STRUCTURES OF $R(f)-\overline{P(f)}$ FOR GRAPH MAPS f

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ABSTRACT. Let G be a graph and $f: G \to G$ be a continuous map. We establish a structure theorem which describes the structures of the set $R(f)-\overline{P(f)}$, where $R(f)$ and $P(f)$ are the recurrent point set and the periodic point set of f respectively. Roughly speaking, the set $R(f)-\overline{P(f)}$ is covered by finitely many pairwise disjoint f-invariant open sets $U_1, \cdots, U_n$; each U_i contains a unique minimal set W_i which absorbs each point of U_i; each W_i lies in finitely many pairwise disjoint circles each of which is contained in a connected closed set; all of these connected closed sets are contained in U_i and permutated cyclically by f. As applications of the structure theorem, several known results are improved or reproved.

1. INTRODUCTION AND PRELIMILARIES

In this section, we will state the main theorem obtained. We will also introduce the backgrounds of the study, the notions and notations used in the paper, and the organizations of the paper.

1.1. **Backgrounds and the aim of the paper.** The study of the dynamics of graph maps can date back to the work of A. Blokh in 1980's ([7, 8, 9]). Since then, lots of literatures appeared in this area. One may consult [5] for a systematic introduction to the combinatorial dynamics and chaotic phenomena for graph maps before 2000 and consult [3, 4, 15, 16, 18, 19, 20, 27] and their references for later investigations.

Recurrence is one of the most fundamental notions in the theory of dynamical system. For a compact metric space X and a continuous map $f: X \to X$, there are several f-invariant subsets of X which exhibit various recurrence behaviors, such as the periodic point set $P(f)$, the almost periodic point set $AP(f)$, and the recurrent point

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set $R(f)$. The ω-limit point set $\omega(f)$ and the nonwandering point set $\Omega(f)$ are also f-invariant and exhibit some weaker dynamical behaviors than recurrence. The structures of these sets have been intensively studied during the development of dynamical systems. One may consult [2, 14] for the introductions to the abstract theory of recurrence and its applications in number theory.

Due to the simplicity of the phase spaces in topology, finer and more interesting results around recurrence can be obtained for graph maps. Blokh constructed the spectral decomposition of the sets $\overline{P(f)}$, $\omega(f)$ and $\Omega(f)$ for any graph map f, and obtained a series of applications of the spectral decomposition ([10]). In [12] and [28], the authors showed that $\overline{R(f)} = \overline{P(f)}$ for any interval map f. This was extended by Ye to tree maps ([29]). For any graph map f, the authors proved that $\overline{R(f)} = R(f) \cup \overline{P(f)}$ and $\overline{R(f)} = \overline{AP(f)}$ ([17, 22]). We suggest the readers to refer to [1, 11, 23, 24] for the study of recurrence for maps on phase spaces beyond graphs.

The aim of this paper is to study the structure of $R(f) - \overline{P(f)}$ for any graph map $f : G \to G$. We will show that $R(f) - \overline{P(f)}$ is contained in $G - \overline{EP(f)}$ which has only finitely many connected components $U_1, \cdots, U_n$, where $EP(f)$ is the eventually periodic point set of f. Then we describe the dynamical behavior of f on the intersection of each U_i with $R(f)$. In the last of this section, we will give an explicit statement of the main theorem.

1.2. **Notions and notations.** Let (X,d) be a metric space with metric d. For any $Y \subset X$, denote by $\mathrm{Int}_X(Y)$, $\partial_X Y$, and $\mathrm{Clos}_X(Y)$ the interior, the boundary, and the closure of Y in X, respectively. If there is no confusion, we also write $\overline{Y}$ for $\mathrm{Clos}_X(Y)$. For any $y \in Y \subset X$ and any $r > 0$, write $B(y,r) = \{x \in X : d(x,y) < r\}$ and $B(Y,r) = \{x \in X : d(x,Y) < r\}$.

By a dynamical system, we mean a pair (X, f), where X is a compact metric space and $f : X \to X$ is a continuous map. Denote by $C^0(X)$ the set of all continuous maps from X to X. Let $\mathbb{N}$ be the set of all positive integers, and let $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. For any $n \in \mathbb{N}$, write $\mathbb{N}_n = \{1, \cdots, n\}$. For any $f \in C^0(X)$, let f^0 be the identity map of X, and let $f^n = f \circ f^{n-1}$ be the composition map of f and f^{n-1}. For $x \in X$, the set $O(x, f) \equiv \{f^n(x) : n \in \mathbb{Z}_+\}$ is called the *orbit* of x under f. A point $x \in X$ is called a *fixed point* of f if $f(x)=x$; is called a *periodic point* of f if $f^n(x)=x$ for some $n\in\mathbb{N}$; is called an *eventually periodic point* of f if the orbit $O(x,f)$ is a finite set; is called a *non-wandering point* of f if for any neighborhood U of x in X there is an $n\in\mathbb{N}$ such that $f^n(U)\cap U\neq\emptyset$. The set $\omega(x,f)\equiv\bigcap_{i=0}^{\infty}\overline{O(f^i(x),f)}$ is called the *ω-limit set* of x under f. Write $\omega(f)=\bigcup_{x\in X}\omega(x,f)$, called the *$\omega$-limit set* of f. The point $x\in X$ is called a *recurrent point* of f if $x\in\omega(x,f)$ and is called an *almost periodic point* of f if for any neighborhood U of x in X there exists an $m\in\mathbb{N}$ such that $\{f^{n+i}(x): i\in\mathbb{N}_m\}\cap U\neq\emptyset$ for every $n\in\mathbb{Z}_+$. A subset W of X is said to be *f-invariant* if $f(W)\subset W$; is said to be *strongly f-invariant* if $f(W)=W$; is said to be a *minimal set* of f if it is non-empty, closed and f-invariant and if no proper subset of W has these three properties. A minimal set W of f is said to be *totally minimal* if it is a minimal set of f^n for all $n\in\mathbb{N}$. Denote by $\mathrm{Fix}(f)$, $P(f)$, $EP(f)$, $AP(f)$, $R(f)$ and $\Omega(f)$ the sets of fixed points, periodic points, eventually periodic points, almost periodic points, recurrent points and non-wandering points of f, respectively. From the definitions it is easy to see that $P(f)\subset EP(f)$ and $\mathrm{Fix}(f)\subset P(f)\subset AP(f)\subset R(f)\subset\omega(f)\subset\Omega(f)$.

A non-degenerate metric space X is called an *arc* (resp. an *open arc*, a *circle*) if it is homeomorphic to the interval $[0,1]$ (resp. the open interval $(0,1)$, the unit circle S^1). A compact connected metric space G is called a (topological) *graph* if there exists a finite subset $V(G)$ of G such that each connected component of $G-V(G)$ is an open arc, and any circle in G contains at least three points in $V(G)$. Every point in the given finite subset $V(G)$ is called a *vertex* of G. Every connected component of $G-V(G)$ is called an *edge* of G. A graph containing no circle is called a *tree*. A continuous map from a graph (resp. a tree, a circle, an interval) to itself is called a *graph map* (resp. a *tree map*, a *circle map*, an *interval map*). Note that if X is a non-degenerate connected closed subset of a graph G then X itself is also a graph.

Let G be a graph. We may assume that the metric d on G satisfies the following two conditions: (1) for any $x\in G$ and any $r>0$, the open ball $B(x,r)$ in G is a connected set; (2) $d(u,v)\geq 1$, for any two different vertices u and v of G. For any finite set S, denote by $|S|$ the number of elements of S. For any $x\in G$, write $\mathrm{val}_G(x)=\lim_{r\to 0}|\partial_G B(x,r)|$, called the *valence* of x in G; x is called a *branching point* (resp. an *endpoint*) of G if $\mathrm{val}_G(x)>2$ (resp. $\mathrm{val}_G(x)=1$). Denote by $\mathrm{Br}(G)$ and $\mathrm{End}(G)$

the sets of branching points and endpoints of G, respectively. For any arc A in G and any two points $a,b \in A$, denote by $[a,b]_A$ the smallest connected closed subset of A containing a and b. If there is no confusion, we also write $[a,b]$ for $[a,b]_A$. In addition, we write $(a,b] = [b,a) = [a,b] - \{a\}$ and $(a,b) = (a,b] - \{b\}$. Note that $[a,a] = \{a\}$ and $(a,a] = (a,a) = \emptyset$.

For any metric space (even for any topological space) X, any $f \in C^0(X)$ and any $n \in \mathbb{N}$, by the definitions, it is easy to see that $P(f) = P(f^n)$ and $\omega(f) = \omega(f^n)$. Erdös and Stone proved that $R(f) = R(f^n)$ and $AP(f) = AP(f^n)$ also hold ([13]). It is well known that if X is a compact metric space and $f \in C^0(X)$, then a point $x \in AP(f)$ if and only if $\overline{O(x,f)}$ is a minimal set of f (see e.g. [6, Proposition V.5]).

1.3. **Organizations and the statement of the main theorem.** In section 2, for a graph map f on a graph G and for a subset K of G, we introduce the notions of absorbed set and main absorbed set of K, and use them to analyse the structures of the orbits of some specified connected subsets of G under f. In the beginning of Section 3, we will recall the structure theorem of graph maps without periodic points obtained by Mai and Shao in [21], which is a key ingredient in the proof of the main theorem. Then, based on these preparations, we prove the following main theorem (Theorem 3.8).

Theorem 1.1. *Let G be a connected graph and $f : G \to G$ be a continuous map such that $P(f) \neq \emptyset$ and $R(f) - \overline{P(f)} \neq \emptyset$. Then there exist pairwise disjoint nonempty open subsets $U_1, \cdots, U_n$ of G with $n \in \mathbb{N}$ such that*

(1) *$f(U_i) \subset U_i$, for each $i \in \mathbb{N}_n$.*

(2) *Write $U = \bigcup_{i=1}^n U_i$ and $U_0 = G - U$. Then $\overline{P(f)} \subset U_0$, $\overline{U} - U \subset EP(f)$, and $\Omega(f) - U_0 = R(f) - U_0 = R(f) - \overline{P(f)} \subset U$.*

(3) *For each $i \in \mathbb{N}_n$, U_i has k_i connected components $U_{i1}, \cdots, U_{ik_i}$ with $k_i \in \mathbb{N}$, which satisfy $f(U_{ik_i}) \subset U_{i1}$ and $f(U_{ij}) \subset U_{i,j+1}$ for $1 \leq j < k_i$.*

(4) *For each $i \in \mathbb{N}_n$, write $W_i = R(f) \cap U_i$. Then W_i is a unique minimal set of f contained in U_i.*

(5) *For each $i \in \mathbb{N}_n$ and $j \in N_{k_i}$, write $W_{ij} = W_i \cap U_{ij}$. Then W_{ij} is a unique minimal set of f^{k_i} contained in U_{ij}, and there is a connected closed subset G_{ij} of G and a circle*

C_{ij} such that $W_{ij} \subset C_{ij} \subset G_{ij} \subset U_{ij}$, $f(W_{ik_i}) = W_{i1}$, $f(G_{ik_i}) = G_{i1}$, and $f(W_{ij}) = W_{i,j+1}$ and $f(G_{ij}) = G_{i,j+1}$ for $1 \le j < k_i$.

(6) *For each $i \in \mathbb{N}_n$, $j \in \mathbb{N}_{k_i}$, and for each $x \in U_{ij}$, one has $\lim_{m\to\infty} d(f^m(x), W_i) = 0$, and $\lim_{m\to\infty} d(f^{mk_i}(x), W_{ij}) = 0$.*

In Section 4, as applications of the main theorem, we give several propositions part of which improve or reprove some known results .

2. Absorbed sets and orbits of subsets of graphs

Definition 2.1. Let $f : G \to G$ be a graph map. For any subset K of G and any $n \in \mathbb{Z}_+$, write

$$O(K,f) = \bigcup_{i=0}^{\infty} f^i(K), \quad \text{and} \quad O_n(K,f) = \bigcup_{i=0}^{n} f^i(K). \tag{2.1}$$

Then $O(K,f)$ is an f-invariant set, called the *orbit of the set K under f*, and the set $O_n(K,f)$ is called a *segment of the orbit of K under f*. Write

$$O_-(K,f) = \bigcup_{i=0}^{\infty} f^{-i}(K), \tag{2.2}$$

called the *inverse orbit of K under f*, or called the *absorbed set by K under f*. Let

$$\mathrm{Ab}(K,f) = \bigcup \{Y : Y \text{ is a connected component of } O_-(K,f), \text{ and } Y \cap K \neq \emptyset\}, \tag{2.3}$$

called the *main absorbed set by K under f*.

For any connected open subset U of G, define a function

$$\xi(U) = \xi_G(U) = \sum \{ \mathrm{val}_G(v) - 2 : v \in U \cap \mathrm{Br}(G) \}, \tag{2.4}$$

called the *total branching number* of U in G. The following lemma gives the supremum of numbers of boundary points of connected sets in G.

Lemma 2.2. ([26, Lemma 4.1]) *Let G be a graph. Then $|\partial_G X| \le \xi(G) + 2$ for any connected subset X of G, and there is a subtree T of G such that $|\partial_G T| = \xi(G) + 2$.*

Corollary 2.3. *Let G be a graph, and U be a connected open subset of G containing a circle C. Then $|\partial_G U| \le \xi(U)$.*

Proof. Let $r = \min\{d(x,y)/3 : x \text{ and } y \text{ are two different points in } \partial_G U \cup V(G)\}$, and let $Z = U - \bigcup\{\overline{B(x,r)} : x \in \partial_G U\}$. Then Z is a connected open subset of G containing C, $U \cap \mathrm{Br}(G) = Z \cap \mathrm{Br}(G)$, and $|\partial_G U| \le |\partial_G Z|$. Let $X = \overline{Z}$. Then X is a subgraph of

G, $\mathrm{Br}(X)=Z\cap \mathrm{Br}(G)$, and $\partial_G Z=\partial_X Z$. Take an arc $A\subset C-V(G)$ and let $Y=Z-A$. Then Y is a connected open subset of Z. From Lemma 2.2 we get $|\partial_X Z|+2=|\partial_X Y|\le \xi_X(X)+2=\xi_G(Z)+2=\xi_G(U)+2$. Thus $|\partial_G U|\le \xi_G(U)$. □

Lemma 2.4. *Let $f:G\to G$ be a graph map, and $L\subset G$ be an f-invariant connected set. Let $W=O_-(L,f)$ and $U=\mathrm{Ab}(L,f)$ be defined as in (2.2) and (2.3). Then*

(1) *Both W and U are f-invariant*;

(2) *Further, if L is open, then W and U are open, $f(\partial_G U)\subset\partial_G U\subset EP(f)$, and $\partial_G U\cap P(f)\neq\emptyset$.*

Proof. (1) Since $f(L)\subset L$ and $f\big(f^{-i}(L)\big)\subset f^{1-i}(L)$ for any $i\in\mathbb{N}$, from the definition of $W=O_-(L,f)$ we get $f(W)\subset W\cup f(L)\subset W\cup L=W$. Since U is the connected component of W containing L, it follows that $f(U)$ is connected, and $f(U)\cap U\supset f(L)\cap L=f(L)\neq\emptyset$. Thus $f(U)\cup U$ is a connected subset of W containing L and hence $f(U)\subset U\cup f(U)=U$. Therefore, W and U are f-invariant.

(2) Further, if L is open, then $f^{-i}(L)$ is open, for any $i\in\mathbb{Z}_+$. Thus W is open, so is the connected component U of W. From $f(U)\subset U$ we get $f(\overline{U})\subset\overline{U}$. If there is a point $x\in\partial_G U$ such that $f(x)\notin\partial_G U$, then we will have $f(x)\in U$, and there will be a connected open neighborhood Z of x such that $f(Z)\subset U$. This means that $Z\subset f^{-1}(U)\subset f^{-1}(W)\subset W$. Therefore, $Z\cup U$ is a connected open subset of W, and hence we have $x\in Z\cup U=U$. However, this contradicts that $x\in\partial_G U$. Thus we must have $f(\partial_G U)\subset\partial_G U$, which with $|\partial_G U|\le\xi(G)+2$ implies that $\partial_G U\subset EP(f)$ and $\partial_G U\cap P(f)\neq\emptyset$. □

Lemma 2.5. *Let $f:G\to G$ be a graph map, and K be an f-invariant connected closed set with $\partial_G K\cap P(f)=\emptyset$. Then*

(1) *There exists an f-invariant connected open set $L\supset K$ such that the absorbed set $O_-(L,f)=O_-(K,f)$, and the main absorbed set $\mathrm{Ab}(L,f)=\mathrm{Ab}(K,f)$*;

(2) *$O_-(K,f)$ and $\mathrm{Ab}(K,f)$ are open sets in G*;

(3) *For any $y\in O_-(K,f)$, there exist an $m\in\mathbb{N}$ and a neighborhood Z of y in G such that $f^i(Z)\subset K$ for all $i\ge m$*;

(4) *$\omega(f)\cap O_-(K,f)=\omega(f)\cap\mathrm{Ab}(K,f)=\omega(f)\cap K=\omega(f|K)$*;

(5) *$\Omega(f)\cap O_-(K,f)=\Omega(f)\cap\mathrm{Ab}(K,f)=\Omega(f)\cap K\subset\Omega(f|K)\cup\partial_G K$.*

Proof. (1) By Lemma 2.2, $\partial_G K$ is a finite subset of K. Write $L_0=\mathrm{Int}_G(K)$. Since $\partial_G K\cap P(f)=\emptyset$, there is an $n\in\mathbb{N}$ such that $f^n(\partial_G K)\subset L_0$. Let $L_1=\bigcup_{i=0}^n f^{-i}(L_0)$. Then L_1 is an open set containing K. Let L be the connected component of L_1 containing K. Then L is also open. Since $f(L)\cap L\supset f(K)\cap K=f(K)\neq\emptyset$ and $f(L)\subset f(L_1)\subset\bigcup_{i=0}^n f^{1-i}(L_0)\subset K\cup\big(\bigcup_{i=0}^{n-1} f^{-i}(L_0)\big)\subset L_1$, we have $f(L)\subset f(L)\cup L=L$. From $O_-(L,f)\subset O_-(L_1,f)=\bigcup_{j=0}^\infty f^{-j}\big(\bigcup_{i=0}^n f^{-i}(L_0)\big)=\bigcup_{j=0}^\infty f^{-j}(L_0)=O_-(L_0,f)\subset O_-(K,f)\subset O_-(L,f)$ we get $O_-(L,f)=O_-(K,f)$, and hence $\mathrm{Ab}(L,f)=\mathrm{Ab}(K,f)$.

(2) follows from (1) of this lemma and (2) of Lemma 2.4.

(3) For any $y\in O_-(K,f)$, there is a $p\in\mathbb{N}$ such that $f^p(y)\in K$. If $f^p(y)\in\partial_G K$, we can take an $n\in\mathbb{N}$ such that $f^{p+n}(y)\in\mathrm{Int}_G(K)$ and put $m=p+n$. If $f^p(y)\in\mathrm{Int}_G(K)$, we put $m=p$. By the continuity of f^m, there is a neighborhood Z of y in G such that $f^m(Z)\subset\mathrm{Int}_G(K)$. Since $f(K)\subset K$, we have $f^i(Z)\subset K$ for all $i\geq m$.

(4) It suffices to show $\omega(f)\cap O_-(K,f)\subset\omega(f|K)$, since $\omega(f)\cap O_-(K,f)\supset\omega(f)\cap\mathrm{Ab}(K,f)\supset\omega(f)\cap K\supset\omega(f|K)$ is clear. Given a point $x\in\omega(f)\cap O_-(K,f)$. Since $O_-(K,f)$ is open, there is a point $y\in O_-(K,f)$ such that $x\in\omega(y,f)$. Take $n\in\mathbb{N}$ such that $f^n(y)\in K$. Then we have $x\in\omega(y,f)=\omega\big(f^n(y),f\big)\subset\omega(f|K)$. Thus $\omega(f)\cap O_-(K,f)\subset\omega(f|K)$.

(5) By (3) of this lemma, we have $\big(O_-(K,f)-K\big)\cap\Omega(f)=\emptyset$, which implies that $\Omega(f)\cap O_-(K,f)=\Omega(f)\cap\mathrm{Ab}(K,f)=\Omega(f)\cap K$. For any $x\in\mathrm{Int}_G(K)$, it is clear that $x\in\Omega(f)$ if and only if $x\in\Omega(f|K)$. Thus we have $\Omega(f)\cap K\subset\Omega(f|K)\cup\partial_G K$. □

Lemma 2.6. *Let $f:G\to G$ be a graph map, and K be a connected closed subset of G with $f(K)\cap K\neq\emptyset$. Let $X=O(K,f)$ and $X_n=O_n(K,f)$ be defined as in* (2.1). *Suppose that $X-X_n\neq\emptyset$ for any $n\in\mathbb{N}$. Write $S=\bigcap_{n=0}^\infty\overline{X-X_n}$. Then*

(1) *$P(f)\supset f(S)=S\neq\emptyset$, and S contains at most $\xi(G)+2$ points* ;

(2) *Further, if $K\cap EP(f)=\emptyset$, then there exists an open arc $(u,v)\subset X-V(G)$ such that $S=\overline{X}-X=\{v\}\subset\mathrm{Fix}(f)$, $f(x)\in(x,v)$ for any $x\in[u,v)$, and X is contained in the main absorbed set $\mathrm{Ab}\big((u,v),f\big)$.*

Proof. (1) By the conditions of the lemma, all the sets $K=X_0\subset X_1\subset X_2\subset X_3\subset\cdots\subset X$ are connected, and X_n is closed in G for all $n\in\mathbb{Z}_+$. Since $X-X_n\neq\emptyset$, for any $n \in \mathbb{N}$, we have $X_{n+1} - X_n \neq \emptyset$. Since $V(G)$ is a finite set and since the valence of any vertex of G is finite, there is a sufficiently large $\beta \in \mathbb{N}$ such that

(a) $X \cap V(G) = X_n \cap V(G) = X_\beta \cap V(G)$ for any $n \geq \beta$;

(b) $\mathrm{val}_{X_n}(w) = \mathrm{val}_{X_\beta}(w)$ for any $w \in X_\beta \cap V(G)$ and any $n \geq \beta$.

From property (a) we can derive

(c) For any $n \geq \beta$, every connected component of $X - X_n$ is contained in an edge of G, and every edge of G contains at most two connected components of $X - X_n$. Thus $X - X_n$ has only finitely many connected components;

(d) For any $n \geq \beta$, every connected component of $X - X_{n+1}$ is contained in a connected component of $X - X_n$, and every connected component of $X - X_n$ contains at most one connected components of $X - X_{n+1}$.

Let λ_n be the number of connected component of $X - X_n$. By property (d) we have $\lambda_\beta \geq \lambda_{\beta+1} \geq \lambda_{\beta+2} \geq \cdots$. Let $\lambda = \lim_{n\to\infty} \lambda_n$. Then $\lambda \geq 1$, and there is an $m \geq \beta$ such that $\lambda_n = \lambda$ for all $n \geq m$. If $m > \beta$, we can replace β by m. Thus we may assume that $\lambda_n = \lambda$ for all $n \geq \beta$. This means that, for $n \geq \beta$, every connected component of $X - X_n$ contains exactly one connected components of $X - X_{n+1}$.

Since X is connected and X_β is closed, no connected component of $X - X_\beta$ is a closed arc. If there is a connected component J of $X - X_\beta$ such that $J = (u, v]$ is a semi-open arc, then $u \in X_\beta$ and there is a neighborhood U of v in G such that $X \cap U \subset (u, v]$. Since $v \in X$, there is $m > \beta$ such that $v \in X_m$, which with $X_m \cap U \subset X \cap U \subset (u, v]$ and $(u, v) \cap V(G) = \emptyset$ implies that $[u, v] \subset X_m$, and hence $J = (u, v]$ contains no connected component of $X - X_m$. This leads to a contradiction. Thus every connected component of $X - X_\beta$ must be an open arc.

Let $J_1, \cdots, J_\lambda$ be the λ connected components of $X - X_\beta$ with $J_i = (u_i, v_i)$. For each $i \in \mathbb{N}_\lambda$, by means of a homeomorphism from $[u_i, v_i]$ to $[0, 1]$ we can define a linear order $<$ on $[u_i, v_i]$ such that $u_i < v_i$. For any $n \geq \beta$, write $J_{in} = J_i - X_n$. Then J_{in} is an open arc, and J_{in} is the connected component of $X - X_n$ contained in J_i. Suppose that $J_{in} = (u_{in}, v_{in})$ with $u_{in} < v_{in}$. Then we have $\lim_{n\to\infty} d(u_{in}, v_{in}) = 0$ and

$$u_i = u_{i\beta} \leq u_{in} \leq u_{im} < v_{im} \leq v_{in} \leq v_{i\beta} = v_i \quad \text{for any} \quad m > n \geq \beta.$$

Let $z_i = \lim_{n\to\infty} u_{in}$. If $u_{in} < z_i < v_{in}$ for all $n \geq \beta$, then $z_i \notin X_n$ for all $n \geq \beta$. This contradicts that $z_i \in J_i \subset X$. Thus there exists $m \geq \beta$ such that $u_{im} < z_i = v_{im}$ or $u_{im} = z_i < v_{im}$. By symmetry, we may assume that the case $u_{im} < z_i = v_{im}$ occurs. In addition, if $m > \beta$ then we can replace β by m. Hence we may assume that,

(e) for any $i \in \mathbb{N}_\lambda$ and any $n \geq \beta$, $J_{in} = (u_{in}, v_i)$, and $\lim_{n\to\infty} u_{in} = v_i$.

From property (e) we get $X - X_n = \bigcup_{i=1}^{\lambda}(u_{in}, v_i)$, $S = \bigcap_{n=0}^{\infty}\overline{X - X_n} = \{v_1, \cdots, v_\lambda\}$, and $\overline{X} = X \cup S$. By property (b), we have $u_i \notin V(G)$. Note that it is possible that $v_i \in V(G) \cup X_\beta$ for some $i \in \mathbb{N}_\lambda$, or that $v_i = v_j$ for some $1 \leq i < j \leq \lambda$. Let E_i be the edge of G containing (u_i, v_i) and let w_i be the endpoint of E_i such that $u_i \in (w_i, v_i)$. Let $\varepsilon = \min\{d(u_i, w_i) : i \in \mathbb{N}_\lambda\}$. Take a $\delta > 0$ such that $d\big(f(x), f(y)\big) < \min\{\varepsilon, 1\}$ for any $x, y \in G$ with $d(x,y) \leq \delta$. We may assume that β is so large that

(f) all the diameters of $J_1, \cdots, J_\lambda$ are less than δ.

For any $i \in \mathbb{N}_\lambda$, it is clear that

$$J_{i,\beta+1} = J_i - X_{\beta+1} \subset X - X_{\beta+1} \subset f(X - X_\beta) = f\big(\textstyle\bigcup_{k=1}^{\lambda} J_k\big) = \bigcup_{k=1}^{\lambda} f(J_k).$$

So for each $i \in \mathbb{N}_\lambda$ there is a $k_i \in \mathbb{N}_\lambda$ such that $f(J_{k_i}) \cap J_{i,\beta+1} \neq \emptyset$, which with property (f) implies that $w_i \notin f(J_{k_i})$. If $v_i \in f(J_{k_i})$, then there exist $x_i \in J_i$ and $m > \beta$ such that $\{x_i, v_i\} \subset f(X_m \cap J_{k_i})$, which leads to $[x_i, v_i] \subset f(X_m \cap J_{k_i}) \subset f(X_m) \subset X_{m+1}$. But this contradicts that $J_i - X_{m+1} = (u_{i,m+1}, v_i)$. So we must have $v_i \notin f(J_{k_i})$, which with $w_i \notin f(J_{k_i})$ implies that $f(J_{k_i}) \subset (w_i, v_i)$. Hence, for any $i, j \in \mathbb{N}_\lambda$ with $j \neq i$, we have $f(J_{k_j}) \cap f(J_{k_i}) \subset (w_j, v_j) \cap (w_i, v_i) = \emptyset$, which implies that $k_j \neq k_i$. Thus we have

(g) $\{k_i : i \in \mathbb{N}_\lambda\} = \mathbb{N}_\lambda$, and $J_{i,\beta+1} \subset f(J_{k_i}) \subset (w_i, v_i)$ for each $i \in \mathbb{N}_\lambda$.

For $n \geq \beta$ and $i \in \mathbb{N}_\lambda$, since $J_{i,n+1} \subset X - X_{n+1} \subset f(X - X_n) = \bigcup_{k=1}^{\lambda} f(J_{kn})$, from property (g) we get $J_{i,n+1} \subset f(J_{k_i n})$, which with

$$\lim_{n\to\infty} \mathrm{diam}\big(f(J_{k_i n})\big) = \lim_{n\to\infty} \mathrm{diam}(J_{k_i n}) = 0$$

implies that $f(v_{k_i}) = v_i$. Hence we have $f(S) = S$. Noting that $S = \{v_1, \cdots, v_\lambda\}$ is a finite set, we have $S \subset P(f)$. By Lemma 2.2, we have

$$|S| \leq \lambda = |\{u_1, \cdots, u_\lambda\}| = |\partial_X X_\beta| \leq |\partial_G X_\beta| \leq \xi(G) + 2.$$

(2) Further, if $K \cap EP(f) = \emptyset$, then $X \cap S \subset X \cap P(f) \subset X \cap EP(f) = \emptyset$, which with $\overline{X} = X \cup S$ implies that $\overline{X} - X = S$.

For any $i \in \mathbb{N}_\lambda$ and any $n \geq \beta$, write $L_{in} = L(i,n) = J_i \cap (X_{n+1} - X_n)$. Then $L_{in} = (u_{in}, u_{i,n+1}] \subset J_{in}$ and $X_{n+1} - X_n = \bigcup_{i=1}^{\lambda} L_{in}$. Noting that $L_{i,n+1} \subset X_{n+2} - X_{n+1} \subset f(X_{n+1} - X_n) = \bigcup_{k=1}^{\lambda} f(L_{kn})$, from property (g) we get $L_{i,n+1} \subset f(L_{k_i n})$.

Define a map $\psi : \mathbb{N}_\lambda \to \mathbb{N}_\lambda$ by $\psi(i) = k_i$ for any $i \in \mathbb{N}_\lambda$. Then ψ is a bijection and $f(v_{\psi(i)}) = v_i$. Let $t_i \in \mathbb{N}_\lambda$ be the least positive integer such that $\psi^{t_i}(i) = i$. Then we have $f^{t_i}(v_i) = v_i$. Choose an $n > \beta + t_i$ such that $L_{in} \neq \emptyset$ and take a point $z_0 \in L_{in}$. Then there exist points $z_1, z_2, \cdots, z_{t_i}$ such that $f(z_j) = z_{j-1}$ and $z_j \in L\big(\psi^j(i), n-j\big)$ for each $j \in \mathbb{N}_{t_i}$. Noting that $z_{t_i} \in L(i, n-t_i) \subset (u_i, u_{in}]$ and $z_0 \in (u_{in}, u_{i,n+1}]$, we have $f^{t_i}(z_{t_i}) = z_0 \in (z_{t_i}, v_i) \subset (u_i, v_i)$.

If $f^{t_i}\big((u_i, v_i)\big) \not\subset (u_i, v_i)$, then there exist $m > n$ and $z \in (u_i, u_{im}] = (u_i, v_i) \cap X_m$ such that $f^{t_i}(z) \in \{u_i, v_i\}$ and $f^{t_i}\big([z_{t_i}, z)\big) \subset (u_i, v_i)$. However, if $f^{t_i}(z) = u_i$ then $(z_{t_i}, z) \cap \mathrm{Fix}(f^{t_i}) \neq \emptyset$, which contradicts that $[z_{t_i}, z] \cap P(f) \subset X \cap EP(f) = \emptyset$. If $f^{t_i}(z) = v_i$ then $X_{m+t_i} \supset f^{t_i}\big([z_{t_i}, z]\big) \supset [z_0, v_i]$, which also contradicts that $J_i - X_{m+t_i} = (u_{i,m+t_i}, v_i)$. Thus we must have $f^{t_i}\big((u_i, v_i)\big) \subset (u_i, v_i)$, which with $f^{t_i}(z_{t_i}) \in (z_{t_i}, v_i)$ and $[u_i, v_i) \cap P(f) = \emptyset$ implies that $f^{t_i}(x) \in (x, v_i)$ for any $x \in [u_i, v_i)$.

Write $t = \prod_{j=1}^{\lambda} t_j$. Then for any $i \in \mathbb{N}_\lambda$, the open arc $J_i = (u_i, v_i)$ is f^t-invariant. Let $U_i = \mathrm{Ab}(J_i, f^t)$ be the main absorbed set by J_i under f^t. Then by Lemma 2.4 we get $\partial_G U_i \subset EP(f^t) = EP(f)$. Hence, if $X \not\subset U_i$ then $X \cap EP(f) \supset X \cap \partial_G U_i \neq \emptyset$. But this will leads to a contradiction. Thus we must have $X \subset U_i$.

If $\lambda \geq 2$, then we have both $X \subset U_1$ and $X \subset U_2$. On the other hand, from $J_1 \cap J_2 = \emptyset$ we get $U_1 \cap U_2 = \emptyset$. These will lead to a contradiction. Thus we must have $\lambda = 1$, which implies that $t = t_1 = 1$. Let $u = u_1$ and $v = v_1$. Then the open arc (u, v) satisfies the all conditions in Lemma 2.6, and the proof is complete. □

As a corollary of Lemma 2.6, we have the following

Proposition 2.7. *Let $f : G \to G$ be a graph map, and K be a connected closed subset of G with $f(K) \cap K \neq \emptyset$. Let $X = O(K, f)$ be the orbit of K under f. If $\overline{X} - X$ contains more than one point, then $X \cap P(f) \neq \emptyset$.*

Proof. Let $X_n = O_n(K, f)$ be defined as in (2.1). Then X_n is closed. Since X is not closed, we have $X - X_n \neq \emptyset$ for any $n \in \mathbb{N}$. If $X \cap P(f) = \emptyset$, then $X \cap EP(f) = \emptyset$, and by Lemma 2.6, $\overline{X}-X$ will contain only one point. Therefore, if $\overline{X}-X$ contains more than one point then we must have $X\cap P(f)\neq\emptyset$. □

Lemma 2.8. *Let $f:G\to G$ be a graph map, and K be a connected closed subset of G with $f(K)\cap K\neq\emptyset$. Let $X=O(K,f)$ and $X_n=O_n(K,f)$ be defined as in* (2.1)*. If $X-X_n\neq\emptyset$ for any $n\in\mathbb{N}$, then the following eight conditions are equivalent:*

(1) $K\cap EP(f)=\emptyset$;

(2) $X\cap EP(f)=\emptyset$;

(3) $X\cap P(f)=\emptyset$;

(4) $X\cap R(f)=\emptyset$;

(5) $X\cap\omega(f)=\emptyset$;

(6) $X\cap\Omega(f)=\emptyset$;

(7) $\overline{X}-X\neq\emptyset$, *and* $\lim_{n\to\infty}d\big(f^n(x),\overline{X}-X\big)=0$ *for any* $x\in X$;

(8) *There exists an open arc* $(u,v)\subset X-V(G)$ *such that* $\overline{X}-X=\{v\}\subset\mathrm{Fix}(f)$, $f(x)\in(x,v)$ *for any* $x\in[u,v)$, *and* $X\subset\mathrm{Ab}\big((u,v),f\big)$.

Proof. (6) $\Rightarrow$ (5) $\Rightarrow$ (4) $\Rightarrow$ (3) are trivial, since $\Omega(f)\supset\omega(f)\supset R(f)\supset P(f)$. By the definition of the orbit $X=O(K,f)$, (3) $\Leftrightarrow$ (2) $\Leftrightarrow$ (1) are clear. Since $\overline{X}-X\subset\partial_G X$ is a finite set, from the definition of recurrent points we can directly derive (7) $\Rightarrow$ (4). In Lemma 2.6 we have proved (1) $\Rightarrow$ (8). Thus it suffices to show that (8) $\Rightarrow$ (6) and (8) $\Rightarrow$ (7).

Suppose that (8) is true. Then $(u,v)\cap\Omega(f)=\emptyset$, and $\lim_{n\to\infty}f^n(x)=v$ for any $x\in[u,v)$. Since $X\subset\mathrm{Ab}\big((u,v),f\big)$, we also have $X\cap\Omega(f)=\emptyset$, and $\lim_{n\to\infty}f^n(x)=v$ for any $x\in X$. Hence (6) and (7) are true. □

From Lemma 2.8 we obtain the following corollary at once.

Corollary 2.9. *Let $f:G\to G$ be a graph map, and K be a connected closed subset of G with $f(K)\cap K\neq\emptyset$. Let $X=O(K,f)$ and $X_n=O_n(K,f)$ be defined as in* (2.1)*. If $K\cap EP(f)=\emptyset$ and $K\cap\Omega(f)\neq\emptyset$, then there exists an $n\in\mathbb{N}$ such that $X=X_n$, and hence X is closed in G.*

The following lemma is given in [22].

Lemma 2.10. ([22, Lemma 2.3]) *Let $f : G \to G$ be a graph map, and $A = [w,z]$ be an arc in G. If $(w,z) \cap \big(V(G) \cup P(f)\big) = \emptyset$ and $(w,z) \cap R(f) \neq \emptyset$, then $O(w,f) \cap (w,z) \neq \emptyset$.*

Corollary 2.11. *Let $f : G \to G$ be a graph map, and $A = [x,y]$ be an arc in G. If $(x,y) \cap \big(V(G) \cup P(f)\big) = \emptyset$ and $(x,y) \cap R(f) \neq \emptyset$, then $A \cap EP(f) = \emptyset$.*

Proof. If $A \cap EP(f) \neq \emptyset$, then there exist $w \in A$ and $n \in \mathbb{N}$ such that $f^n(w) \in P(f)$, which with $(x,y) \cap P(f) = \emptyset$ implies that $f^i(w) \notin (x,y)$ for any $i \geq n$. On the other hand, take a point $z \in [x,y] - \{w\}$ such that $(w,z) \cap R(f) \neq \emptyset$. Then $(w,z) \cap R(f^n) \neq \emptyset$, and by Lemma 2.10 we get $O(w, f^n) \cap (w,z) \neq \emptyset$. But this contradicts that $f^i(w) \notin (x,y)$ for any $i \geq n$. Thus we must have $A \cap EP(f) = \emptyset$. □

3. STRUCTURES OF $R(f) - \overline{P(f)}$ FOR GRAPH MAPS f

Now we list several results coming from [21] in the following theorem, which is a key ingredient in the proof of the main theorem.

Theorem 3.1. *Let $f : G \to G$ be a graph map without a periodic point. Then there exist an $n \in \mathbb{N}$, a subgraph X of G and a circle $Q \subset X$ such that the following items hold:*

(1) $G \supsetneq f(G) \supsetneq \cdots \supsetneq f^{n-2}(G) \supsetneq f^{n-1}(G) = X = f(X)$;

(2) *$f|X$ has a unique minimal set M which is totally minimal;*

(3) $\Omega(f|X) = AP(f|X) = M \subset Q$;

(4) *for each $x \in X$, we have $\lim_{m \to \infty} d(f^m(x), M) = 0$;*

(5) *f is topologically semi-conjugate to an irrational rotation of the unit circle S^1.*

Remark 3.2. The conclusion (1) of Theorem 3.1 follows from Theorem [21, Theorem 4.2] and Definitions 3.3 and 3.4 in [21]; the conclusions (2) and (3) follow from Claim 24 and Corollary 4.4 in [21]; the conclusion (5) is implied by [21, Theorem 4.3]. Though the conclusion (4) of Theorem 3.1 is not explicitly stated in [21], it can be seen easily from [21, Theorem 4.2] and the constructions in Section 3 of [21].

Definition 3.3. Let $f : G \to G$ be a graph map. A subset X of G is called a *component-cyclic f-invariant set* if X has only finitely many connected components $X_1, \cdots, X_k$ and $f(X_i) \subset X_{i+1\,(\mathrm{mod}\,k)}$ for every $i \in \mathbb{N}_k$. A component-cyclic f-invariant set X is called a *component-cyclic strongly f-invariant set* if $f(X) = X$.

Lemma 3.4. *Let $f : G \to G$ be a graph map, and X, Y be component-cyclic f-invariant sets. If $X \cap Y \neq \emptyset$, then $X \cup Y$ is also a component-cyclic f-invariant set.*

Proof. Suppose that X has n connected components $X_1, \cdots, X_n$ with $f(X_i) \subset X_{i+1\,(\mathrm{mod}\,n)}$ for $i \in \mathbb{N}_n$, and Y has m connected components $Y_1, \cdots, Y_m$ with $f(Y_i) \subset Y_{i+1\,(\mathrm{mod}\,m)}$ for $i \in \mathbb{N}_m$. Since $X \cap Y \neq \emptyset$, we may assume that $X_1 \cap Y_1 \neq \emptyset$. For any $i \in \mathbb{N}$, write $X_{i+n} = X_i$ and $Y_{i+m} = Y_i$. Then $X_i \cap Y_i \neq \emptyset$. Let $Z = X \cup Y$ and let Z_i be the connected component of Z containing $X_i \cup Y_i$. Then $f(Z_i) \subset Z_{i+1}$, and there is a common factor k of n and m such that $Z_{i+k} = Z_i$ for all $i \in \mathbb{N}$. Thus $Z = X \cup Y$ is also a component-cyclic f-invariant set. □

Lemma 3.5. *Let $f : G \to G$ be a graph map. Then for any $x \in R(f) - \overline{P(f)}$ there is a component-cyclic f-invariant closed set Y such that $x \in Y \subset G - EP(f)$.*

Proof. Since $x \in R(f) - \overline{P(f)}$, there is an arc $A = [x, y] \subset G - P(f)$ such that $(x, y] \cap V(G) = \emptyset$ and $(x, y) \cap R(f) \supset (x, y) \cap O(x, f) \neq \emptyset$. Hence, there is an $n \in \mathbb{N}$ such that $f^n(A) \cap A \neq \emptyset$. Let $Y = O(A, f)$ and $Z = O(A, f^n)$ be the orbits of A under f and f^n, respectively. Then $f(Y) \subset Y = \bigcup_{i=0}^{n-1} f^i(Z)$, $f^n(Z) \subset Z$, and Z is connected. By Corollary 2.11, we have $A \cap EP(f) = \emptyset$. Hence $Y \subset G - EP(f)$. By Corollary 2.9, Z is closed in G. So Y is also closed. Let k be the number of connected components of Y. Then k is a factor of n. Let Y_1 be the connected component of Y containing Z. Then $x \in Y_1$ and $f^k(Y_1) \subset Y_1$. Write $Y_i = f^{i-1}(Y_1)$ for $i = 2, \cdots, k$. Then $Y_1, Y_2, \cdots, Y_k$ are just the k connected components of Y, and $f(Y_i) \subset Y_{i+1\,(\mathrm{mod}\,k)}$ for every $i \in \mathbb{N}_k$. Thus Y is a component-cyclic f-invariant set. □

For any graph map $f : G \to G$ and any $n \in \mathbb{N}$, write

$$EP_n(f) = \{x \in G : \text{ the orbit } O(x, f) \text{ contains at most } n \text{ points}\}.$$

Then $EP_n(f)$ is a closed subset of G, and $EP_n(f) \subset EP(f)$. For any connected open set U in G, let $\xi(U)$ be defined as in (2.4). The following proposition describes the structures of component-cyclic f-invariant closed sets without periodic points.

Proposition 3.6. *Let $f : G \to G$ be a graph map, and $Y \subset G - P(f)$ be a component-cyclic f-invariant closed set with k connected components $Y_1, \cdots, Y_k$ such that $f(Y_i) \subset Y_{i+1\,(\mathrm{mod}\,k)}$ for each $i \in \mathbb{N}_k$. Then*

(1) *There exists a component-cyclic strongly f-invariant closed set X with k connected components $X_1, \cdots, X_k$ such that $X_i \subset Y_i$ for each $i \in \mathbb{N}_k$;*

(2) *For any $i \in \mathbb{N}_k$, let $U_i = \mathrm{Ab}(X_i, f^k)$ be the main absorbed set by X_i under f^k, and let $U = \bigcup_{i=1}^k U_i$. Then $Y_i \subset U_i$, $f(U_i) \subset U_{i+1 \,(\mathrm{mod}\, k)}$, $Y \subset U = \mathrm{Ab}(X, f)$, and U is a component-cyclic f-invariant open set with k connected components $U_1, \cdots, U_k$;*

(3) *For any $i \in \mathbb{N}_k$, X_i contains at least one circle C_i, and $f^k|X_i$ has a unique minimal set M_i, which satisfy $\omega(f) \cap U_i = AP(f^k|X_i) = M_i \subset C_i$ and $\Omega(f) \cap U_i = \Omega(f^k) \cap X_i \subset M_i \cup \partial_G X_i$. This M_i is also a unique minimal set of $f^k|Y_i$ and of $f^k|U_i$;*

(4) *Write $M = \bigcup_{i=1}^k M_i$. Then $\omega(f) \cap U = AP(f|X) = AP(f^k|X) = M$, $\Omega(f) \cap U = \Omega(f) \cap X = \Omega(f^k) \cap X \subset M \cup \partial_G X$, and M is a unique minimal set of any of the three maps $f|X$, $f|Y$ and $f|U$;*

(5) *For any $i \in \mathbb{N}_k$, the main absorbed set $U_i = \mathrm{Ab}(X_i, f^k)$ is a connected component of any of the three sets $G - EP(f)$, $G - \overline{EP(f)}$ and $G - EP_{\xi(G)}(f)$.*

Proof. Since Y is f-invariant, from $Y \subset G - P(f)$ we get $Y \subset G - EP(f)$.

(1) Note that Y_1 itself is a graph, and $f^k|Y_1 : Y_1 \to Y_1$ is a graph map without periodic point. By (1) of Theorem 3.1, there exist an $n \in \mathbb{N}$ and a connected closed set $X_1 \subset Y_1$ such that $f^{k(n-1)}(Y_1) = X_1 = f^k(X_1)$. Let $X_i = f^{i-1}(X_1)$ for $i = 2, \cdots, k$. Put $X = \bigcup_{i=1}^k X_i$. Then X satisfies the conditions mentioned in (1) of this theorem.

(2) For any $i \in \mathbb{N}_k$, it follows from (2) of Lemma 2.5 that $U_i = \mathrm{Ab}(X_i, f^k)$ is open in G. Since $f^{kn}(Y_i) = f^{i-1} f^{k(n-1)} f^{k+1-i}(Y_i) \subset f^{i-1} f^{k(n-1)}(Y_1) = f^{i-1}(X_1) = X_i$, we have $Y_i \subset U_i$, and hence $Y \subset U$. Write $X_{k+1} = X_1$ and $U_{k+1} = U_1$. Since $f(U_i)$ is a connected set containing X_{i+1} and $f(U_i) \subset f\big(O_-(X_i, f^k)\big) \subset O_-(X_{i+1}, f^k)$, we have $f(U_i) \subset U_{i+1}$. Since $X_1, \cdots, X_k$ are pairwise disjoint, the main absorbed sets $U_1, \cdots, U_k$ are also pairwise disjoint. Thus U is a component-cyclic f-invariant open set with k connected components $U_1, \cdots, U_k$. From (2.3) and (2.2) it is easy to check that $U = \bigcup_{i=1}^k U_i$ is just the main absorbed set $\mathrm{Ab}(X, f)$.

(3) For any $i \in \mathbb{N}_k$, from Theorem 3.1 we see that X_i contains at least one circle C_i, and $f^k|X_i$ has a unique minimal set M_i, which satisfy $\Omega(f^k|X_i) = \omega(f^k|X_i) = AP(f^k|X_i) = M_i \subset C_i$. It is well known that $\omega(f) = \omega(f^k)$. By (2) of this theorem we have $\Omega(f) \cap U_i = \Omega(f^k) \cap U_i$. Hence, from (4) and (5) of Lemma 2.5 we get

$\omega(f)\cap U_i = \omega(f^k)\cap U_i = \omega(f^k|X_i) = M_i$ and $\Omega(f)\cap U_i = \Omega(f^k)\cap U_i = \Omega(f^k)\cap X_i \subset \Omega(f^k|X_i)\cup \partial_G X_i = M_i \cup \partial_G X_i$, and from $\omega(f^k)\cap U_i = M_i$ we see that M_i is also a unique minimal set of $f^k|Y_i$ and of $f^k|U_i$.

(4) follows from (2) and (3) of this theorem and the fact that $AP(\varphi) = AP(\varphi^n)$ for any $n\in\mathbb{N}$ and any continuous map φ from a topological space to itself ([13]).

(5) For any $i\in\mathbb{N}_k$, from $X_i\subset G-EP(f)$ we get $U_i = \mathrm{Ab}(X_i, f^k)\subset G-EP(f)$. By (1) of Lemma 2.5 and (2) of Lemma 2.4, we have $\partial_G U_i\subset EP(f)$. Thus U_i is just a connected component of $G-EP(f)$. Since U_i is open, it is also a connected component of $G-\overline{EP(f)}$. By Corollary 2.3, we have $|\partial_G U_i|\le \xi(U_i)$. So $|\bigcup_{j=1}^k \partial_G U_j|\le \sum_{j=1}^k \xi(U_j)\le \xi(G)$. From $f(U_i)\subset U_{i+1}$ and $f(\partial_G U_i)\subset f\big(EP(f)\big)\subset EP(f)\subset G-U_{i+1}$ we get $f(\partial_G U_i)\subset \partial_G U_{i+1}$. Thus $\bigcup_{j=1}^k \partial_G U_j$ is f-invariant, and $\partial_G U_i\subset \bigcup_{j=1}^k \partial_G U_j\subset EP_{\xi(G)}(f)$. Hence U_i is also a connected component of $G-EP_{\xi(G)}(f)$.

All together, we complete the proof. □

Corollary 3.7. *Let $f: G\to G$ be a graph map, and Y and Y' be component-cyclic f-invariant closed sets contained in $G-P(f)$. If $Y\cap Y'\neq\emptyset$, then $\omega(f|Y)=\omega(f|Y')$.*

Proof. By (4) of Proposition 3.6, $f|Y$ and $f|Y'$ have unique minimal sets M and M', respectively, which satisfy $\omega(f|Y)=M$ and $\omega(f|Y')=M'$. Obviously, both M and M' are minimal sets of $f|(Y\cup Y')$. Since $Y\cap Y'\neq\emptyset$, by Lemma 3.4, $Y\cup Y'$ is also a component-cyclic f-invariant closed set. Hence, by (4) of Proposition 3.6, $f|(Y\cup Y')$ has only one minimal set, which must be $M=M'$. Thus $\omega(f|Y)=\omega(f|Y')$. □

The following theorem is a main result of this paper, which describes the dynamical behavior of f on the intersection of each connected component of $G-\overline{EP(f)}$ with $R(f)$.

Theorem 3.8. *Let G be a connected graph, and $f: G\to G$ be a continuous map such that $P(f)\neq\emptyset$ and $R(f)-\overline{P(f)}\neq\emptyset$. Then there exist pairwise disjoint nonempty open subsets $U_1,\cdots,U_n$ of G with $n\in\mathbb{N}$ such that*

(1) *$f(U_i)\subset U_i$, for each $i\in\mathbb{N}_n$.*

(2) *Write* $U=\bigcup_{i=1}^{n}U_i$ *and* $U_0=G-U$. *Then* $\overline{P(f)}\subset U_0$, $\overline{U}-U\subset EP(f)$, *and* $\Omega(f)-U_0=R(f)-U_0=R(f)-\overline{P(f)}\subset U$.

(3) *For each* $i\in\mathbb{N}_n$, U_i *has* k_i *connected components* $U_{i1},\cdots,U_{ik_i}$ *with* $k_i\in\mathbb{N}$, *which satisfy* $f(U_{ik_i})\subset U_{i1}$ *and* $f(U_{ij})\subset U_{i,j+1}$ *for* $1\le j<k_i$.

(4) *For each* $i\in\mathbb{N}_n$, *write* $W_i=R(f)\cap U_i$. *Then* W_i *is a unique minimal set of* f *contained in* U_i.

(5) *For each* $i\in\mathbb{N}_n$ *and* $j\in N_{k_i}$, *write* $W_{ij}=W_i\cap U_{ij}$. *Then* W_{ij} *is a unique minimal set of* f^{k_i} *contained in* U_{ij}, *and there is a connected closed subset* G_{ij} *of* G *and a circle* C_{ij} *such that* $W_{ij}\subset C_{ij}\subset G_{ij}\subset U_{ij}$, $f(W_{ik_i})=W_{i1}$, $f(G_{ik_i})=G_{i1}$, *and* $f(W_{ij})=W_{i,j+1}$ *and* $f(G_{ij})=G_{i,j+1}$ *for* $1\le j<k_i$.

(6) *For each* $i\in\mathbb{N}_n$, $j\in\mathbb{N}_{k_i}$, *and for each* $x\in U_{ij}$, *one has* $\lim_{m\to\infty}d(f^m(x),W_i)=0$, *and* $\lim_{m\to\infty}d(f^{mk_i}(x),W_{ij})=0$.

Proof. Since $R(f)-\overline{P(f)}\neq\emptyset$, by Lemma 3.5, there exists at least one component-cyclic f-invariant closed set $Y^{(1)}$ in $G-EP(f)$. By Proposition 3.6, $Y^{(1)}\subset G-\overline{EP(f)}$, and if $Y^{(1)}$ has k_1 connected components then $Y^{(1)}$ contains at least k_1 pairwise disjoint circles. Thus we can assume that there exist n pairwise disjoint component-cyclic f-invariant closed sets $Y^{(1)},\cdots,Y^{(n)}$ in $G-\overline{EP(f)}$ with $n\in\mathbb{N}_m$ but $G-\overline{EP(f)}$ cannot admit $n+1$ pairwise disjoint component-cyclic f-invariant closed sets, where m is the maximal number of pairwise disjoint circles in G. Suppose that $Y^{(i)}$ has k_i connected components. Then we have $n\le\sum_{i=1}^{n}k_i\le m$.

If $R(f)-\overline{P(f)}\not\subset\bigcup_{i=1}^{n}Y^{(i)}$, then there is a point $x\in R(f)-\overline{P(f)}-\bigcup_{i=1}^{n}Y^{(i)}$. By Lemma 3.5 and Proposition 3.6, there exists a component-cyclic f-invariant closed set Y such that $x\in Y\subset G-\overline{EP(f)}$. For each $i\in\mathbb{N}_n$, since $\omega(f|Y)-\omega(f|Y^{(i)})\supset R(f|Y)-Y^{(i)}\supset\{x\}\neq\emptyset$, by Corollary 3.7 we have $Y\cap Y^{(i)}=\emptyset$. So $Y,Y^{(1)},\cdots,Y^{(n)}$ are $n+1$ pairwise disjoint component-cyclic strongly f-invariant closed sets in $G-\overline{EP(f)}$. But this will leads to a contradiction. Hence we must have $R(f)-\overline{P(f)}\subset\bigcup_{i=1}^{n}Y^{(i)}$.

For each $i\in\mathbb{N}_n$, let $X^{(i)}$ be the component-cyclic strongly f-invariant closed set contained in Y_i and let $G_{i1},\cdots,G_{ik_i}$ be the connected components of $X^{(i)}$ (see (1) of Proposition 3.6). Set $U_{ij}=\mathrm{Ab}(G_{ij},f^{k_i})$, $U_i=\bigcup_{j=1}^{k_i}U_{ij}$, and $U=\bigcup_{i=1}^{n}U_i$. Then, by

(2) of Proposition 3.6 and by Lemma 2.4, each U_i is an f-invariant open set, $f(U_{ik_i}) \subset U_{i1}$ and $f(U_{ij}) \subset U_{i,j+1}$ for $1 \le j < k_i$, and $\overline{U}_i - U_i \subset \bigcup_{j=1}^{k_i}(\overline{U}_{ij} - U_{ij}) \subset EP(f^{k_i}) = EP(f)$. So, $\overline{U} - U \subset \bigcup_{i=1}^{n}(\overline{U}_i - U_i) \subset EP(f)$. From the definition of U_{ij}, we see that $U_{ij} \cap P(f) = \emptyset$, which means that $P(f) \cap U = \emptyset$. Thus $\overline{P(f)} \subset G - U$. Let $U_0 = G - U$. Then $R(f) - U_0 \subset R(f) - \overline{P(f)}$. Since $R(f) - \overline{P(f)} \subset \bigcup_{i=1}^{n} Y^{(i)} \subset U = G - U_0$, we have $R(f) - U_0 \supset R(f) - \overline{P(f)}$. So, $R(f) - U_0 = R(f) - \overline{P(f)}$. Thus (1), (2), and (3) are proved except for the relation $\Omega(f) - U_0 = R(f) - U_0$.

As in the statement of the theorem, for each $i \in \mathbb{N}_n$ and $j \in N_{k_i}$, let $W_i = R(f) \cap U_i$ and $W_{ij} = W_i \cap U_{ij}$. From (3) and (4) of Proposition 3.6, we have W_i and W_{ij} are the unique minimal sets of $f|U_i$ and $f^{k_i}|U_{ij}$ respectively, and there exist circles C_{ij} with $W_{ij} \subset C_{ij} \subset G_{ij} \subset U_{ij}$. This together with (2) and the strong invariance of $X^{(i)}$ implies that $f(W_{ik_i}) = W_{i1}$, $f(G_{ik_i}) = G_{i1}$, and $f(W_{ij}) = W_{i,j+1}$ and $f(G_{ij}) = G_{i,j+1}$ for $1 \le j < k_i$. Thus (4) and (5) are proved. The conclusions of (6) follow from (4) of Theorem 3.1, which clearly implies the equation $\Omega(f) - U_0 = R(f) - U_0 = \bigcup_{i=1}^{n} W_i$. Thus the proof of (2) is complete. □

4. Applications of the main theorem

As applications of Theorem 3.8, we will prove several propositions part of which improve or reprove some known results.

The following theorem is also implied by [7, Theorem 4].

Theorem 4.1. *Let $f : G \to G$ be a graph map, and let m be the greatest number of pairwise disjoint circles in G. Then there exist minimal sets $M_1, \cdots, M_n$ of f in $G - \overline{EP(f)}$ with $0 \le n \le m$ such that $\overline{R(f)} = \overline{P(f)} \cup \big(\bigcup_{i=1}^{n} M_i\big)$.*

Proof. By [22, Theorem 2.1], we get $\overline{R(f)} = R(f) \cup \overline{P(f)} = \overline{P(f)} \cup \big(R(f) - \overline{P(f)}\big)$. If $R(f) - \overline{P(f)} = \emptyset$ then we can put $n = 0$. Otherwise, $R(f) - \overline{P(f)} \neq \emptyset$. Let n, U_i and the minimal sets W_i be the same as in Theorem 3.8. Then, by Theorem 3.8, we have $1 \le n \le m$ and

$$R(f) - \overline{P(f)} = \bigcup_{i=1}^{n} \big(R(f) \cap U_i\big) = \bigcup_{i=1}^{n} W_i \subset G - \overline{EP(f)}.$$

Hence, writing M_i for W_i, we obtain $\overline{R(f)} = \overline{P(f)} \cup \big(\bigcup_{i=1}^{n} M_i\big)$. □

The following proposition indicates that, for a graph map $f: G \to G$ and $x \in G$, if every neighborhood of x contains both a recurrent point and an eventually periodic point of f, then every neighborhood of x must contain a periodic point.

Proposition 4.2. *Let f be a graph map. Then $\overline{R(f)} \cap \overline{EP(f)} = \overline{P(f)}$.*

Proof. By [22, Theorem 2.1] we get $\overline{R(f)} = R(f) \cup \overline{P(f)} = \overline{P(f)} \cup \big(R(f) - \overline{P(f)}\big)$, and by Theorem 3.8 we get $R(f) - \overline{P(f)} \subset G - \overline{EP(f)}$. Hence

$$\big(\overline{R(f)} \cap \overline{EP(f)}\big) - \overline{P(f)} = \big(\overline{R(f)} - \overline{P(f)}\big) \cap \overline{EP(f)} = \big(R(f) - \overline{P(f)}\big) \cap \overline{EP(f)} = \emptyset.$$

In addition, $\overline{R(f)} \cap \overline{EP(f)} \supset \overline{P(f)}$ follows from $R(f) \supset P(f)$ and $EP(f) \supset P(f)$. Thus we have $\overline{R(f)} \cap \overline{EP(f)} = \overline{P(f)}$. □

Example 4.3. *For special graph maps, by Theorem 3.8 or Theorem 4.1 we can obtain some further detailed information. For example, let S^1 be the unit circle in the complex plane $\mathbb{C}$. For $n \in \mathbb{N}$, let $T_n = \{z \in \mathbb{C} : z^n \in [0, 2^n]\}$ and let $G_n = S^1 \cup T_n$. Then any two circles in G_n intersect. Hence, from Theorem 3.8 we see that, for any $f \in C^0(G_n)$, if $\overline{R(f)} \neq \overline{P(f)}$ then there exist a unique connected component U of $G - \overline{EP(f)}$, a unique strongly f-invariant connected closed set X, a circle C in G and a unique minimal set M of f such that $U \cap R(f) \neq \emptyset$ and $M \subset C \subset X \subset U$. By Theorem 4.1, we have $\overline{R(f)} = \overline{P(f)} \cup M$. By Theorem 3.1, this minimal set M is totally minimal.*

Noting that $AP(f) = \bigcup\{M : M \text{ is a minimal set of } f\}$ and $P(f) \subset AP(f) \subset R(f)$, by Theorem 4.1, we get $\overline{R(f)} \subset AP(f) \cup \overline{P(f)} \subset \overline{R(f)}$. Then we have

Theorem 4.4. *Let $f: G \to G$ be a graph map. Then $\overline{R(f)} = AP(f) \cup \overline{P(f)}$.*

For any graph map f, Hawete showed that $\overline{R(f)} = \overline{AP(f)}$ (see [17, Lemma 3.1]). In general, if X is topological space and $f \in C^0(X)$, then from the relation $P(f) \subset AP(f) \subset R(f)$ we can easily get that the condition $\overline{R(f)} = AP(f) \cup \overline{P(f)}$ implies $\overline{R(f)} = \overline{AP(f)}$ and $\overline{R(f)} = R(f) \cup \overline{P(f)}$. The following examples show that neither $\overline{R(f)} = \overline{AP(f)}$ nor $\overline{R(f)} = R(f) \cup \overline{P(f)}$ implies $\overline{R(f)} = AP(f) \cup \overline{P(f)}$. Thus Theorem 4.4 is an essential improvement of [17, Lemma 3.1] and [22, Theorem 2.1].

Example 4.5. *(1) In* [25, Example 3.3] *the authors constructed an isometric homeomorphism f from a complex Hilbert space X to itself, which satisfies $R(f) = X$ and*

$AP(f)=\emptyset$. For this f, we have $\overline{R(f)}=R(f)\cup\overline{P(f)}$ but have neither $\overline{R(f)}=AP(f)\cup\overline{P(f)}$ nor $\overline{R(f)}=\overline{AP(f)}$.

(2) Let $g:[0,1]\to[0,1]$ be the tent map defined by $g(x)=\min\{2x,2-2x\}$ for all $x\in[0,1]$, and let $h:S^1\to S^1$ be an irrational rotation. Put $X=[0,1]\times S^1$. Then X is a cylinder. Define $f:X\to X$ by $f(x,y)=\big(g(x),h(y)\big)$ for any $(x,y)\in X$. Then $P(f)=\emptyset$, and both $AP(f)$ and $X-R(f)$ are dense subsets of X. For this f, we have $\overline{R(f)}=\overline{AP(f)}$ but have neither $\overline{R(f)}=AP(f)\cup\overline{P(f)}$ nor $\overline{R(f)}=R(f)\cup\overline{P(f)}$.

From [22, Theorem 2.1], we know that $\overline{R(f)}-\overline{P(f)}=\big(R(f)\cup\overline{P(f)}\,\big)-\overline{P(f)}=R(f)-\overline{P(f)}$. By Theorem 4.1, $\overline{R(f)}-\overline{P(f)}=\bigcup_{i=1}^{n}M_i$ which is closed. Hence we have

Proposition 4.6. *Let $f:G\to G$ be a graph map. If $P(f)\neq\emptyset$ and $R(f)-\overline{P(f)}\neq\emptyset$, then the distance $d\big(R(f)-\overline{P(f)}\,,\,\overline{P(f)}\,\big)=d\big(\overline{R(f)}-\overline{P(f)}\,,\,\overline{P(f)}\,\big)>0$.*

The following theorem is first given by Zhang, Liu and Qin in [30], which can be also obtained as a corollary of Proposition 4.6.

Theorem 4.7. ([30, Theorem 2.8]) *Let $f:G\to G$ be a graph map. If $\overline{R(f)}=G$ and $P(f)\neq\emptyset$, then $\overline{P(f)}=G$.*

Proof. Assume to the contrary that $\overline{P(f)}\neq G$. Then, by Proposition 4.6, there will be a neighborhood U of $\overline{P(f)}$ in G such that $U-\overline{P(f)}\neq\emptyset$ and $U\cap\big(R(f)-\overline{P(f)}\,\big)=\emptyset$. But these will imply that $\overline{R(f)}\neq G$. This is a contradiction. □

By Theorem 3.8, for any $x\in\overline{R(f)}-\overline{P(f)}=R(f)-\overline{P(f)}$, there exist a component-cyclic strongly f-invariant closed set $X_i\subset G-\overline{EP(f)}\subset G-\overline{P(f)}$ and a circle $C_{ij}\subset X_i$ such that $x\in C_{ij}$. Then, we obtain the following

Proposition 4.8. *Let $f:G\to G$ be a graph map and W be a subset of $G-\overline{P(f)}$. If $W\cap C=\emptyset$ for any circle $C\subset G-\overline{P(f)}$, then $W\cap\overline{R(f)}=\emptyset$. Specially, if W is a connected component of $G-\overline{P(f)}$ which contains no circle, then $W\cap\overline{R(f)}=\emptyset$; if $G-\overline{P(f)}$ contains no circle, then $\overline{R(f)}=\overline{P(f)}$.*

From Proposition 4.8 we can directly derive the following theorem, which is first given by Ye in [29].

Theorem 4.9. *([29, Theorem 2.6]) Let $f:T\to T$ be a tree map. Then $\overline{R(f)}=\overline{P(f)}$.*

Proposition 4.10. *Let $f:G\to G$ be a graph map, and $A=[x,y]$ be an arc in G.*

(1) *If $(x,y)_A\cap \mathrm{Br}(G)=\emptyset$ and $A\cap\overline{P(f)}\neq\emptyset$, then $(x,y)_A\cap\big(\overline{R(f)}-\overline{P(f)}\big)=\emptyset$.*

(2) *If $x\in\overline{P(f)}$ and $y\in\overline{R(f)}-\overline{P(f)}$, then $(x,y]_A\cap\mathrm{Br}(G)\neq\emptyset$.*

Proof. (1) Under the given conditions, for any circle $C\subset G-\overline{P(f)}$, we have $(x,y)_A\cap C=\emptyset$. Let $W=(x,y)_A-\overline{P(f)}$. By Proposition 4.8, we have $W\cap\overline{R(f)}=\emptyset$. This means that $(x,y)_A\cap\big(\overline{R(f)}-\overline{P(f)}\big)=\emptyset$.

(2) If $x\in\overline{P(f)}$ and $(x,y]_A\cap\mathrm{Br}(G)=\emptyset$, then $y\notin C$ for any circle $C\subset G-\overline{P(f)}$, and it follows from Proposition 4.8 that $y\notin\overline{R(f)}-\overline{P(f)}$. Therefore, if $x\in\overline{P(f)}$ and $y\in\overline{R(f)}-\overline{P(f)}$ then we must have $(x,y]_A\cap\mathrm{Br}(G)\neq\emptyset$. □

Example 4.11. *Let $G_n=S^1\cup T_n\subset\mathbb{C}$ be the same as in Example 4.3, and let $f\in C^0(G_n)$. If the origin $0\in\overline{P(f)}$, and $S^1\cap\overline{P(f)}\neq\emptyset$, then $G_n-\overline{P(f)}$ contains no circle, and from Proposition 4.8 we get $\overline{R(f)}=\overline{P(f)}$.*

Theorem 4.12. *Let $f:G\to G$ be a graph map, and U be a connected component of $G-\overline{EP(f)}$ with $U\cap R(f)\neq\emptyset$. Then there exists $k\in\mathbb{N}$ such that $f^k(U)\subset U$, and $f^k|U$ is topologically semi-conjugate to an irrational rotation of the unit circle S^1.*

Proof. Use the all notations in Theorem 3.8. From Theorem 3.8 we see that there exist $i\in\mathbb{N}_n$ and $j\in\mathbb{N}_{k_i}$ such that $U=U_{ij}=\mathrm{Ab}(X_{ij},f^{k_i})$. Let $k=k_i$ and write $X=X_{ij}$. Then $f^k(X)=X$ and $f^k(U)\subset U$. By Theorem 3.1, $f^k|X$ is topologically semi-conjugate to some irrational rotation $h:S^1\to S^1$, that is, there is a continuous surjection $\varphi:X\to S^1$ such that $h\varphi=\varphi f^k|X$. By means of φ we define a map $\psi:U\to S^1$ as follows. For any $x\in U$, taking an $n\in\mathbb{Z}_+$ such that $f^{kn}(x)\in X$, and then we put $\psi(x)=h^{-n}\varphi f^{kn}(x)$. Write $x_n=f^{kn}(x)$. For any $i\in\mathbb{N}$, we have $h^{-n-i}\varphi f^{kn+ki}(x)=h^{-n-i}\varphi f^{ki}(x_n)=h^{-n-i}h^i\varphi(x_n)=h^{-n}\varphi f^{kn}(x)$. This means that the definition of $\psi(x)$ is independent of the choice of n. So we obtain a map $\psi:U\to S^1$. This ψ is surjective, since $\varphi=\psi|X$ is surjective. By (3) of Lemma 2.5, for any $x\in U$, there exist an $m\in\mathbb{N}$ and an open neighborhood Z of x in U such that $f^{km}(Z)\subset X$, which implies that $\psi|Z=h^{-m}\varphi f^{km}|Z$ is continuous. Thus ψ is continuous. From $h\psi(x)=hh^{-n}\varphi f^{kn}(x)=h^{-n+1}\varphi f^{kn-k}(f^k(x))=\psi f^k(x)$ we get

$h\psi = \psi f^k|U$. Hence $f^k|U$ is topologically semi-conjugate to the irrational rotation $h: S^1 \to S^1$. Thus the theorem is proven. □

Remark 4.13. Theorem 4.12 and (5) of Theorem 3.1 seem similarly, but there are some differences between them. In (5) of Theorem 3.1, since the graph map $f: G \to G$ has no periodic point, there exist a unique strongly f-invariant compact set X and an $n \in \mathbb{N} \cup \{0\}$ such that $f^n(G) \subset X$. However, in Theorem 4.12, if f has periodic points, then the f^k-invariant set U is open in G and is not compact, and there is no $n \in \mathbb{N}$ such that $f^{kn}(U)$ is contained in the strongly f^k-invariant compact set $X = X_{ij}$.

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