theorem 16]). Bronšteĭn proved that if X is a connected and locally connected finitely dimensional compact metric space

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which admits a distal minimal group action, then X must be a manifold and the fundamental group $\pi_1(X)$ is virtually nilpotent ([7]); this implies that if X is a closed surface except for the sphere $\mathbb{S}^2$, the real projective plane $\mathbb{RP}^2$, the torus $\mathbb{T}^2$, and the Klein bottle $\mathbb{K}^2$, then it admits no distal minimal actions by any group.

In this paper, we are aiming at further study on some obstructions on minimal distal actions, particular in actions on surfaces.

Our first result is to consider minimal distal action by amenable groups which is an analogy of Furtenberg's result that simply connectivity of the phase space is an obstruction for a locally compact abelian group acting minimally and distally.

Theorem 1.1. *Let X be a non-degenerate compact connected metric space. If X admits a minimal distal action by a finitely generated amenable group, then the first Čech cohomology group $\check{H}^1(X)$ with integer coefficients is nontrivial. In particular, if X is homotopically equivalent to a CW complex, then the fundamental group of X cannot be finite.*

Then the following corollary is immediate.

Corollary 1.2. *The n-sphere $\mathbb{S}^n$ ($n \geq 2$) does not admit any minimal distal actions by finitely generated amenable groups.*

If we focus on $\mathbb{S}^2$ or $\mathbb{RP}^2$ we can show more. Recall that a group G is a *small group* if it contains no free nonabelian subgroups.

Theorem 1.3. *Let X be the 2-sphere $\mathbb{S}^2$ or the real projective plane $\mathbb{RP}^2$. If Γ is a finitely generated group acting minimally and distally on X, then Γ contains a nonabelian free subgroup. Equivalently, X admits no distal minimal actions by a small group.*

Here we remark that the class of small groups is strictly larger than that of amenable groups, so this theorem is not implied by Theorem 1.1 and it is easy to construct distal minimal actions on $\mathbb{S}^2$ and $\mathbb{RP}^2$ by $\mathbb{Z} * \mathbb{Z}$. However, the theorem does not hold when X is either the torus $\mathbb{T}^2$ or the Klein bottle $\mathbb{K}^2$, since they admit distal minimal actions by abelian groups (see section 4.1).

Compared to amenable groups, groups having Kazhdan's property (T) lie in the other extreme end. In [27, Corollary 1.2], Zimmer shows that every regular distal ergodic action on compact manifolds under groups having Kazhdan's property (T) is isometric and it is conjectured that whether it also holds for distal action. Here regular distality is stronger than distality (see [27]). However, we will show this is the case for surfaces (2-dimensional compact connected manifolds). Considering the distal minimal actions of some higher rank lattices on closed surfaces, we obtain the following result.

Theorem 1.4. *Let Γ be a lattice in $\mathrm{SL}(n,\mathbb{R})$ with $n \geq 3$ and X be a closed surface. Then Γ has no distal minimal action on X.*

Here we remark that this theorem does not hold for the case $n = 2$, since the free nonabelian group $\mathbb{Z}*\mathbb{Z}$ is a lattice in $\mathrm{SL}(2,\mathbb{R})$ and we have mentioned that $\mathbb{Z}*\mathbb{Z}$ can act on $\mathbb{S}^2$ minimally and distally. In addition, there do exist isometric minimal actions of some lattices in $\mathrm{SL}(n,\mathbb{R})$ on compact manifolds with dimension > 2 (see e.g. [17] or [7, p.59]). We should notice that there are nondistal minimal actions of higher rank lattices on surfaces, such as the action of $\mathrm{SL}(3,\mathbb{Z})$ on $\mathbb{RP}^2$ induced by the linear action on $\mathbb{R}^3$.

Now we summarize all the known results around the existence of distal minimal group actions on closed surfaces in the following tabular.

Closed surfaces	Existence	Non-existence
$\mathbb{S}^2, \mathbb{RP}^2$	$\mathbb{Z}*\mathbb{Z}$	small groups, $\mathrm{SL}(n,\mathbb{Z})(n \geq 3)$
$\mathbb{T}^2, \mathbb{K}^2$	$\mathbb{Z}$	$\mathrm{SL}(n,\mathbb{Z})(n \geq 3)$
Others		Any groups

Organization of the paper. Section 2 is devoted to give some necessary notions and classical results used in the sequel. Then we prove Theorem 1.1, Theorem 1.3 and Theorem 1.4 in section 3, 4 and 5 repectively.

2. Preliminaries

In this section, we will recall some basic notions and introduce some results which will be used in the proof of the main theorems.

2.1. **Distal actions.** Let X be a topological space and let $\mathrm{Homeo}(X)$ be the homeomorphism group of X. Suppose G is a topological group. A group homomorphism $\phi : G \to \mathrm{Homeo}(X)$ is called a *continuous action* of G on X if $(x,g) \mapsto \phi(g)(x)$ is continuous; we use the symbol (X,G,ϕ) to denote this action. The action ϕ is said to be *faithful* if it is injective. For brevity, we usually use gx or $g(x)$ instead of $\phi(g)(x)$ and use (X,G) instead of (X,G,ϕ) if no confusion occurs.

For $x \in X$, the *orbit* of x is the set $Gx \equiv \{gx : g \in G\}$; $K \subset X$ is called G *invariant* if $Gx \in K$ for every $x \in X$; (X,G,ϕ) is called *minimal* if Gx is dense in X for every $x \in X$, which is equivalent to that G has no proper closed invariant set; is called *transitive* if $Gx = X$ for some $x \in X$. If $K \subset X$ is G invariant, then we naturally get a restriction action $\phi|K$ of G on X; if K is closed and nonempty, and the restriction action $(K,G,\phi|K)$ is minimal, then we call K a *minimal set* of X or of the action. It is well known that (X,G,ϕ) always has a minimal set when X is a compact metric space.

Suppose (X,G,ϕ) and (Y,G,ψ) are two actions. If there is a continuous surjection $f : X \to Y$ such that $f(\phi(g)x) = \psi(g)f(x)$ for every $g \in G$ and every $x \in X$, then we say f is a *homomorphism* and (Y,G,ψ) is a *factor* of (X,G,ϕ). If Y is a single point, then we call (Y,G,ψ) a *trivial factor* of (X,G,ϕ).

Assume further that X is a compact metric space with metric d. The action (X,G,ϕ) is called *equicontinuous* if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $d(gx,gy) < \varepsilon$ for any

$g \in G$ whenever $d(x,y) < \delta$; is called *distal*, if for every $x \neq y \in X$, $\inf_{g\in G} d(gx,gy) > 0$. Clearly, equicontinuity implies distality.

The following results can be found in [2].

Theorem 2.1 ([2], p.98)**.** *Let (X,G,ϕ) and (Y,G,ψ) be distal minimal actions, and let $f : X \to Y$ be a homomorphism. Then f is open.*

Theorem 2.2 ([2], p.104)**.** *Let (X,G,ϕ) be a distal minimal action. If X is not a single point, then (X,G,ϕ) has a nontrivial maximal equicontinuous factor.*

Theorem 2.3 ([2], p.52)**.** *Let (X,G,ϕ) be equicontinuous. Then the closure $\overline{\phi(G)}$ in* $\mathrm{Homeo}(X)$ *with respect to the uniform convergence topology is a compact topological group.*

Theorem 2.4. [7, Theorem 3.17.12] *Let $\pi : (X,G,\phi) \to (Y,G,\psi)$ be a homomorphism between minimal systems. Suppose that π is open and G is finitely generated. If (Y,G,ψ) is equicontinuous and there is some $y \in Y$ such that $\pi^{-1}(y)$ is of* 0*-dimension. Then (X,G,ϕ) is also equicontinuous.*

Theorem 2.5. [21] *Let (X,G,ϕ) and (Y,G,ψ) be distal minimal actions, and let $f : X \to Y$ be a homomorphism. Then for every $y \in Y$, we have* $\dim(Y) + \dim(f^{-1}(y)) = \dim(X)$.

We remark here that Rees also showed in [22] that fibers are the same for distal extension under some mild conditions. Let (X,G) be a quasi-separable minimal system and $\pi : (X,G) \to (Y,G)$ be a distal extension. If Y is arcwise connected, then for any $y_1, y_2 \in Y$, the fibers $\pi^{-1}(y_1)$ and $\pi^{-1}(y_2)$ are homeomorphic. In addition, she also constructed a counterexample there to show that the arcwise connectedness of Y cannot be dropped.

Theorem 2.6. [7, Theorem 3.17.10] *Let (X,G,ϕ) be a distal minimal system with X being a connected and locally connected compact metric space of finite dimension, then X is a manifold.*

It is well known that a compact connected Hausdorff space is locally connected metrizable if and only if it is a continuous image of the closed interval $[0,1]$ ([19, Theorem 8.18]). Thus the following result is direct.

Lemma 2.7. *Let X be a compact metric space which is connected and locally connected and Y be a Hausdorff space. If there is a continuous surjection $f : X \to Y$, then Y is connected, locally connected and metrizable.*

Now combining the above results, we can have the following proposition.

Proposition 2.8. *Let G be a finitely generated group and X be a connected compact manifold with* $\dim(X) \geq 2$. *If G acts on X minimally and distally, then the maximal equicontinuous factor Y is also a connected manifold with* $1 \leq \dim(Y) \leq \dim(X)$. *Further, if (X,G) is not equicontinuous then* $\dim(Y) < \dim(X)$.

Proof. First by Theorem 2.2, (X,G) has a nontrivial maximal equicontinuous factor (Y,G). Then it follows from Theorem 2.4 and Theorem 2.5 that $\dim(Y) \leq \dim(X)$ and $\dim(Y) < \dim(X)$ whenever (X,G) is not equicontinuous. By Lemma 2.7, Y is connected and locally connected. Then Theorem 2.6 implies that Y is also a connected compact manifold, since $\dim(Y) \leq \dim(X)$. Clearly, $\dim(Y) \geq 1$ since Y is nontrivial and connected. □

2.2. **Compact Lie groups.** Let G be a connected Lie group and let $\mathrm{Lie}(G)$ be the Lie algebra of G. Recall that G is said to be *solvable* if its Lie algebra is solvable, that is there is a sequence of ideals $\mathrm{Lie}(G) = \mathfrak{I}_0 \rhd \mathfrak{I}_1 \rhd \cdots \rhd \mathfrak{I}_n = \{0\}$ such that $\mathfrak{I}_i/\mathfrak{I}_{i+1}$ is commutative for each i; this is equivalent to the existence of a sequence of closed normal subgroups $G = G_0 \rhd G_1 \rhd \cdots \rhd G_n = 0$ such that G_i/G_{i+1} is commutative for each i. If the Lie algebra $\mathrm{Lie}(G)$ of G contains no nontrivial solvable ideal, then G is said to be *semisimple*.

The following theorems are classical in the theory of Lie groups.

Theorem 2.9 ([12], Corollary 4.25)**.** *Let G be a compact Lie group and let $\mathfrak{I}$ be the Lie algebra of G. Then $\mathfrak{I} = Z(\mathfrak{I}) \bigoplus [\mathfrak{I},\mathfrak{I}]$, where $Z(\mathfrak{I})$ is the center of $\mathfrak{I}$ and $[\mathfrak{I},\mathfrak{I}]$ is semisimple.*

Theorem 2.10 ([12], Corollary 1.103)**.** *Let G be a compact connected commutative Lie group of dimension n. Then G is isomorphic to the n-torus $\mathbb{T}^n$.*

Corollary 2.11. *Let G be a connected compact Lie group. If G is solvable, then G is isomorphic to the n-torus $\mathbb{T}^n$.*

Proof. Let $\mathfrak{I}$ be the Lie algebra of G and let $Z(\mathfrak{I})$ be its center. If $[\mathfrak{I},\mathfrak{I}] \neq 0$, then $\mathfrak{I}/Z(\mathfrak{I})$ is semisimple by Theorem 2.14. However, this is impossible since $\mathfrak{I}/Z(\mathfrak{I})$ is also solvable. So $[\mathfrak{I},\mathfrak{I}] = 0$ and hence $\mathfrak{I}$ is commutative. This implies G is commutative, since G is connected. It follows from Theorem 2.10 that G is isomorphic to the n-torus $\mathbb{T}^n$, where n is the dimension of G. □

Theorem 2.12 ([12], Corollary 4.22)**.** *Let G be a compact Lie group. The G is isomorphic to a closed linear group.*

Theorem 2.13 ([16], p.99)**.** *Let G be a compact group and let U be an open neighborhood of the identity e. Then U contains a normal subgroup H of G such that G/H is isomorphic to a Lie group.*

Theorem 2.14. [12, Corollary 4.25] *Let G be a compact Lie group and let $\mathfrak{g}$ be the Lie algebra of G. Then $\mathfrak{g} = Z(\mathfrak{g}) \bigoplus [\mathfrak{g},\mathfrak{g}]$, where $Z(\mathfrak{g})$ is the center of $\mathfrak{g}$ and $[\mathfrak{g},\mathfrak{g}]$ is semisimple.*

Theorem 2.15. [24, Theorem 3.50] *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Then the center of G is a closed Lie subgroup of G with Lie algebra the center of $\mathfrak{g}$.*

2.3. **Compact transformation group.** Let (X,G,ϕ) be a group action and H be a closed subgroup of G. Then we use X/H to denote the orbit space under the H action, which is endowed with the quotient space topology. We use G/H to denote the coset space with the quotient topology, which is also the orbit space obtained by the left translation action on G by H. If H is a normal closed subgroup of G, then G/H is a topological group.

The following theorems can be seen in [16]. We only state them in some special cases which are enough for our uses.

Theorem 2.16 ([16], p.65)**.** *Let X be a compact metric space and let (X,G) be an action of group G on X. Suppose G is compact. Then for every $x\in X$, G/G_x is homeomorphic to Gx, where $G_x=\{g\in G: gx=x\}$.*

Theorem 2.17 ([16], p.61)**.** *Let X be a compact metric space and let (X,G) be an action of group G on X. Suppose G is compact and H is a closed normal subgroup of G. Then G/H can act on X/H by letting $gH\cdot H(x)=H(gx)$ for $gH\in G/H$ and $H(x)\in X/H$.*

2.4. **Čech cohomology group.** First we will recall an equivalent definition of the first Čech cohomology group with integer coefficients. Let $\mathbb{S}^1$ be the unit circle in the complex plane. For any paracompact normal space X, let $C(X,\mathbb{S}^1)$ be the set of all continuous functions from X to $\mathbb{S}^1$, and let $I(X,\mathbb{S}^1)$ be the set of all $f\in C(X,\mathbb{S}^1)$ which is inessential (i.e. f is homotopoic to a constant map from X to $\mathbb{S}^1$). Then under pointwise complex multiplication, $C(X,\mathbb{S}^1)$ becomes a commutative group and $I(X,\mathbb{S}^1)$ is a subgroup of $C(X,\mathbb{S}^1)$. Define the first cohomology group $\check{H}^1(X)$ of X by $\check{H}^1(X)=C(X,\mathbb{S}^1)/I(X,\mathbb{S}^1)$.

Suppose $f:X\to Y$ is continuous. Then f naturally induced a group homomorphism $f^*:H^1(Y)\to H^1(X)$ by letting $f^*([g])=[g\circ f]$ for any $[g]\in H^1(Y)$. The map f is called *confluent* if for any subcontinuum B of Y and any component A of $f^{-1}(B)$, we have $f(A)=B$.

The following theorem is due to Lelek (see [14] or [19, Theorem 13.45]).

Theorem 2.18. *Let $f:X\to Y$ be a confluent map from continuum X onto continuum Y. Then $f^*:H^1(Y)\to H^1(X)$ is injective.*

The following theorem is due to Whyburn (see [25] or [19, Theorem 13.14]).

Theorem 2.19. *Every open map of one compact metric space onto another is confluent.*

From Theorem 2.18 and Theorem 2.19, we immediately get the following corollary.

Corollary 2.20. *Let $f:X\to Y$ be an open map from continuum X onto continuum Y. Then $f^*:H^1(Y)\to H^1(X)$ is injective.*

3. Amenable group actions

This section is devoted to prove Theorem 1.1.

3.1. **Amenable groups.** *Amenability* was first introduced by von Neumann. Recall that a countable group G is *amenable* if there is a sequence of finite sets F_i ($i = 1,2,3, \dots$) such that $\lim\limits_{i\to\infty} \frac{|gF_i \triangle F_i|}{|F_i|} = 0$ for every $g \in G$, where $|F_i|$ is the number of elements in F_i; the sequence (F_i) is called a Følner *sequence*. For an abstract group G, if there is a sequence of normal subgroups $G = G_0 \rhd G_1 \rhd \cdots \rhd G_n = \{e\}$ such that G_i/G_{i+1} is commutative for each i, then G is called *solvable*.

Now we list some well known facts about amenable groups and solvable groups. One may consult [20] for the details.

Theorem 3.1. *(1) Solvable groups and finite groups are amenable; (2) any group containing a free noncommutative subgroup cannot be amenable; (3) every subgroup of an amenable group (resp. solvable group) is amenable (resp. solvable); (4) every quotient group of an amenable group (resp. solvable group) is amenable (resp. solvable).*

Lemma 3.2. *Let G be a Lie group and Γ be a dense subgroup of G. If Γ is solvable as an abstract group, then G is a solvable Lie group.*

Proof. Since Γ is solvable, we have a sequence of normal subgroups Γ_i such that $\Gamma = \Gamma_0 \rhd \Gamma_1 \rhd \cdots \rhd \Gamma_n = \{e\}$ such that Γ_i/Γ_{i+1} is commutative for each i. Let $G_i = \overline{\Gamma}_i$. Then we get a decreasing sequence of closed normal subgroups $G = G_0 \rhd G_1 \rhd \cdots \rhd G_n = \{e\}$. Since $\Gamma_{i+1} \subset G_{i+1}$ and Γ_i/Γ_{i+1} is commutative, $\Gamma_i/(\Gamma_i \cap G_{i+1})$ is commutative. So G_i/G_{i+1} is commutative by the continuity of group operations. Then G is solvable. □

Theorem 3.3. [4, Theorem 1.3] *Let G be a connected non-solvable real Lie group of dimension d. Then any finitely generated dense subgroup of G contains a dense free subgroup of rank* $2d$.

The following remarkable result is known as Tits Alternative (see [23]).

Theorem 3.4. *Let Γ be a finitely generated subgroup of a linear group. Then either Γ contains a free nonabelian subgroup or Γ has a solvable subgroup with finite index.*

3.2. **Proof of Theorem 1.1.**

Lemma 3.5. *Let G be a compact Lie group and G_0 be the connected component of the unit $e \in G$. Suppose G acts transitively on a connected compact manifold M. Then the G_0 action on M is also transitive.*

Proof. It is well known that G_0 is a clopen normal subgroup of G. Since G is compact, G_0 has finite index in G. Let $G = g_1G_0 \cup \ldots \cup g_kG_0$ be the coset decomposition, where $k = [G : G_0]$. Fix an $x_0 \in M$. Then $\cup_{i=1}^k g_iG_0x_0 = Gx_0 = M$, since the G action is transitive. So, G_0x_0 contains a nonempty open set, which implies that G_0x_0 is open by the homogeneous of the orbit G_0x_0. Thus G_0x_0 is clopen in M. Hence $G_0x_0 = M$ by the connectedness of M. □

Proof of Theorem 1.1. Let X be a connected compact metric space and let Γ be a finitely generated amenable group. Suppose X admits a minimal distal action $\phi : \Gamma \to \text{Homeo}(X)$. We will show that the first Čech cohomology group $\check{H}^1(X)$ with integer coefficients is nontrivial.

By Theorem 2.2, there is a nontrivial equicontinuous factor (Y, Γ, ψ) of (X, Γ, ϕ), which is still minimal and connected. Set $H = \overline{\psi(\Gamma)}$. From Theorem 2.3, H is a compact subgroup of $\text{Homeo}(Y)$ with respect to the uniform convergence topology. Applying Theorem 2.13, we can take a small normal subgroup H' of H such that H/H' is a Lie group and $H'y$ is a proper subset of Y for every $y \in Y$.

Then we get an equicontinuous action ψ' of the Lie group H/H' on the quotient space Y/H' by Theorem 2.17; in particular, Y/H' is homeomorphic to the quotient space $(H/H')/Ker(\psi')$ by Theorem 2.16, which is a connected compact manifold of dimension ≥ 1 (see [24, Theorem 3.58]). Then H/H' is a compact Lie group of dimension ≥ 1. From Theorem 3.1-(4), $\psi(\Gamma)H'$ is an amenable subgroup of H/H' (as abstract groups); from Theorem 2.12, Theorem 3.4, and Theorem 3.1-(2), we see that $\psi(\Gamma)H'$ has a solvable subgroup Γ' of finite index. Then $\overline{\Gamma'}$ is a closed subgroup of H/H' with finite index, and hence contains the connected component $(H/H')_0$ of H/H'. Since Γ' is solvable as a abstract group, $\overline{\Gamma'}$ is a solvable Lie group by Lemma 3.2. So, $(H/H')_0$ is a connected compact solvable Lie group, and hence is isomorphic to $\mathbb{T}^n$ with $n \geq 1$, by Corollary 2.11.

It follows from Lemma 3.5 that the $(H/H')_0$ action on Y/H' is still transitive. So, Y/H' is homeomorphic to $(H/H')_0/(Ker(\psi') \cap (H/H')_0)$. Since $(H/H')_0$ is isomorphic to $\mathbb{T}^n$, Y/H' is homeomorphic to $\mathbb{T}^m$ for some $1 \leq m \leq n$. Thus the first Čech cohomology group $\check{H}^1(Y/H') \cong \check{H}^1(\mathbb{T}^m) \cong \mathbb{Z}^m \neq 0$. Noting that $(Y/H', \Gamma)$ is a minimal equicontinuous factor of (X, Γ), we denote by π the factor map between them. Then π is open and surjective by Theorem 2.1. Applying Theorem 2.20, we have $\pi^* : \check{H}^1(Y/H') \to \check{H}^1(X)$ is injective; in particular, $\check{H}^1(X) \neq 0$.

Since the first Čech cohomology group coincides with the first singular cohomology group $H^1(X, \mathbb{Z})$ when X is homotopically equivalent to a CW complex. By the universal coefficient theorem, we have

$$0 \to Ext(H_0(X), \mathbb{Z}) \to H^1(X, \mathbb{Z}) \to Hom(H_1(X, \mathbb{Z}), \mathbb{Z}) \to 0.$$

Since $H_0(X)$ is always free, $Ext(H_0(X), \mathbb{Z}) = 0$. Thus $H^1(X, \mathbb{Z}) \cong Hom(H_1(X, \mathbb{Z}), \mathbb{Z})$. Hence $H_1(X, \mathbb{Z})$ is not finite. Finally, it follows from $H_1(X, \mathbb{Z}) = \pi_1(X)/[\pi_1(X), \pi_1(X)]$ that $\pi_1(X)$ is not finite. □

Note that an ingredient used in the proof of Theorem 1.1 is the existence of a non-trivial equicontinuous factor. It is well known that a minimal system with an invariant probability measure is weakly mixing if and only if it has no non-trivial equicontinuous factor(see [2, p. 132]). Thus if Γ is a finitely generated amenable group and X is continuum with $\check{H}^1(X)$

is nontrivial then every minimal action by Γ on X is weakly mixing since Γ preserves a probability measure on X.

4. Small group actions on $\mathbb{S}^2$ or $\mathbb{RP}^2$

Lemma 4.1. *Let X be the 2-sphere $\mathbb{S}^2$ or the real projective plane $\mathbb{RP}^2$. Let Γ be a finitely generated group and $\phi : \Gamma \to \mathrm{Homeo}(X)$ be a distal minimal action on X. Then (X, Γ, ϕ) is equicontinuous.*

Proof. Assume to the contrary that ϕ is not equicontinuous. From Theorem 2.2, we let (Y, Γ, ψ) be the maximal equicontinuous factor of (X, Γ, ϕ) with a factor map π. By Proposition 2.8, we conclude that Y is a connected compact manifold of 1-dimension. Further, since Γ acts on Y minimally, Y is the circle $\mathbb{S}^1$. Since π is open and $\check{H}^1(\mathbb{S}^1)$ is the integer group, from Theorem 2.1 and Theorem 2.20, we have that $\check{H}^1(X)$ is infinite, which is a contradiction. □

Lemma 4.2. *Let G be a connected compact Lie group acting faithfully and transitively on a closed surface X with finite fundamental group. Then G is semisimple.*

Proof. Assume to the contrary that G is not semisimple. Then by Theorem 2.14, Theorem 2.10, and Theorem 2.15, the connected component $Z(G)_0$ of the center of G is isomorphic to some torus $\mathbb{T}^n$ with $n > 0$. Set $K = Z(G)_0$. For $x \in X$, let $\mathrm{Stab}(x) := \{k \in K : kx = x\}$ be the stabilizer of x in K. Then from Theorem 2.16, Kx is homeomorphic to $K/\mathrm{Stab}(x)$ which is also a torus. Thus Kx is either a point or a circle. If for every $x \in X$, Kx is a circle, then similar to the arguments in Lemma 4.1, X/K is a circle and the Čech cohomology group $\check{H}^1(X)$ with integer coefficients is infinite. This is a contradiction. So, there is some $x_0 \in X$ with $Kx_0 = x_0$. Since K is in the center of G, we have $Kgx_0 = gKx_0 = gx_0$ for every $g \in G$. Noting that $Gx_0 = X$, the action of K on X is trivial, which contradicts the faithfulness of the action. So, G is semisimple. □

Proof of Theorem 1.3. Let $\phi : \Gamma \to \mathrm{Homeo}(X)$ be the distal minimal action. From Lemma 4.1, we see that the (X, Γ, ϕ) is equicontinuous. Let K be the closure of $\phi(\Gamma)$ with respect to the uniform topology on $\mathrm{Homeo}(X)$. It follows from Theorem 2.3 that K is a compact metric group acting transitively on X. By Theorem 2.13, we can take a closed normal subgroup N of K such that K/N is a Lie group and X/N is not a single point. Then it canonically induces an action of K on X/N and the natural quotient map $X \to X/N$ is a equicontinuous extension. Thus it follows from Theorem 2.5, Theorem 2.6, and Theorem 2.7 that X/N is a manifold of dimension ≤ 2. Similar to the arguments in Lemma 4.1, we have $\dim(X/N) = 2$.

Set $p : \mathbb{S}^2 \to X$ be a covering and $q : X \to X/N$ be the quotient map. Let $\pi : Y \to X/N$ be the universal covering and $\widetilde{qp} : \mathbb{S}^2 \to Y$ be the lifting of qp. Since π is open and p and q are local homeomorphisms, we see that $\widetilde{qp}$ is open. Thus $\widetilde{qp}(\mathbb{S}^2)$ is open and closed in

Y. Thus $Y = \widetilde{qp}(\mathbb{S}^2)$ by the connectedness. So Y is compact (homeomorphic to $\mathbb{S}^2$). Thus π is a finite cover and then the fundamental group of X/N is finite.

Now consider the natural action of K/N on X/N (see Theorem 2.17). Since the connected component $(K/N)_0$ has finite index in K/N, the $(K/N)_0$ action on X/N is still transitive and $\phi(\Gamma)N \cap (K/N)_0$ is dense in $(K/N)_0$. Applying Lemma 4.2, we see that a quotient group of $(K/N)_0$ is semisimple. This together with Theorem 3.3 implies the existence of free nonabelian subgroups in Γ. □

4.1. **Minimal distal homeomorphism on the Klein bottle.** In this subsection, we show that there is a minimal distal homeomorphism on the Klein bottle.

Let the torus $\mathbb{T}^2$ be $\mathbb{R}^2/\mathbb{Z}^2$. Then there is a $\mathbb{Z}_2$ action on $\mathbb{T}^2$ by

$$h(x,y) = (x + \frac{1}{2}, 1-y) \mod \mathbb{Z}^2,$$

where h is the nonidentity element in $\mathbb{Z}_2$. It is easy to see that this action is free and properly discontinuous and the quotient space $\mathbb{T}^2/\mathbb{Z}_2$ is the Klein bottle $\mathbb{K}^2$.

Now for a homeomorphism T of $\mathbb{T}^2$, if it commutes with h, i.e., $Th = hT$, then it induces a homeomorphism $\widetilde{T}$ of $\mathbb{K}^2$.

Let α be an irrational number and $\phi : \mathbb{T} \to \mathbb{T}$ be a continuous mapping. Further, define a homeomorphism $T : \mathbb{T}^2 \to \mathbb{T}^2$ by

$$T(x,y) = (x+\alpha, y+\phi(x)) \mod \mathbb{Z}^2.$$

The system $(\mathbb{T}^2, T)$ is a skew product system and it is well known that this system is distal, since it is a group extension of a minimal equicontinuous system. The following theorem characterizes the minimality of such skew product system.

Theorem 4.3. [2, Chapter 5, Theorem 10] *The above defined system $(\mathbb{T}^2, T)$ is minimal if and only if for each $k \in \mathbb{Z} \setminus \{0\}$, there is no continuous function $f : \mathbb{T} \to \mathbb{T}$ such that $f(x+\alpha) = f(x) + k\phi(x)$ for each $x \in \mathbb{T}$.*

Now we take $\phi : \mathbb{T} \to \mathbb{T}$ to be $\phi(x) = 1 - |1-2x|$ for $x \in [0,1)$. By comparing the Fourier coefficients, there is no continuous function $f : \mathbb{T} \to \mathbb{T}$ such that $f(x+\alpha) = f(x) + k\phi(x)$ for each $k \in \mathbb{Z} \setminus \{0\}$. Thus the system $(\mathbb{T}^2, T)$ is minimal by Theorem 4.3. It is straightforward to calculate that for each $(x,y) \in \mathbb{T}^2$, $Th(x,y) = (x+\alpha+\frac{1}{2}, 1-y+\phi(x+\frac{1}{2}))$ and $hT(x,y) = (x+\alpha+\frac{1}{2}, 1-y-\phi(x))$. Note that

$$\phi(x+\frac{1}{2}) = \begin{cases} 1-2x, & x \in [0,1/2) \\ 2x-1, & x \in [1/2,1) \end{cases} \text{ and } -\phi(x) = \begin{cases} -2x, & x \in [0,1/2) \\ 2x-2, & x \in [1/2,1) \end{cases}.$$

Therefore, it follows that T commutes with h and thus the induced homeomorphism $\widetilde{T}$ on $\mathbb{K}^2$ is also minimal and distal.

5. Higher rank lattice actions

5.1. **Higher rank lattices.** A subgroup Γ of a Lie group G is called a lattice in G if Γ is discrete and G/Γ has finite volume. If Γ is a lattice of G and G/Γ is compact, then Γ is called a *cocompact* lattice of G. It is well known that $\mathrm{SL}(n,\mathbb{R})$ with $n \geq 3$ always has cocompact and non-cocompact lattices; $\mathrm{SL}(n,\mathbb{Z})$ is a non-cocompact lattice of $\mathrm{SL}(n,\mathbb{R})$. One may consult [26] for the examples of cocompact lattices in $\mathrm{SL}(n,\mathbb{R})$.

The following theorem can be deduced from [15, Theorem VII.6.5] and [26, Corollary 16.4.2] (see also [7, Theorem 12.4]).

Theorem 5.1. *Let Γ be a lattice in $\mathrm{SL}(n,\mathbb{R})$ with $n \geq 3$ and let H be a compact Lie group. Suppose $\phi : \Gamma \to H$ is a group homomorphism. If $\phi(\Gamma)$ is infinite, then $\dim(H) \geq n^2 - 1$.*

5.2. **Dimension of compact subgroups of** $\mathrm{Homeo}(\mathscr{S})$**.** It is well known that if M is a connected compact Riemannian manifold of dimension n and $I(M)$ is the isometric transformation group of M, then $I(M)$ is a compact Lie group of dimensional at most $n(n+1)/2$ (see e.g. [13]). In general, for a compact group G of $\mathrm{Homeo}(M)$, it no longer preserves the Riemannian metric on M. However, G can still preserve a compatible metric on M. In fact, let μ be the Haar measure of G and ρ be any compatible metric on M. Then the metric d on M defined by $d(x,y) = \int_{g\in G} \rho(gx,gy)d\mu(g)$ is G invariant.

The following theorem is [11, Proposition 4.1].

Theorem 5.2. *Up to conjugacy, the rotation group $\mathrm{SO}(2,\mathbb{R})$ is the only maximal compact subgroup of $\mathrm{Homeo}_+(\mathbb{S}^1)$.*

Theorem 5.3. *Let G be a compact Lie group acting faithfully and transitively on a closed surface $\mathscr{S}$. Then $\dim(G) \leq 3$.*

Proof. Fix a G invariant metric d on $\mathscr{S}$ and fix a point $p \in \mathscr{S}$. Since G acts on $\mathscr{S}$ transitively, G/G_p is homeomorphic to $\mathscr{S}$ by Theorem 2.16, where G_p is the closed subgroup of G which fix p. Note that G_p is also a Lie subgroup of G, since a closed subgroup of a Lie group is a Lie group (see [24, Theorem 3.42]). Thus $\dim(G) = \dim(G_p) + \dim(\mathscr{S}) = \dim(G_p) + 2$. We need only to show that $\dim(G_p) \leq 1$. Let F be the connected component of e in G_p. Then F is a closed and hence a Lie subgroup of G_p which has the same dimension as that of G_p. If $F = e$, then $\dim(G_p) = \dim(F) = 0 \leq 1$ and we are done. So, we assume that F is a nontrivial connected Lie group in the sequel.

Take a closed disk D in $\mathscr{S}$ such that the interior $\mathrm{Int}(D)$ contains p, and take an $\varepsilon > 0$ such that the closed ball $\overline{B_d(p,\varepsilon)} \subset \mathrm{Int}(D)$. Since F preserves the metric d and leaves p invariant, $B_d(p,\varepsilon)$ is F invariant. Let D_1 be a closed disk such that $p \in \mathrm{Int}(D_1) \subset D_1 \subset B_d(p,\varepsilon)$. Let $K = \cup_{g\in F} gD_1$. Then K is an F invariant locally connected continuum by 2.7, which is the closure of the connected open set $\cup_{g\in F} g(\mathrm{Int}(D_1))$. Similar to the argument in [5, corollary 4.7], we then get an F invariant closed disk $D_3 \subset \mathrm{Int}(D)$ with $p \in \mathrm{Int}(D_3)$.

Then the boundary ∂D_3 is an F invariant simple closed curve; in fact, ∂D_3 is the boundary of the (unique) unbounded component of $\mathrm{Int}(D) \setminus K$.

Define a map $R : F \to \mathrm{Homeo}(\partial D_3)$ by letting $R(g) = g|\partial D_3$. If there is some $g \neq e \in F$ such that g fixes every point of ∂D_3, then g fixes every point of $\mathscr{S}$ (see the proof of [5, Lemma 4.8]), which contradicts the assumption that the action of G on $\mathscr{S}$ is faithful. Therefore, R is a continuous isomorphism between F and $R(F)$; particularly, $R(F)$ is a compact connected subgroup of $\mathrm{Homeo}(\partial D_3)$. Then we get $\dim(F) = \dim(R(F)) = 1$ by Theorem 5.2.

All together, we have $\dim(F) \leq 1$ and hence $\dim(G) \leq 3$. □

5.3. **Proof of Theorem 1.4.** First we show that every discrete group having Kazhdan's property (T) acts on a closed surface minimally and distally must be equicontinuous.

Proposition 5.4. *Let Γ be a discrete group having Kazhdan's property (T) and X be a closed surface. If Γ acts on X minimally and distally, then it is equicontinuous.*

Proof. To the contrary, suppose that (X,Γ) is not equicontinuous. Then, by Theorem 2.2, (X,Γ) has a maximally equicontinuous factor (Y,Γ) and it follows from Proposition 2.8 that Y is a connected compact manifold of 1-dimension. Since Γ acts on Y minimally, Y is the circle $\mathbb{S}^1$. Then by Theorem 2.3, it will extend to an action by compact group. Further, it follows from Theorem 5.2 that there is a group homomorphism $\phi : \Gamma \to \mathrm{SO}(2,\mathbb{R})$. It is well known that the abelianisationa of a discrete group having Kazhdan's property (T) is finite(see [3, Corollary 1.3.6]). Thus the homomorphism ϕ has finite image since $\mathrm{SO}(2,\mathbb{R})$ is abelian. In other words, $(\mathbb{S}^1,\Gamma)$ factors through a finite group action. But this cannot be a minimal action. This contradiction shows that Γ acts on X equicontinuously. □

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. Assume to the contrary that there is a distal minimal action $\phi : \Gamma \to \mathrm{Homeo}(X)$. Recall that Γ has Kazhdan's property (T) (see [3, §1.4]). Thus it follows from Proposition 5.4 that Γ acts on X equicontinuously. Set $H = \overline{\phi(\Gamma)}$. From Theorem 2.3, H is a compact subgroup of $\mathrm{Homeo}(X)$. Applying Theorem 2.13, we can take a small normal subgroup H' of H such that H/H' is a Lie group and $H'x$ is a proper subset of K for every $x \in X$. Then we get a continuous transitive action ψ of the Lie group H/H' on the quotient space X/H' by Theorem 2.17; in particular, X/H' is a connected compact manifold of dimension ≥ 1. Since $(X/H',\Gamma,\psi\phi)$ is a nontrivial factor of $(\mathscr{S},\Gamma,\phi)$, by Theorem 2.5, we have $\dim(X/H') \leq \dim(X) \leq 2$. It follows from Theorem 5.2 and Theorem 5.3 that $\dim(H/H') \leq 3$. Since Γ acts on X/H' minimally and X/H' is of dimension ≥ 1, $\psi\phi(\Gamma)$ cannot be finite. Thus $\dim(H/H') \geq 8$ by Theorem 5.1, which is a contradiction.

□

REFERENCES

[1] H. Abel, *Which group act distally*? Ergodic Theory Dynam. Systems **3** (1983), no. 2, 167-185.

[2] J. Auslander, *Miniml flows and their extensions*. North-Holland Mathematics Studies, 153. Notas de Matemática [Mathematical Notes], 122. North-Holland Publishing Co., Amsterdam, 1988.

[3] B. Bekka, P. de la Harpe and A. Valette, *Kazhdan's Property (T)*, Cambridge University Press, 2008.

[4] E. Breuillard, T. Gelander, *On dense free subgroups of Lie groups*. J. Algebra 261 (2003), 448-467.

[5] C. Bonatti, B. Kolev, Surface homeomorphisms with zero-dimensional singular set. Topology Appl. 90 (1998), no. 1-3, 69-95.

[6] A. Brown, Entropy, Lyapunov exponents, and rigidity of group actions. arXiv:1809.09192v4 [math.DS], 11 Jun 2019.

[7] I. Bronšteĭn, *Extensions of minimal transformation groups*, Suthoff & Noordhoff, 1979.

[8] R. Ellis, Distal transformation Groups, Pacific J. Math. 8 (1958) 401-405.

[9] R. Engelking, Dimension theory. North-Holland Mathematical Library, 19. North-Holland Publishing Co., Amsterdam-Oxford-New York; Polish Scientific Publishers, Warsaw, 1978, x+314 pp.

[10] H. Furstenberg, *The Structure of distal flows*. Amer. J. Math. 85 (1963), 477-515.

[11] É. Ghys. Groups acting on the circle. L'Enseign. Math. 47 (2001), 329-407.

[12] A. Knapp, *Lie groups beyaond an introduction* (Second Edition). PM 140, Birkhauser, 2002.

[13] S. Kobayashi, Transformation Groups in Differential Geometry. Springer-Verlag Berlin Heidelberg, 1995.

[14] A. Lelek, On confluent mappings. Colloq. Math. 15 (1966), 223-233.

[15] G. Margulis, Discrete subgroups of semisimple Lie groups. Springger-Verlag, Berlin, 1991.

[16] D. Montgomery, L. Zippin, *Topological transformation groups*. Interscience Publishers, New York-London, 1955.

[17] R. Moore, Concerning upper semi-continuous collections of continua. Trans. Amer. Math. Soc. 27 (1925), no. 4, 416-428.

[18] C. Moore and R. Zimmer, *Groups admitting ergodic actions with generalized discrete spectrum*. Invent. Math. **51** (1979), no. 2, 171-188.

[19] S. Nadler, *Continuum theory*, volume 158 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 1992. An introduction.

[20] A. Paterson, Amenability, American Mathematical Society, Providence, RI, 1988.

[21] M. Rees, *On the structure of minimal distal transformation groups with manifolds as phase spaces*, thesis, University of Warwick, Coventry, England.

[22] M. Rees, *On the fibres of a minimal distal extension of a transformation group*, Bull. London Mth. Soc., 10 (1978), 97-104.

[23] J. Tits, Free subgroups in linear groups, J. Algebra 20 (1972), 250-270.

[24] F. Warner, *Foundations of differentiable manifolds and Lie groups*. Springer-Verlag, GTM 94, 1983.

[25] G. Whyburn, Interior Transformations on compact sets. Duke Math. J. 3(1937), 370-381.

[26] D. Witte-Morris, Introduction to arithmetic groups. Deductive Press, 2015.

[27] R. Zimmer, *Lattices in semisimple groups and distal geometric structures*. Invent. math. **80**, 123-137 (1985).

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