RIGIDITY FOR NON-LEFT-ORDERABLE GROUP ACTIONS ON DENDRITES

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ABSTRACT. We establish two rigidity theorems for non-left-orderable group actions on dendrites. Explicitly, letting X be a dendrite without infinite order points, we show that if Γ is a non-virtually-left-orderable small group, then no Γ-action on X is almost free; and if Γ is a higher rank lattice, then every Γ-action on X is a highly proximal extension of an almost finite action.

1. INTRODUCTION

The notion of left-orderable group is closely related to the study of rigidity for higher rank lattice actions. Recall that a group is *left-orderable* if it admits a total ordering which is invariant by left translations; and is a *higher rank lattice* if it is an irreducible lattice in a connected real semisimple Lie group with finite center, no compact factors and with $\mathbb{R}$-rank at least two. It was conjectured that every continuous action on the circle by a higher rank lattice must factor through a finite group action (the so called 1-dimensional Zimmer's rigidity conjecture); this is equivalent to saying that every orbit of such actions is finite. Burger-Monod, and Ghys proved independently the existence of finite orbits for higher rank lattice actions on the circle ([5, 12]). This translates the conjecture into an equivalent form: no higher rank lattice is left-orderable. The latter was answered positively by Witte-Morris in [31] for finite index subgroups of $SL_n(\mathbb{Z})$ with $n \geq 3$ and by Deroin-Hurtado for any higher rank lattice in [7]. Recently there has been a great progress on the Zimmer program for smooth higher rank lattice actions on manifolds with dimensions ≥ 2. We do not plan to list all the related results here and just suggest the readers to consult [4, 11] for the surveys.

A *dendrite* is a Peano curve containing no simple closed curves. There has been intensively studied around group actions on dendrites very recently. One motivation is that dendrites can appeare as the limit sets of some Klein groups, the structures of which are closely related to the geometric properties of 3-dimensional hyperbolic manifolds (see e.g. [3, 21]). Also, the compactifications of the Cayley graphs of free groups are dendrites, which are important for understanding the algebraic properties of free groups. Group actions on the circle have been systematically investigated during the past few decades [13, 23]. However, group actions on general curves lack of the same depth of understanding. Dendrites and the circle lie on two opposite ends of Peano curves in topologies. So, studying group actions on dendrites is a starting point for better understanding group actions on curves or continua of higher dimensions. Some people studied

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the Ghys-Margulis' alternative for group actions on dendrites ([25, 18, 10]) and on totally regular curves ([26]). One may consult [1, 14, 27] for the discussions around the structures of minimal sets for group actions on dendrites. The algebraic structures of dendrite homeomorphism groups were investigated in [9].

The aim of the paper is to investigate the rigidity phenomena for non-left-orderable group actions on dendrites. However, even for higher rank lattice actions on dendrites, the exact analogy to the rigidity results by Witte-Morris and Deroin-Hurtado mentioned above do not hold anymore. In fact, every countable infinite group can act faithfully on a starlike dendrite of infinite order and every residually finite group admits a faithful action on a dendrite without infinite order points (see Section 7 for examples). Despite of these, the results we obtained indicate that if Γ is either a non-virtually-left-orderable small group or a higher rank lattice and X is a dendrite without infinite order points, then the actions of Γ on X are still very restrictive (a *small group* is a group containing no free non-abelian subgroups).

Now, we start to introduce the main results explicitly. We call an action of a group G on a topological space X is *almost free* if for every $g \in G \setminus \{e_G\}$, the fixed point set $\mathrm{Fix}(g)$ of g is totally disconnected. A group G is *virtually left-orderable* if there is a finite index subgroup of G that is left-orderable.

Theorem 1.1. *Let Γ be a finitely generated small group and X be a nondegenerate dendrite without infinite order points. If G is not virtually left-orderable, then Γ does not admit an almost free action on X.*

The proof of Theorem 1.1 relies on the existence of finite orbits for small group actions on dendrites by Malyutin [18], Duchesne-Monod [10], and Glasner-Megrelishvili [14] etc. and on the the following proposition which characterizes the left-orderability of a finitely generated group via its actions on dendrites.

Proposition 1.2. *Let Γ be a finitely generated group. Then Γ is left-orderable if and only if it admits an almost free action on a nondegenerate dendrite leaving an end point fixed.*

Duchesne-Monod proved the existence of finite orbits for higher rank lattice actions on dendrites ([10]). Based on this result, we further describe the structures of higher rank lattice actions on dendrites without infinite order points. We say a group action is *almost finite* if it is the inverse limit of an inverse system consisting of finite actions (see Section 2.1 for the definition).

Theorem 1.3. *Let Γ be a higher rank lattice acting on a nondegenerate dendrite X with no infinite order points. Then there exists a nondegenerate subdendrite Y which is Γ-invariant and satisfies the following items:*

(1) There is an inverse system of finite actions $\{(Y_i,\Gamma) : i = 1,2,3,\dots\}$ with monotone bonding maps $\phi_i : Y_{i+1} \to Y_i$ and with each Y_i being a dendrite, such that $(Y,\Gamma|Y)$ is topologically conjugate to the inverse limit $(\varprojlim(Y_i,\Gamma),\Gamma)$; particularly, the action is almost finite.

(2) The first point map $r : X \to Y$ is a highly proximal extension; that is, for each $y \in Y$, there is a sequence $g_i \in \Gamma$ with $\mathrm{diam}(g_i r^{-1}(y)) \to 0$. In addition, if $x \in X \setminus Y$, then $r(x)$ is an end point of Y with infinite orbit.

The strategy to prove Theorem 1.3 is to analyze the semilinear left preoders on the acting groups. It is known that the topology on $\mathbb{R}$ is determined by the unique complete separable continuous linear order on it. Fixing an end point on a dendrite X, there is a canonical semilinear order on X derived by the linear order on each arc connecting the fixed end point. While every semilinear order with a countable dense subset and supremum for each chain has a completion isomorphic to a dendrite ([9, Theorem 5.19]). The definition of semilinear order and the relation with dendrites can be seen in [9, Section 5] for details. So, considering the group actions on dendrites, it will naturally lead to study the semilinear order on groups. This helps us establish the following local property that is crucial in proving Theorem 1.3.

Proposition 1.4. *Let Γ be a higher rank lattice acting on a nondegenerate dendrite X. If Γ fixes some end point z of X, then there is another point $s \in X$ such that Γ fixes the arc $[z,s]$ pointwise.*

We will prove Proposition 1.4 for Γ being a finite index subgroup of $SL_n(\mathbb{Z})$ with $n \geq 3$ and being a general higher rank lattice respectively. Though the former is only a special case of the latter, the proofs rely on distinct techniques which have their own interests.

As a byproduct, we also get the following result on groups which generalizes a theorem of Kopytov in [17, Theorem 2.7] that a group admitting a semilinear left partial order is left-orderable.

Proposition 1.5. *Let G be a group admitting a semilinear left preorder. Then there is a canonical associated quotient group of G which is left orderable. In particular, if G admits a semilinear left partial order, then G is left-orderable.*

The paper is organized as follows. In section 2, we introduce some notions and results around the structures of dendrites and orderings on groups. In section 3, we first give a new characterization of left-orderability of a group via its actions on dendrites and then by which afford a proof of Theorem 1.1. In section 4, assuming Proposition 1.4, we prove Theorem 1.3. In section 5, inspired by some key ideas of Witte-Morris in [31], we show Proposition 1.4 for finite index subgroups of $SL_n(\mathbb{Z})$ with $n \geq 3$. In section 6, we first study some properties of groups admitting semilinear left preorders and establish Proposition 1.5. Then we show Proposition 1.4 in general case and complete the proof of Theorem 1.3. Finally, in section 7, we give some examples for better understanding the theorems obtained and restate some related open questions.

2. Preliminaries

In this section, we will recall some notions and results around group actions, dendrites, and left-orderability of a countable group. Particularly, we will introduce the dynamical realization technique which is very useful in proving the left-orderability of a group.

2.1. **Notions around group actions.** Let G be a group and X be a topological space. Recall that an *action* of G on X is a group homomorphism $\phi : G \to \mathrm{Homeo}(X)$, where $\mathrm{Homeo}(X)$ denotes the homeomorphism group of X; we use the pair (X,G) to denote this action and use gx or $g.x$ to denote $\phi(g)(x)$ for $g \in G$ and $x \in X$. If $\ker(\phi) = e_G$, we say the action (X,G) is *faithful*. We also call the action (X,G) a *system*. For $x \in X$, the set $Gx := \{gx : g \in G\}$ is called the *orbit* of x; a subset A of X is called G*-invariant* if $Gx \subset A$

for any $x \in A$; if $Gx = \{x\}$, then x is called a *fixed point* of G; we use $\mathrm{Fix}(G)$ to denote the fixed point set of G. Suppose A is closed and G-invariant. Then we naturally have a restriction action of G on A, which is denoted by $(A, G|A)$ and is called a *subsystem* of (X,G). If $\phi(G)$ is finite, we say the action (X,G) is *finite*. For two actions (X,G) and (Y,G), if there is a continuous surjection $\phi : X \to Y$ with $\phi(gx) = g\phi(x)$ for each $g \in G$ and $x \in X$, then ϕ is said to be a *factor map* or an *extension map* and (Y,G) is said to be a *factor* of (X,G) or (X,G) is an *extension* of (Y,G); if ϕ is additionally a homeomorphism, then ϕ is called a *topological conjugation*, and (X,G), (Y,G) are called *topologically conjugate*. An extension $\phi : (X,G) \to (Y,G)$ is *highly proximal* if for each $y \in Y$, $\phi^{-1}(y)$ is contractible; that is, there is a sequence $g_1, g_2, \ldots$ in G such that $\mathrm{diam}(g_i\phi^{-1}(y)) \to 0$ as $i \to \infty$.

If $(X_i, G), i = 0,1,2,\ldots$, is a sequence of G actions associated to each i a factor map $\phi_i : X_{i+1} \to X_i$, then we say that these (X_i, G) together with ϕ_i's form an *inverse system* and call each ϕ_i a *bonding map*. The inverse limit of this inverse system is defined to be the set

$$\varprojlim (X_i, G) := \left\{ (x_0, x_1, \ldots) \in \prod_{i=0}^{\infty} X_i : \phi_i(x_{i+1}) = x_i, \text{ for each } i \right\}$$

together with a specified action by G: $g.(x_0, x_1, \ldots) = (gx_0, gx_1, \ldots)$ for each $g \in G$; we use $(\varprojlim (X_i, G), G)$ to denote this specified action. It is known that if each X_i is a compact metric space, then so is $\varprojlim (X_i, G)$. We call a group action (X,G) being *almost finite* if it is topologically conjugate to an inverse limit $(\varprojlim (X_i, G), G)$ with each (X_i, G) being a finite action.

2.2. **Dendrites and fixed point properties.** By a *continuum* we mean a connected compact metrizable space. A continuum is *nondegenerate* if it is not a single point. An (nondegerate) *arc* is a continuum homeomorphic to the closed interval $[0,1]$. A *dendrite* is a continuum that is locally connected and contains no simple closed curve. In the case of a dendrite X, the Menger-Urysohn order (*order* for short) of a point $x \in X$ is just the cardinality of the set of connected components of $X \setminus \{x\}$, which is denoted by $\mathrm{ord}_X(x)$. A point of X is an *end point*, *regular point*, and *branch point* if its order is one, two, and ≥ 3, respectively. If the orders of all points in X have an uniform upper bound, then we say X is of *finite order*. For $a, b \in X$, we use $[a,b]$ to denote the unique arc (may be degenerate) connecting a and b; and use $[a,b), (a,b], (a,b)$ to denote the sets $[a,b] \setminus \{b\}, [a,b] \setminus \{a\}, [a,b] \setminus \{a,b\}$ respectively. It is known that the end point set of a nondegenerate dendrite X is nonempty; the regular point set of X is dense and uncountable. If Y is a subdendrite of X, then for each $x \in X$, there is a unique $r(x) \in Y$ with $[x, r(x)] \cap Y = \{r(x)\}$; the map r so defined is called the *first point map* from X to Y. One may refer to [22, Chapter X] for more details around dendrites.

Lemma 2.1. *Let X be a nondegenerate dendrite and $h : X \to X$ be a homeomorphism. If h fixes an end point $e \in X$, then h fixes another point $o \neq e$.*

Proof. Take a point $u \neq e \in X$. Since e is an end point, $h([e,u]) \cap [e,u] = [e,v]$ for some $v \neq e \in X$. Then there is $w \in [e,v]$ such that either $h(w) = v$ or $h^{-1}(w) = v$. WLOG, we assume that $h(w) = v$. Then $[e,w] \subset [e,h(w)] \subset [e,h^2(w)] \subset \cdots$. Let $o = \lim_{i \to \infty} h^i(w)$. Then $o \neq e$ and $h(o) = o$. □

The following proposition will be used later.

Proposition 2.2. *Let G be a finitely generated nilpotent group acting on a nondegenerate dendrite X. Suppose that G fixes an end point z of X. Then G has another fixed point.*

Proof. Take a subnormal series $G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_n \trianglerighteq G_{n+1} = \{e\}$ such that G_i/G_{i+1} is cyclic for each i. Take $g_i \in G_i \setminus G_{i+1}$ such that $G_i/G_{i+1} \cong \langle g_i G_{i+1} \rangle$. Thus $G_i = \langle g_i, \cdots, g_n \rangle$ for each $i = 0, 1, \cdots, n$.

By Lemma 2.1, there is a point in $X \setminus \{z\}$ which is fixed by G_n. Now assume that there is a point $x \in X \setminus \{z\}$ fixed by G_i for some $i \in \{1, \cdots, n\}$. Since z is an end point, there is $y \in X \setminus \{z\}$ such that $g_{i-1}([z,x]) \cap [z,x] = [z,y]$. Then there is some $w \in [z,y]$ such that either $g_{i-1}(w) = y$ or $g_{i-1}^{-1}(w) = y$. WLOG, we may assume that $g_{i-1}(w) = y$. Then $[z,w] \subset [z, g_{i-1}(w)] \subset [z, g_{i-1}^2(w)] \subset \cdots$. Let $u = \lim_{k\to\infty} g_{i-1}^k(w)$. Then u is fixed by g_{i-1}.

We claim that u is fixed by G_i and hence is fixed by G_{i-1}. We may assume that $g_{i-1}(x) \neq x$, otherwise $u = y = x$ is fixed by G_i. Since g_{i-1} normalizes G_i, the image of any G_i-fixed point under g_{i-1} is also fixed by G_i. By the definition of y, it is the point that the arc $[z, g_{i-1}(x)]$ branching away from the arc $[z,x]$. Now that $z, x, g_{i-1}(x)$ are fixed by G_i, so is y. Further, each $g_{i-1}^k(w)$ is fixed by G_i and hence u is fixed by G_i.

By induction, there is some point in $X \setminus \{z\}$ fixed by G. □

The following theorem is due to Duchesne-Monod ([10]).

Theorem 2.3. *Let Γ be a higher rank lattice acting on a dendrite X. Then either Γ has a fixed point or has an invariant arc.*

2.3. **Ordering relations on groups.** Recall that a binary relation $\preceq$ on a set X is a *pre-order* if it satisfies that

(O1) $x \preceq x$ for any $x \in X$;
(O2) if $x \preceq y$ and $y \preceq z$ then $x \preceq z$ for any $x, y, z \in X$.

If, in addition, $\preceq$ satisfies that

(O3) if $x \preceq y$ and $y \preceq x$ then $x = y$ for any $x, y \in X$,

then $\preceq$ is a *partial order* on X.

A preorder (resp. partial order) on X is called a *total preorder* (resp. *total order*) on X if

(O4) any $x, y \in X$ are comparable; that is either $x \preceq y$ or $y \preceq x$.

Let G be a group with e being the unit. A preorder/ partial order/ total order $\preceq$ on G is said to be *left-invariant* if

(O5) for any $x, y \in X$ and $g \in G$, $gx \preceq gy$ whenever $x \preceq y$.

Finally, we say $\preceq$ is a *semilinear left preorder* (resp. *semilinear left partial order*) if it satisfies (O1), (O2), (O5) (resp. (O1),(O2),(O3),(O5)) and

(O6) for any $x \in G$, any two elements of $\{y \in G : y \preceq x\}$ are comparable;
(O7) for any $x, y \in G$, there is some $z \in G$ with $z \preceq x$ and $z \preceq y$.

2.4. **Topologies on left-ordering spaces.** We recall the topology on the ordering space of a countable group introduced by Sikora in [28]. Let Γ be a countable group and let $\Delta = \{(\gamma,\gamma) : \gamma \in \Gamma\}$. A total order relation $\prec$ on Γ corresponds to a unique point $\varphi \in \{-1,1\}^{\Gamma\times\Gamma\setminus\Delta}$ satisfying

(R) (Reflexivity) $\varphi(g,h) = -\varphi(h,g)$;
(T) (Transitivity) if $\varphi(f,g) = \varphi(g,h) = 1$, then $\varphi(f,h) = 1$;

by setting $\varphi(g,h) = 1$ whenever $g \succ h$. Then the set $\mathscr{O}(\Gamma)$ of total orders on Γ corresponds to a subset of $\{-1,1\}^{\Gamma\times\Gamma\setminus\Delta}$. Taking the discrete topology on $\{-1,1\}$ and endowing $\{-1,1\}^{\Gamma\times\Gamma\setminus\Delta}$ with the product topology, the subset consisting of φ satisfying (R) and (T) is closed; this leads to a compact metrizable topology on $\mathscr{O}(\Gamma)$. Furthermore, for $\varphi \in \mathscr{O}(\Gamma)$, if it satisfies additionally that

(L) (Left-invariance) $\varphi(fg, fh) = \varphi(g,h)$ for any $f,g,h \in \Gamma$ with $g \neq h$,

then the ordering $\prec$ corresponding to φ is left invariant, which is called a *left-ordering* on Γ. According to the condition (L), the space $\mathscr{LO}(\Gamma)$ of left-orderings on Γ forms a closed subspace of $\mathscr{O}(\Gamma)$. Clearly, Γ is left-orderable if and only if $\mathscr{LO}(\Gamma) \neq \emptyset$.

2.5. **Dynamical realizations.** The following proposition is a dynamical characterization of left-orderability (see e.g. [13, 23]).

Proposition 2.4. *Let Γ be a countable group. Then Γ is left-orderable if and only if it admits a faithful action on the real line $\mathbb{R}$ by orientation-preserving homeomorphisms.*

The proof of the necessity part of Proposition 2.4 uses the dynamical realization technique, which will be used later. So, we outline the construction process here.

Suppose Γ is a left-orderable group with a left-ordering $\preceq$. We enumerate Γ as $\{g_1, g_2, \ldots\}$. Define a map $t : \Gamma \to \mathbb{R}$ by the induction process: let $t(g_1) = 0$ and suppose that $t(g_1), \ldots, t(g_n)$ have been defined; if g_{n+1} is greater than (resp. smaller than) $t(g_1), \ldots, t(g_n)$, then let $t(g_{n+1}) = \max\{t(g_1), \ldots, t(g_n)\} + 1$ (resp. $\min\{t(g_1), \ldots, t(g_n)\} - 1$); if g_{n+1} lies between g_i, g_j and g_i, g_j are adjacent for some $i \neq j \in \{1, \ldots, n\}$, then let $t(g_{n+1}) = (t(g_i) + t(g_j))/2$. For $g \in \Gamma$, define the action of g on $t(\Gamma)$ by letting $gt(g') = t(gg')$ for each $g' \in G$; then extend this action to the closure $\overline{t(\Gamma)}$ and extend further to the whole line by mapping affinely on the maximal intervals in $\mathbb{R} \setminus \overline{t(\Gamma)}$. Thus we obtain an orientation-preserving faithful action of Γ on $\mathbb{R}$. This construction process is called the *dynamical realization*.

2.6. **Some facts on groups.**

Lemma 2.5. *Let H be a subgroup of G of finite index. Then there is a normal subgroup K of G that is contained in H and has finite index in G. Further, if $[G:H] \leq k$ for some $k > 0$, then we can assume $[G:K] \leq k!$.*

Proof. Assume that the index of H in G is $n > 0$. Then the canonical action of G on the coset space G/H induces a homomorphism $\phi : G \to \mathrm{Sym}(n)$, where $\mathrm{Sym}(n)$ denotes the permutation group of n elements. Then $\ker(\phi) \leq H$ and $\ker(\phi) \trianglelefteq G$. Further, $[G : \ker(\phi)] \leq n! \leq k!$. Thus $\ker(\phi)$ is just what we are looking for. □

Lemma 2.6. [8, Proposition 5.11] *Let G be a finitely generated group. For any $n \in \mathbb{N}$, there are finitely many subgroups of G with indices less than n.*

The following is the well-known Margulis' Normal Subgroup Theorem (see e.g. [19, Chapter IV] or [32, Theorem 8.1.2]).

Theorem 2.7. *Let G be a connected real semisimple Lie group with finite center and no compact factors, let Γ be an irreducible lattice of G. Assume that $\mathbb{R}$-rank$(G) \geq 2$. Then every normal subgroup of Γ either is contained in the center of G and hence is finite or has finite index in Γ.*

The following theorem is due to Deroin-Hurtado ([7]).

Theorem 2.8. *No higher rank lattice is left-orderable. (This is equivalent to saying that every orientation-preserving action on $[0,1]$ by a higher rank lattice is trivial.)*

Finally, note that higher rank lattices are finitely generated since they have Kazhdan's Property (T) (see [16, Proposition 5.7]).

3. Left-orderability and group actions on dendrites

In this section, we will give a characterization of the left-orderability for a group Γ through its actions on dendrites, by which we give the proof of Theorem 1.1.

3.1. **Local conditions for left-orderability.** Let Γ be a countable group and B be a nonempty subset of Γ. If $\varphi \in \{-1,1\}^{\Gamma\times\Gamma\setminus\Delta}$ satisfying

(R_B) $\varphi(g,h) = -\varphi(h,g)$, for any $g,h \in B$ with $g \neq h$;

(T_B) if $\varphi(f,g) = \varphi(g,h) = 1$, then $\varphi(f,h) = 1$, for any $f,g,h \in B$ with $f \neq g, f \neq h, g \neq h$;

then φ defines a total ordering $\preceq_\varphi$ on B by setting $g \succ_\varphi h$ if and only if $\varphi(g,h) = 1$ for any $g \neq h \in B$. Then the set

$$\mathscr{O}(\Gamma;B) := \left\{\varphi \in \{-1,1\}^{\Gamma\times\Gamma\setminus\Delta} : \varphi \text{ satisfies } R_B, T_B\right\}$$

is closed in $\{-1,1\}^{\Gamma\times\Gamma\setminus\Delta}$ with respect to the topology given in Section 2.4. If $(B_n)_{n=1}^\infty$ is an increasing sequence of subsets of Γ with $\Gamma = \bigcup_{n=1}^\infty B_n$, then $\bigcap_{n=1}^\infty \mathscr{O}(\Gamma;B_n) = \mathscr{O}(\Gamma)$.

Now, let F,B,B' be nonempty subsets of Γ with $B \subset B'$ and $FB \subset B'$, where $FB = \{fb : f \in F, b \in B\}$. For $\varphi \in \mathscr{O}(\Gamma;B')$, we say φ is (F,B)-*invariant*, if it satisfies

($\mathrm{L}_{F,B}$) $\varphi(fg,fh) = \varphi(g,h)$ for any $f \in F$ and $g \neq h \in B$.

Then the set

$$\mathscr{L}_F\mathscr{O}(\Gamma;B,B') := \left\{\varphi \in \{-1,1\}^{\Gamma\times\Gamma\setminus\Delta} : \varphi \text{ satisfies } R_{B'}, T_{B'}, L_{F,B}\right\}$$

is also closed in $\{-1,1\}^{\Gamma\times\Gamma\setminus\Delta}$.

Let $\mathscr{L}_F\mathscr{O}(\Gamma)$ denote the set of total orderings on Γ that are invariant under the left translations of F, i.e.,

$$\mathscr{L}_F\mathscr{O}(\Gamma) = \left\{\varphi \in \{-1,1\}^{\Gamma\times\Gamma\setminus\Delta} : \varphi \text{ satisfies } R, T, L_{F,\Gamma}\right\},$$

where the conditions R and T are as in Section 2.4. It is clear that $\mathscr{L}_F\mathscr{O}(\Gamma)$ is also closed in $\{-1,1\}^{\Gamma\times\Gamma\setminus\Delta}$ and

$$\mathscr{L}\mathscr{O}(\Gamma) = \bigcap\{\mathscr{L}_F\mathscr{O}(\Gamma) : F \text{ is a finite subset of } \Gamma\}.$$

If $(B_n)_{n=1}^{\infty}$ is a sequence of subsets of Γ satisfying $FB_n \cup B_n \subset B_{n+1}$ for each $n \geq 1$ and $\Gamma = \bigcup_{n=1}^{\infty} B_n$, then

$$\mathscr{L}_F\mathscr{O}(\Gamma) = \bigcap_{n=1}^{\infty} \mathscr{L}_F\mathscr{O}(\Gamma; B_n, B_{n+1}).$$

According to the above discussion, we have

Lemma 3.1. *Let Γ be a countable group. Suppose F is a nonempty finite subset of Γ and $(B_n)_{n=1}^{\infty}$ is a sequence of subsets of Γ satisfying $FB_n \cup B_n \subset B_{n+1}$ for each $n \geq 1$ and $\Gamma = \bigcup_{n=1}^{\infty} B_n$. If for each $n \geq 1$, there is a total ordering $\preceq_n$ on B_{n+1} that is (F, B_n)-invariant, i.e., $fg \preceq_n fh$ whenever $g \preceq_n h$ for any $f \in F$ and $g, h \in B_n$, then there is an F-invariant total ordering on Γ. Further, if, for each finite subset E of Γ, there is an E-invariant total ordering on Γ, then Γ is left-orderable.*

3.2. **Characterizations of left-orderability via actions on dendrites.** In this section, we will give an equivalent characterization of left-orderability for a finitely generated group through its actions on dendrites. Then we complete the proof of the first main theorem by using this characterization.

Proposition 3.2. *Let Γ be a finitely generated group. Then Γ is left-orderable if and only if it admits an almost free action on a nondegenerate dendrite with an end point fixed.*

Proof. ($\Longrightarrow$) Let $\preceq$ be a left-ordering on Γ and numerate Γ as $\{g_1, g_2, g_3, \ldots\}$. Then by the dynamical realization as in Section 2.5, we get an orientation-preserving faithful action of Γ on $\mathbb{R}$. Extending this action to the two points compactification $I := \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$ by letting $-\infty$ and $+\infty$ fixed by each $g \in \Gamma$, we get an action of Γ on the arc I.

We claim that this extended action on I is almost free. Otherwise, there is some $g \neq e \in \Gamma$ with the fixed point set Fix(g) of g not totally disconnected; so, it contains a nondegenerate arc J. By the definition of dynamical realization, we see that there is some maximal open interval (a, b) of $\mathbb{R} \setminus \overline{t(\Gamma)}$ with $J \subset [a, b]$, where t is as in Section 2.5. Since g fixes a and b, $\{a, b\} \cap t(\Gamma) = \emptyset$. Thus a and b are accumulation points of $t(\Gamma)$. Take $g', g'' \in \Gamma$ with $0 < a - t(g') < (b-a)/3$ and $0 < t(g'') - b < (b-a)/3$. Suppose $g' = g_m$ and $g'' = g_n$ for some indices m, n. Let $k = \max\{n, m\}$ and let $n', m' \in \{1, 2, \ldots, k\}$ be such that $t(g_{m'})$ is maximal in $\{t(g_1), t(g_2), \ldots, t(g_k)\} \cap (-\infty, a)$ and $t(g_{n'})$ is minimal in $\{t(g_1), t(g_2), \ldots, t(g_k)\} \cap (b, +\infty)$ respectively. Let i be the first index so that $i > \max\{m', n'\}$ and $g_{m'} \prec g_i \prec g_{n'}$. Then $t(g_i) = (t(g_{m'}) + t(g_{n'}))/2 \in (a, b)$, which is a contradiction.

Thus the action of Γ on the arc I is almost free and I is also a nondegenerate dendrite with an end point fixed by each $g \in \Gamma$.

($\Longleftarrow$) Let $z \in X$ be an end point of X fixed by Γ. Let $\{g_1^{\pm}, \ldots, g_k^{\pm}\}$ be a set of generators for Γ. By Lemma 2.1, for each $i \in \{1, \ldots, k\}$, there is a point $t_i \in X \setminus \{z\}$ fixed by g_i. Let T be the smallest subcontinuum of X containing $\{z, t_1, \ldots, t_n\}$, which is a subtree of X. Let I denote the intersection of all arcs $[z, t_i]$, $i = 1, \ldots, k$. Since z is an end point, the arc I is not reduced to a point. We write $I = [z, t]$ and give a canonical ordering $<$ on $[z, t]$ with $t > z$.

Fix a finite subset F of Γ. Choose a sequence $(B_n)_{n=1}^{\infty}$ of finite subsets of Γ satisfying $FB_n \cup B_n \subset B_{n+1}$ for each $n \geq 1$ and $\Gamma = \cup_{n=1}^{\infty} B_n$. It is clear that such sequence exists.

Given $n \geq 1$, there is a point $s \in (z,t)$ such that $g([z,s]) \subset [z,t)$ for each $g \in B_{n+1}$. Now choose a dense sequence $(x_i)_{i=1}^{\infty}$ in (z,s). For each pair of two distinct $g,h \in B_{n+1}$, define $g \prec h$ if the smallest $j \geq 1$ for which $g(x_j) \neq h(x_j)$ is such that $g(x_j) < h(x_j)$ with respect to the canonical ordering $<$ on $[z,t]$. Indeed, such j exists by the almost freeness of the Γ-action. It is easy to verify that $\prec$ is a total ordering defined on B_{n+1}. Note that for each $g \in B_{n+1}$, the restriction of f to (z,s) is increasing with respect to $<$ on $[z,t]$. Thus for every $f \in F$ and $g,h \in B_n$, we have $fg \prec fh$ whenever $g \prec h$. Hence the ordering $\preceq$ on B_{n+1} is (F,B_n)-invariant.

According to Lemma 3.1, we conclude that G is left-orderable. □

3.3. **Proof of Theorem 1.1.** Now, we are ready to prove the first main theorem of the paper. We need the following alternative for the group actions on dendrites.

Lemma 3.3. [10, 18, 25] *Let G be a countable group acting on a dendrite X. Then either G preserves an arc (which may degenerate to a singleton) or G contains a non-abelian free subgroup.*

Proof of Theorem 1.1. Assume to the contrary that the action (X,Γ) is almost free. According to Lemma 3.3, Γ either has a fixed point or preserves a nondegenerate arc, since Γ is a small group.

We discuss it into two cases:

Case 1. Γ preserves a nondegenerate arc I. Fix an orientation on I. Then there is a subgroup Γ' of Γ with index at most two such that the restriction of Γ' to I preserves the orientation of I. Let $\phi : \Gamma' \to \mathrm{Homeo}_+(I)$ be the restriction of the Γ'-action on I, where $\mathrm{Homeo}_+(I)$ be the orientation preserving homeomorphism group of I. If the ϕ is faithful then Γ' is left-orderable by Proposition 2.4, which is a contradiction. Otherwise, the kernel $\ker(\phi)$ is nontrivial and every element in $\ker(\phi)$ fixes I pointwise. This contradicts the almost freeness of the action (X,Γ).

Case 2. Γ has a fixed point $z \in X$. Since the order of z is finite, there are finitely many connected components of $X \setminus \{z\}$. Fix a connected component C of $X \setminus \{z\}$. Let Γ'' be the subgroup of Γ that preserves C, i.e.

$$\Gamma'' = \{\gamma \in \Gamma : \gamma(C) = C\}.$$

Then Γ'' has finite index in Γ. Consider the restriction action of Γ'' to $Y = \overline{C} = C \cup \{z\}$. Noting that z is an end point of Y fixed by Γ'' and Γ'' is finitely generated, Γ'' is left-orderable by Proposition 3.2. This contradicts our assumption that Γ is not virtually left-orderable. □

Here, we give some remarks on the virtual left-orderability condition in Theorem 1.1. One may wonder that whether there are finitely generated small groups that are not virtually left-orderable. The answer is yes. Indeed, the Tarski monster group is such an example, which is an infinite simple group and every element of which is a torsion. However, we do not know whether there exist such examples of torsion-free. We should note that a finitely generated torsion-free solvable group is virtually left-orderable. This can be seen as follows. On the one hand, a solvable group G is left-orderable if and only if it is *locally indicable*, which means that every nontrivial finitely generated subgroup of G has a homomorphism onto $\mathbb{Z}$ ([6, Theorem A]). On the other hand, every finitely generated solvable group is either finite or has a finite index subgroup that has a homomorphism onto $\mathbb{Z}$ ([2, Lemma 5.3]). Combining these two aspects, if a finitely generated solvable group is torsion-free, then it has a finite index subgroup that is left-orderable.

4. Proof of Theorem 1.3

In this section, we prove Theorem 1.3 assuming Proposition 1.4. We need the following simple lemma.

Lemma 4.1. *Let Γ be a higher rank lattice acting on a dendrite X. Then the fixed point set $\mathrm{Fix}(\Gamma)$ of Γ is a subdendrite of X.*

Proof. We first claim that $\mathrm{Fix}(\Gamma) \neq \emptyset$. If not, Γ leaves an arc I invariant by Theorem 2.3. Then there is $\gamma \in \Gamma$ and a subgroup $\Gamma' \leq \Gamma$ such $\Gamma = \Gamma' \cup \gamma\Gamma'$ and Γ' preserves the orientation of I. By Theorem 2.8, the restriction action of Γ' to I is trivial. Note that γ has a fixed point p in I, which is then a fixed point of Γ. Thus the claim holds.

Now we show that $\mathrm{Fix}(\Gamma)$ is connected. Let p, q be two points in $\mathrm{Fix}(\Gamma)$. Then the arc $[p,q]$ is preserved by Γ. It follows from Theorem 2.3 that $[p,q] \subset \mathrm{Fix}(\Gamma)$. Thus $\mathrm{Fix}(\Gamma)$ is connected.

Clearly, $\mathrm{Fix}(\Gamma)$ is closed. Therefore, $\mathrm{Fix}(\Gamma)$ is a subdendrite of X. □

Proof of Theorem 1.3. In the following, we will define, for each $i = 0, 1, 2, \ldots$, a subdendrite X_i of X and a normal subgroup Γ_i of Γ with finite index, satisfying that:

(1) $\{p\} = X_0 \subset X_1 \subset X_2 \subset \cdots$ with $p \in \mathrm{Fix}(\Gamma)$, and $\Gamma = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \cdots$;
(2) For each $i \geq 1$, $X_i = \mathrm{Fix}(\Gamma_i)$ that is a nondegenerate Γ-invariant subdendrite of X, and $(X_i, \Gamma|X_i)$ is a finite action;
(3) The first point map $\phi_i : X_{i+1} \to X_i$ is a factor map from $(X_{i+1}, \Gamma|X_{i+1})$ to $(X_i, \Gamma|X_i)$;
(4) For each $i \geq 0$, the image of $X \setminus X_{i+1}$ under the first point map $r_{i+1} : X \to X_{i+1}$ is contained in $X_{i+1} \setminus \{x \in X_i : \mathrm{ord}_X(x) \leq i+1\}$.

First, according to Lemma 4.1, we can take a fixed point p of Γ. Let $X_0 = \{p\}$ and let $\Gamma_0 = \Gamma$; then X_0 is Γ_0-invariant. Since the order of p is finite, the number of connected components of $X \setminus \{p\}$ is finte; thus there is a finite index normal subgroup Γ_1 of Γ which leaves each component of $X \setminus \{p\}$ invariant. For each such component C, applying Proposition 1.4 to $(C \cup \{p\}, \Gamma_1 | C \cup \{p\})$, there is a point $s_C \in C$ such that the arc $[p, s_C]$ is fixed pointwise by Γ_1 (note that p is an end point of $C \cup \{p\}$).

Let $X_1 = \mathrm{Fix}(\Gamma_1)$. Then, by Lemma 4.1, X_1 is a nondegenerate subdendrite of X. Since Γ_1 is normal in Γ, X_1 is Γ-invariant. Thus the action of Γ on X_1 factors through Γ/Γ_1 action, which is finite. It is clear that the first point map $\phi_0 : X_1 \to X_0$ satisfies $\phi_0(gx) = g\phi_0(x) = p$ for each $g \in G$ and $x \in X_1$.

Now suppose that we have defined $X_0, \ldots, X_m$ and $\Gamma = \Gamma_0 \trianglerighteq \Gamma_1 \trianglerighteq \cdots \trianglerighteq \Gamma_m$, for some positive integer m, satisfying (1)-(4) above. Let $B_m = \{x \in X_m : \mathrm{ord}_X(x) \leq m+1\}$ and let $r_m : X \to X_m$ be the first point map. Since X_m is a nondegenerate subdendrite of X, there must be some point $x \in X_m$ with $\mathrm{ord}_X(x) = 2$ and hence $B_m \neq \emptyset$. For each point $x \in B_m$, the preimage $Y_x := r_m^{-1}(\{x\})$ is a sub-dendrite of X and has a unique intersecting point x with X_m. By assumption, every $x \in X_m$ is fixed by Γ_m. For each $x \in B_m$, the cardinality of the set $\mathrm{Comp}(Y_x - x)$ of components of $Y_x \setminus \{x\}$ is no greater than $m+1$. Thus there is a finite index subgroup $\Gamma_{m,x}$ of Γ_m such that $\Gamma_{m,x}$ leaves each component of $Y_x \setminus \{x\}$ invariant and $[\Gamma_m : \Gamma_{m,x}] \leq m+1$. By Lemma 2.5, for each $x \in B_m$, there is a subgroup $\Gamma'_{m,x} \leq \Gamma_{m,x}$ which is normal in Γ_m and $[\Gamma_m : \Gamma'_{m,x}] \leq (m+1)!$. Applying Lemma 2.6, the subgroup $\Gamma_{m+1} = \bigcap_{x\in B_m} \Gamma'_{m,x} \leq \Gamma_m$ has finite index in Γ and is normal in Γ.

By the definition of Γ_{m+1}, X_m is contained in the fixed point set $\mathrm{Fix}(\Gamma_{m+1})$ of Γ_{m+1}. Let X_{m+1} be the connected component of $\mathrm{Fix}(\Gamma_{m+1})$ that contains X_m. Since $p \in X_{m+1}$ is fixed by Γ and Γ_{m+1} is normal in Γ, we have that Γ leaves X_{m+1} invariant. By Proposition 1.4, X_{m+1} satisfies (4). It is clear that (1)-(3) hold for X_{m+1}.

In such way, we inductively define the desired sequence $X_0 \subset X_1 \subset \cdots$ and $\Gamma = \Gamma_0 \trianglerighteq \Gamma_1 \trianglerighteq \cdots$ satisfying (1)-(4). Let $Y = \overline{\cup_{i=0}^{\infty} X_i}$. Then Y is a Γ-invariant dendrite and topologically conjugate to the inverse limit $(\underleftarrow{\lim}(X_i,\Gamma),\Gamma))$ by [22, Theorem 10.36]. Since each $(X_i, \Gamma|X_i)$ is a finite action, (Y,Γ) is almost finite.

It remains to show that the first point map $r : X \to Y$ is a highly proximal extension. It is obvious in case of $Y = X$. So we suppose that $Y \neq X$.

Claim 1. For each $m \geq 1$ and $x \in X_{m+1} \setminus X_m$, there are two distinct connected components $C, C'$of $X \setminus \{r_m(x)\}$ and $g \in \Gamma_m$ such that $x \in C, g(x) \in C'$ and $C \cup C' \subset X \setminus X_m$, where $r_m : X \to X_m$ is the first point map.

Proof of Claim 1. Since $x \in X_{m+1} \setminus X_m$ and $X_m = \mathrm{Fix}(\Gamma_m)$, we have $[x, r_m(x)) \cap \mathrm{Fix}(\Gamma_m) = \emptyset$. This together with Proposition 1.4 implies that there exists $g \in \Gamma_m$ such that $g[x, r_m(x)) \cap [x, r_m(x)) = \emptyset$. Let C and C' be the components of $X \setminus \{r_m(x)\}$ containing x and $g(x)$ respectively. Clearly, $C \neq C'$ and $C \cup C' \subset X \setminus X_m$, since $X_m = \mathrm{Fix}(\Gamma_m)$.

Claim 2. For any $x \in X \setminus Y$, the orbit of $r(x)$ is infinite.

Proof of Claim 2. By the definition of X_m and Proposition 1.4, we see that $r(x) \in Y \setminus \cup_{i=0}^{\infty} X_i$. From Property (4), there is a subsequence (m_j) with $r_{m_{j+1}}(x) \in X_{m_{j+1}} \setminus X_{m_j}$, where $r_{m_{j+1}} : X \to X_{m_{j+1}}$ is the first point map. It follows from Claim 1 that there are $g_{m_j} \in \Gamma_{m_j}$ and $C_{m_j} \neq C'_{m_j} \in \mathrm{Comp}(X \setminus \{r_{m_j}(x)\})$ such that $r_{m_{j+1}}(x) \in C_{m_j}$ and $g_{m_j}(r_{m_{j+1}}(x)) \in C'_{m_j}$. Note that $C_{m_1} \supset C_{m_2} \supset C_{m_3} \supset \cdots$ and $C'_{m_j} \subset C_{m_{j-1}}$ for each j. Thus we have

$$\begin{gathered} r_{m_{j+1}}(x) \in C_{m_j}, \\ g_{m_j} r_{m_{j+1}}(x) \in C'_{m_j} \subset C_{m_{j-1}}, \\ g_{m_{j-1}} g_{m_j} r_{m_{j+1}}(x) \in C'_{m_{j-1}} \subset C_{m_{j-2}}, \\ \cdots\cdots\cdots \\ g_{m_1} g_{m_2} \cdots g_{m_j} r_{m_{j+1}}(x) \in C'_{m_1}, \end{gathered}$$

which imply that the arcs

$$[r(x), r_{m_{j+1}}(x)],\ g_{m_j}[r(x), r_{m_{j+1}}(x)], \ldots, g_{m_1} g_{m_2} \ldots g_{m_j}[r(x), r_{m_{j+1}}(x)]$$

are mutually disjoint. In particular,

$$r(x), g_{m_j} r(x), g_{m_{j-1}} g_{m_j} r(x), \ldots, g_{m_1} g_{m_2} \cdots g_{m_j} r(x)$$

are pairwise distinct. By the arbitrariness of j, we see that the orbit of $r(x)$ is infinite.

Noting that any sequence of mutually disjoint subcontinua of a dendrite forms a null sequence (see [30, V.2.6]), it follows from Claim 2 that $r : X \to Y$ is highly proximal. □

5. Proof of Proposition 1.4 for finite index subgroups of $SL_n(\mathbb{Z})$

Let G be a countable group. Recall that a *left total preorder* $\preceq$ on G is a binary relation on G satisfying

(1) for any $g,h \in G$, either $g \preceq h$ or $h \preceq g$;
(2) for any $f,g,h \in G$, if $f \preceq g$ and $g \preceq h$, then $f \preceq h$;
(3) for any $f,g,h \in G$, if $g \preceq h$, then $fg \preceq fh$.

Let $\preceq$ be a left total preorder on G and $g,h \in G$. We say $g \ll h$ if either $g^k \preceq h$ for all $k \in \mathbb{Z}$ or $g^k \preceq h^{-1}$ for all $k \in \mathbb{Z}$. We write $g \succ h$ if $g \not\preceq h$. By (1), there is no element $g \in G$ satisfying $g \succ g$.

The following lemma is similar to Lemma 3.2 in [31].

Lemma 5.1. *Let $\preceq$ be a left total preorder on a group G. If $a,b,c \in G$ satisfies $[a,b] = a^{-1}b^{-1}ab = c^r$ for some $r \in \mathbb{Z} \setminus \{0\}$ and c commutes with both a and b, then either $c \ll a$ or $c \ll b$.*

Proof. By the definition of the relation $\ll$, for any $g,h \in G$, $g \ll h$ is equivalent to $g^{\pm 1} \ll h^{\pm 1}$. Thus we may assume that $e := e_G \preceq a,b,c$ and $r > 0$; and assume to the contrary that $c \not\ll a$ and $c \not\ll b$. Thus there are some $p,q \in \mathbb{Z}_+$ such that $c^p \succ a$ and $c^q \succ b$. According to the left invariance, we have

$$e \prec a^{-1}c^p, \quad e \prec b^{-1}c^q.$$

Noting that $e \preceq a, \quad e \preceq b, \quad e \preceq c$, we have for sufficiently large positive integer m:

$$\begin{aligned} e &\prec (b^{-1}c^q)^m(a^{-1}c^p)^m a^m b^m \\ &= [b^m, a^m]c^{m(p+q)} \\ &= c^{-m^2r+m(p+q)} \\ &\preceq e. \end{aligned}$$

This is a contradiction. □

Lemma 5.2. *Suppose that Γ is a finite index subgroup of $SL_3(\mathbb{Z})$. If Γ acts on a nondegenerate dendrite X and fixes an end point z, then there is a point $s \in X \setminus \{z\}$ such that the arc $[z,s]$ is fixed by Γ pointwise.*

Proof. Since Γ has finite index in $SL_3(\mathbb{Z})$, there is some positive integer r such that Γ contains the following six elements

$$a_1 = \begin{bmatrix} 1 & r & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 1 & 0 & r \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & r \\ 0 & 0 & 1 \end{bmatrix},$$

$$a_4 = \begin{bmatrix} 1 & 0 & 0 \\ r & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad a_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ r & 0 & 1 \end{bmatrix}, \quad a_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & r & 1 \end{bmatrix}.$$

Let $\Gamma_i = \langle a_{i-1}, a_i, a_{i+1} \rangle$ for each $i \in \mathbb{Z}/6\mathbb{Z}$. A straightforward verification shows that $[a_i, a_{i+1}] = e$ and $[a_{i-1}, a_{i+1}] = a_i^{\pm r}$ for each $i \in \mathbb{Z}/6\mathbb{Z}$. Now, by Lemma 2.2, there is a point s_i different from z that is fixed by Γ_i for each $i \in \mathbb{Z}/6\mathbb{Z}$.

For each $i \in \mathbb{Z}/6\mathbb{Z}$, fix a linear order $\leq_{[z,s_i]}$ on $[z,s_i]$ with $z < s_i$. For every $x \in (z,s_i)$, define a left total preorder $\preceq_x^{(i)}$ on Γ_i by setting

$$\gamma_1 \preceq_x^{(i)} \gamma_2 \text{ if and only if } \gamma_1(x) \leq_{[z,s_i]} \gamma_2(x), \text{ for any } \gamma_1, \gamma_2 \in \Gamma_i.$$

Set $I = \cap_{i=1}^{6}[z,s_i)$ and take a point $y \neq z \in I$ such that $a_j^{\pm 1}[z,y] \subset I$ for each $j \in \{1,\cdots,6\}$. By Lemma 5.1, for each i,

$$\text{either } a_i \ll_y^{(i)} a_{i-1} \text{ or } a_i \ll_y^{(i)} a_{i+1}.$$

Claim. $a_i(y) = y$ for each $i \in \{1,\cdots,6\}$.

Proof of the Claim. To the contrary, we may assume that $a_1(y) \neq y$. Note that either $a_1 \ll_y^{(1)} a_6$ or $a_1 \ll_y^{(1)} a_2$. We discuss into two cases.

Case 1. $a_1 \ll_y^{(1)} a_2$. Then, by definition, either $a_1^k \preceq_y^{(1)} a_2$ for all $k \in \mathbb{Z}$ or $a_1^k \preceq_y^{(1)} a_2^{-1}$ for all $k \in \mathbb{Z}$. In either case, we have $a_1^k(y) \leq_{[z,s_1]} \max\{a_2(y), a_2^{-1}(y)\}$, for all $k \in \mathbb{Z}$. If $a_2(y) = y$ then $a_1(y) = y$ as well; this contradicts the assumption. Thus we have that $a_2(y) \neq y$ and hence $a_2(y) \neq a_2^{-1}(y)$. So $a_2 \not\ll_y^{(2)} a_1$. Further we have that $a_2 \ll_y^{(2)} a_3$. Similarly, if $a_3(y) = y$ then $a_2(y) = y$, which implies that $a_1(y) = y$; this is a contradiction. Thus $a_3(y) \neq y$ and then $a_3 \not\ll_y^{(3)} a_2$. Inductively, we have

$$a_1 \ll_y^{(1)} a_2 \ll_y^{(2)} a_3 \ll_y^{(3)} a_4 \ll_y^{(4)} a_5 \ll_y^{(5)} a_6 \ll_y^{(6)} a_1.$$

By the definitions of these quasi-orders and the choice of y, we have

$$\begin{aligned}\sup_{k\in\mathbb{Z}} a_1^k(y) &\leq_I \max\{a_2(y), a_2^{-1}(y)\} \leq_I \max\{a_3(y), a_3^{-1}(y)\} \leq_I \max\{a_4(y), a_4^{-1}(y)\} \\ &\leq_I \max\{a_5(y), a_5^{-1}(y)\} \leq_I \max\{a_6(y), a_6^{-1}(y)\} \leq_I \max\{a_1(y), a_1^{-1}(y)\},\end{aligned}$$

where $\leq_I$ is the natural linear order on I with respect to which z is minimal. Thus we have $a_1(y) = y$, which contradicts our assumption. So the claim holds in this case.

Case 2. $a_1 \ll_y^{(1)} a_6$. Similar to Case 1, we have

$$a_1 \ll_y^{(1)} a_6 \ll_y^{(6)} a_5 \ll_y^{(5)} a_4 \ll_y^{(4)} a_3 \ll_y^{(3)} a_2 \ll_y^{(2)} a_1,$$

and hence

$$\begin{aligned}&\sup_{k\in\mathbb{Z}} a_1^k(y) \leq_I \max\{a_6(y), a_6^{-1}(y)\} \leq_I \max\{a_5(y), a_5^{-1}(y)\} \leq_I \max\{a_4(y), a_4^{-1}(y)\} \\ &\leq_I \max\{a_3(y), a_3^{-1}(y)\} \leq_I \max\{a_2(y), a_2^{-1}(y)\} \leq_I \max\{a_1(y), a_1^{-1}(y)\}.\end{aligned}$$

We also have $a_1(y) = y$ and the claim holds in this case.

Let $\Gamma' = \langle a_1, a_2, a_3, a_4, a_5, a_6\rangle$. By a result of Tits in [29] (also refer to [20]), we know that Γ' has finite index in Γ. Note that the claim holds for any $y \in I$ with $a_j^{\pm 1}[z,y] \subset I$ for each $j \in \{1,\cdots,6\}$. Thus there is a $t \in X \setminus \{z\}$ such that the arc $[z,t]$ is fixed by Γ' pointwise. Now the set $\Gamma[z,t] = \{\gamma[z,t] : \gamma \in \Gamma\}$ consists of finitely many arcs. Since z is an end point fixed by Γ, there is some $s \in X \setminus \{z\}$ with $[z,s] = \cap\Gamma[z,t]$; and $[z,s]$ is then fixed by Γ pointwise. □

Now we can prove Proposition 1.4 for a class of special lattices. We restate it as follow.

Proposition 5.3. *Suppose that Γ is a finite index subgroup of $SL_n(\mathbb{Z})$ with $n \geq 3$. If Γ acts on a nondegenerate dendrite X and fixes an end point z, then there is a point $s \in X \setminus \{z\}$ such that the arc $[z,s]$ is fixed by Γ pointwise.*

Proof. Let $u_{i,j}$ be the matrix in $SL_n(\mathbb{Z})$ with 1's along the diagonal and at the entry (i,j) and 0's elsewhere. Since Γ has finite index in $SL_n(\mathbb{Z})$, there is an $\ell \in \mathbb{Z}_+$ such that $u_{i,j}^\ell \in \Gamma$ for each $i,j \in \{1,\cdots,n\}, i \neq j$. Given $1 \leq i < j \leq n-1$, let

$$a_1 = u_{i,j}^\ell,\ \ a_2 = u_{i,j+1}^\ell,\ \ a_3 = u_{j,j+1}^\ell,\ \ a_4 = u_{j,i}^\ell,\ \ a_5 = u_{j+1,i}^\ell,\ \ a_6 = u_{j+1,j}^\ell.$$

A straightforward check shows that they also satisfy that $[a_i, a_{i+1}] = e$ and $[a_{i-1}, a_{i+1}] = a_i^{\pm r}$ for each $i \in \mathbb{Z}/6\mathbb{Z}$. Applying the proof of Lemma 5.2 to the group $\Gamma_{i,j} = \langle a_1, \cdots, a_6\rangle$, there is some $s_{i,j} \in X \setminus \{z\}$ such that $\Gamma_{i,j}$ fixes the arc $[z, s_{i,j}]$ pointwise. Let $t \in X \setminus \{z\}$ be such that $[z,t] = \bigcap_{1 \leq i<j\leq n-1}[z, s_{i,j}]$. Then the arc $[z,t]$ is fixed pointwise by $\Gamma_{i,j}$, for each $1 \leq i < j \leq n-1$. Now let $\Gamma' = \langle u_{i,j}^\ell : 1 \leq i,j \leq n, i \neq j\rangle$. Then $[z,t]$ is fixed by Γ' pointwise, by noting that $\Gamma' = \langle \Gamma_{i,j} : 1 \leq i < j \leq n-1\rangle$. Recall that Γ' has finite index in $SL_n(\mathbb{Z})$ and hence in Γ (see [29] or [20]). Thus the set $\Gamma[z,t] = \{\gamma[z,t] : \gamma \in \Gamma\}$ consists of finitely many arcs. Since z is an end point fixed by Γ, there is some $s \in X \setminus \{z\}$ such that $[z,s] = \cap\Gamma[z,t]$ and $[z,s]$ is fixed by Γ pointwise. □

6. Proof of Proposition 1.4 for general case

6.1. **Some basic facts of semilinear left preorder.** The following characterization of semilinear left preorder is similar to [17, Theorem 1.1]. Recall the definition of semilinear left preorder is given in section 2.3.

Proposition 6.1. *Let $\preceq$ be a semilinear left preorder on a group G. Then the positive cone $P := \{g \in G : e \preceq g\}$ of $\preceq$ has the following properties:*

- (P1) *P is a semigroup;*
- (P2) *$P \cap P^{-1}$ is a subgroup of G;*
- (P3) *$G = P^{-1} \cdot P$;*
- (P4) *$P \cdot P^{-1} \subseteq P \cup P^{-1}$.*

If $\preceq$ is further a partial order, then

- (P2') *$P \cap P^{-1} = \{e\}$;*

is further a total preorder, then

- (P5) *$G = P \cup P^{-1}$.*

Conversely, a subset P of G satisfying (P1)-(P4) (resp. (P1)(P2')(P3)(P4)) determines a semilinear left preorder (resp. semilinear left partial order) on G.

Proof. ($\Longrightarrow$) (P1) and (P2) are direct from the definition of semilinear left preorder.

To prove (P3), let $g \in G$. Take $u \in G$ with $u \preceq e$ and $u \preceq g$ by (O7). Then $g = u(u^{-1}g)$ and $e \preceq u^{-1}g$. So, $G \subset P^{-1} \cdot P$. The inclusion $P^{-1} \cdot P \subset G$ is clear.

To prove (P4), let $x, y \in P$. Then $x^{-1} \preceq e$ and $y^{-1} \preceq e$. From (O6), either $x^{-1} \preceq y^{-1}$ or $y^{-1} \preceq x^{-1}$, which implies $xy^{-1} \in P \cup P^{-1}$.

(P2') and (P5) are clear.

($\Longleftarrow$) Define $x \preceq y$ if $x^{-1}y \in P$. We only check (O6) and (O7), the others are direct.

For (O6), let $x,y,z \in G$ be such that $x \preceq z$ and $y \preceq z$. Then $x^{-1}z \in P$ and $z^{-1}y \in P^{-1}$. By (P4), $x^{-1}y = x^{-1}z \cdot z^{-1}y \in P \cup P^{-1}$. This means either $x \preceq y$ or $y \preceq x$.

For (O7), let $x,y \in G$. By (P3), we have $x^{-1}y = a \cdot b$, where $a \in P^{-1}$ and $b \in P$. Then $a \preceq e$ and $a \preceq x^{-1}y$, which implies $xa \preceq x$ and $xa \preceq y$.

□

According to Proposition 6.1, we also say that a subsemigroup P of G satisfying (P1)-(P4) is a semilinear left preorder on G.

Lemma 6.2. *Let P be a semilinear left preorder on G.*

(1) *For any $q \in P$, we have $q(P \cup P^{-1})q^{-1} \subset P \cup P^{-1}$.*
(2) *For any $g \in G$, we have $\{x \in G : x \preceq g\} = gP^{-1}$.*
(3) *Let w be a nontrivial word composed by some elements of P. If w is in $P \cap P^{-1}$ then every letter occurring in w is in $P \cap P^{-1}$.*

Proof. (1) For any $x \in P$, we have $qxq^{-1} \in qPq^{-1} \subset P \cdot P^{-1} \subset P \cup P^{-1}$ by (P4). For any $x \in P^{-1}$, we have $xq^{-1} \in P^{-1}$. Thus $qP^{-1}q^{-1} \subset P \cdot P^{-1} \subset P \cup P^{-1}$.

(2) If $x \preceq g$ then $g^{-1}x \in P^{-1}$ and hence $x = g(g^{-1}x) \in gP^{-1}$. For any $y \in P^{-1}$, we have $gy \preceq g$. Thus (2) holds.

(3) Set $H = P \cap P^{-1}$ and $w = g_1 \cdots g_n$ with $g_1, \cdots, g_n \in P$. To the contrary, assume that $g_i \notin H$ for some $i \in \{1, \cdots, n\}$. Then $g_i \succ e$ and

$$\begin{aligned} w &= g_1 \cdots g_n \succeq g_1 \cdots g_{n-1} \succeq \cdots \succeq g_1 \cdots g_i \\ &\succ g_1 \cdots g_{i-1} \succeq e. \end{aligned}$$

This contradicts that w is in H. Hence each g_i is in H. □

6.2. Left orderability and semilinear left preorders.

Definition 6.3. Let $\preceq$ be a semilinear preorder on a set X. A subset $F \subset X$ is *coinitial* if for any $x \in X$ there is some $y \in F$ with $y \preceq x$.

For a subset S of a group G, we use $\mathrm{sgr}(S)$ to denote the semigroup generated by S. It was shown in [17, Theorem 2.7] that a group admitting a semilinear left partial order is left-orderable. Now we generalize it to the case of preorder.

Proposition 6.4. *Let G be a group admitting a semilinear left preorder $\preceq$ and let P be its positive cone. Set $H = P \cap P^{-1}$ and $\widetilde{H} = \bigcup_{p \in P} \bigcap_{q^{-1} \preceq p^{-1}} q^{-1}Hq$. Then*

(1) *$\widetilde{H}$ is a normal subgroup of G;*
(2) *if G is finitely generated and $H \neq G$, then $\widetilde{H} \neq G$;*
(3) *the quotient $G/\widetilde{H}$ is left-orderable.*

Proof. (1) From (P2), H is a subgroup of G; thus $x \in \widetilde{H}$ implies $x^{-1} \in \widetilde{H}$ by the definition. Let $x,y \in \widetilde{H}$. Suppose that $x \in \bigcap_{q^{-1} \preceq p_1^{-1}} q^{-1}Hq$ and $y \in \bigcap_{q^{-1} \preceq p_2^{-1}} q^{-1}Hq$ for some $p_1, p_2 \in P$. Since p_1^{-1} and p_2^{-1} are comparable by (O6) in section 2.3, we may assume that $p_1^{-1} \preceq p_2^{-1}$. Thus $y \in \bigcap_{q^{-1} \preceq p_2^{-1}} q^{-1}Hq \subseteq \bigcap_{q^{-1} \preceq p_1^{-1}} q^{-1}Hq$ and hence $xy \in \bigcap_{q^{-1} \preceq p_1^{-1}} q^{-1}Hq \subset \widetilde{H}$. So $\widetilde{H}$ is a subgroup of G.

For each $p \in P$ and $g \in G$, from Lemma 6.2 (2),

$$\begin{aligned} & g\left(\cap_{q^{-1}\preceq p^{-1}} q^{-1}Hq\right)g^{-1} = g\left(\cap_{x\in p^{-1}P^{-1}} xHx^{-1}\right)g^{-1} \\ & = \cap_{x\in p^{-1}P^{-1}} gxH(gx)^{-1} = \cap_{x\in gp^{-1}P^{-1}} xHx^{-1} \\ & = \cap_{x\preceq gp^{-1}} xHx^{-1} \subset \cap_{x\preceq y^{-1}} xHx^{-1} \\ & \subset \widetilde{H}, \end{aligned}$$

where y^{-1} is some element in P^{-1} with $y^{-1} \preceq gp^{-1}$. Thus $\widetilde{H}$ is normal in G.

(2) Suppose that G is finitely generated and $\{g_1, \cdots, g_n\}$ is a finite set of generators. To the contrary, assume that $G = \widetilde{H}$. Then, for each $i \in \{1, \cdots, n\}$, there is some $p_i \in P$ such that $g_i \in \cap_{q^{-1}\preceq p_i^{-1}} q^{-1}Hq$. Take some $p \in P$ with $p^{-1} \preceq p_i^{-1}$ for each $i \in \{1, \cdots, n\}$. Then we have

$$\{g_1, \cdots, g_n\} \subset \cap_{q^{-1}\preceq p^{-1}} q^{-1}Hq \subset p^{-1}Hp.$$

Thus $G \subset p^{-1}Hp$ and hence $G = H$. So (2) holds.

(3) We may assume that $H \neq G$; otherwise, $\widetilde{H} = G$ and the conclusion is trivial.

Claim 1. For any finitely many $x_1, \cdots, x_n \in G \setminus \widetilde{H}$, there are some $\varepsilon_1, \cdots, \varepsilon_n \in \{-1, 1\}$ such that

$$\mathrm{sgr}(x_1^{\varepsilon_1}, \cdots, x_n^{\varepsilon_n}) \cap \widetilde{H} = \emptyset.$$

We show the Claim 1 by induction on n. Given $x \in G \setminus \widetilde{H}$, suppose that there is a positive integer k such that $x^k \in \widetilde{H}$. Then $x^k \in \bigcap_{q^{-1}\preceq p^{-1}} q^{-1}Hq$, by the definition of $\widetilde{H}$, for some $p \in P$ with $p^{-1} \preceq x$. Let $q^{-1} \preceq p^{-1}$ be given. Then $q^{-1} \preceq x$ and hence $qxq^{-1} \in P \cup P^{-1}$ by (P4). WLOG, we may assume that $qxq^{-1} \in P$. Then $qx^kq^{-1} \in H$ implies that $qxq^{-1} \in H$, by Lemma 6.2 (3). Thus $x \in \bigcap_{q^{-1}\preceq p^{-1}} q^{-1}Hq$ whence $x \in \widetilde{H}$. This contradicts the assumption. So the Claim 1 holds for $n = 1$.

Now assume that $n \geq 2$ and the Claim 1 holds for any $y_1, \cdots, y_m \in G \setminus \widetilde{H}$ with $m < n$. By (O7), there is some $p \in P$ with

$$p^{-1} \preceq x_1, \cdots, p^{-1} \preceq x_n.$$

Then $qx_1, \cdots, qx_n \in P$, for each $q^{-1} \preceq p^{-1}$. According to (P4) in Proposition 6.1, we have $\{qx_1q^{-1}, \cdots, qx_nq^{-1}\} \subset P \cup P^{-1}$. Thus there are some $\vec{\varepsilon}(q) = (\varepsilon_1(q), \cdots, \varepsilon_n(q)) \in \{-1, 1\}^n$ such that $\{qx_1^{\varepsilon_1(q)}q^{-1}, \cdots, qx_n^{\varepsilon_n(q)}q^{-1}\} \subset P$. For each $\vec{\varepsilon} = (\varepsilon_1, \cdots, \varepsilon_n) \in \{-1, 1\}^n$, let

$$Q(\vec{\varepsilon}) = \left\{q^{-1} \preceq p^{-1} : \{qx_1^{\varepsilon_1}q^{-1}, \cdots, qx_n^{\varepsilon_n}q^{-1}\} \subset P\right\}.$$

Then $\{q^{-1} : q^{-1} \preceq p^{-1}\} = \bigcup_{\vec{\varepsilon}\in\{-1,1\}^n} Q(\vec{\varepsilon})$.

Let $P^{++} = \{g \in G : g \succ e\}$ and $P^{--} = \{g \in G : g \prec e\}$ be the strictly positive and negative cones respectively. Now we discuss into two cases.

Case 1. There is an $\vec{\varepsilon} = (\varepsilon_1, \cdots, \varepsilon_n) \in \{-1, 1\}^n$ and a coinitial set $Q \subset Q(\varepsilon)$ such that for any $q^{-1} \in Q$,

$$\{qx_1^{\varepsilon_1}q^{-1}, \cdots, qx_n^{\varepsilon_n}q^{-1}\} \subset P^{++}.$$

Noting that any word composed of $qx_1^{\varepsilon_1}q^{-1},\cdots,qx_n^{\varepsilon_n}q^{-1}$ will lie in P^{++} for any $q^{-1}\in Q$ and Q is a coinitial set, the semigroup $\mathrm{sgr}(x_1^{\varepsilon_1},\cdots,x_n^{\varepsilon_n})$ has empty intersection with $\widetilde{H}$. Then the Claim holds.

Case 2. For any $\vec{\varepsilon}=(\varepsilon_1,\cdots,\varepsilon_n)\in\{-1,1\}^n$, there is some $p^{-1}(\vec{\varepsilon})\preceq p^{-1}$ such that for any $q^{-1}\in\{q^{-1}\in Q(\vec{\varepsilon}):q^{-1}\preceq p^{-1}(\vec{\varepsilon})\}$, we have

$$\{i\in\{1,\cdots,n\}:qx_i^{\varepsilon_i}q^{-1}\in H\}\neq\emptyset.$$

Thus, for each $q^{-1}\in Q(\vec{\varepsilon})$, there is a partition $\{1,\cdots,n\}=A(q,\varepsilon)\cup B(q,\varepsilon)$ such that

$$\{qx_i^{\varepsilon_i}q^{-1}:i\in A(q,\varepsilon)\}\subset H \quad\text{and}\quad \{qx_i^{\varepsilon_i}q^{-1}:i\in B(q,\varepsilon)\}\subset P^{++}.$$

Since $H\neq G$ by the assumption, we have $P^{++}=P\setminus H\neq\emptyset$ and hence P^{++} is infinite. Note that there are only finitely many partitions of $\{1,\cdots,n\}$. So there is an $\vec{\varepsilon}=(\varepsilon_1,\cdots,\varepsilon_n)\in\{-1,1\}^n$, a coinitial set $Q\subset Q(\vec{\varepsilon})$ and a partition $\{1,\cdots,n\}=A\cup B$ with $A\neq\emptyset,B\neq\emptyset$ such that

$$\{qx_i^{\varepsilon_i}q^{-1}:i\in A\}\subset H \quad\text{and}\quad \{qx_i^{\varepsilon_i}q^{-1}:i\in B\}\subset P^{++},$$

for each $q^{-1}\in Q$. WLOG, we may assume that $A=\{1,\cdots,k\}$ and $B=\{k+1,\cdots,n\}$ for some $k\in\{1,\cdots,n-1\}$.

Now by the induction hypothesis, there is some $\vec{\eta}=(\eta_1,\cdots,\eta_k)\in\{-1,1\}^k$ such that

$$\mathrm{sgr}(x_1^{\eta_1},\cdots,x_k^{\eta_k})\cap\widetilde{H}=\emptyset.$$

Then we conclude that $(\eta_1,\cdots,\eta_k,\varepsilon_{k+1},\cdots,\varepsilon_n)\in\{-1,1\}^n$ satisfies

$$\mathrm{sgr}(x_1^{\eta_1},\cdots,x_k^{\eta_k},x_{k+1}^{\varepsilon_{k+1}},\cdots,x_n^{\varepsilon_n})\cap\widetilde{H}=\emptyset.$$

Indeed, let w be a word composed of $x_1^{\eta_1},\cdots,x_k^{\eta_k},x_{k+1}^{\varepsilon_{k+1}},\cdots,x_n^{\varepsilon_n}$. If $x_{k+1}^{\varepsilon_{k+1}},\cdots,x_n^{\varepsilon_n}$ do not occur in w, then the choice of $\vec{\eta}$ implies that w is not in $\widetilde{H}$. If there are some letters of $x_{k+1}^{\varepsilon_{k+1}},\cdots,x_n^{\varepsilon_n}$ occur in w, then for each $q^{-1}\in Q$, $qwq^{-1}\in P^{++}$ and hence w is not in $\widetilde{H}$ by the coinitiality of Q. Thus we complete the proof of the Claim 1.

Now (3) is followed from some standard arguments (see [15, Lemma 2.2.3]). For convenience of the readers, we afford a detailed proof here.

By the principe of compactness, the Claim 1 implies the following directly.

Claim 2. There is a map $\varepsilon:G\setminus\widetilde{H}\to\{-1,1\}$ such that for any finite $g_1,\cdots,g_n\subset G\setminus\widetilde{H}$, $\mathrm{sgr}\left(g_1^{\varepsilon(g_1)},\cdots,g_n^{\varepsilon(g_n)}\right)\cap\widetilde{H}=\emptyset$.

Let $\widetilde{P}=\{g\in G\setminus\widetilde{H}:\varepsilon(g)=1\}$. Then it is easy to verify that $\widetilde{P}$ is a subsemigroup of G and $G=\widetilde{P}\cup\widetilde{H}\cup\widetilde{P}^{-1}$.

Claim 3. $\widetilde{P}=\widetilde{H}\widetilde{P}\widetilde{H}$.

First $\widetilde{P}=e\widetilde{P}e\subset\widetilde{H}\widetilde{P}\widetilde{H}$. To show the converse, we conclude that for any $h\in\widetilde{H}$ and $q\in\widetilde{P}$, $hq\in\widetilde{P}$. Otherwise, $\varepsilon(hq)=-1$. Then $h^{-1}\in\mathrm{sgr}((hq)^{-1},q)\cap\widetilde{H}$, which contradicts the choice of ε. Similarly, $qh\in\widetilde{P}$. Thus for any $h_1,h_2\in\widetilde{H}$ and $q\in\widetilde{P}$, $h_1qh_2\in\widetilde{P}$ and hence $\widetilde{H}\widetilde{P}\widetilde{H}\subset\widetilde{P}$. Thus Claim 3 holds.

We define an order $\leq$ on the quotient group $G/\widetilde{H}=\{g\widetilde{H}:g\in G\}$ by

$$f\widetilde{H}<g\widetilde{H}\quad\text{if and only if}\quad f^{-1}g\in\widetilde{P}.$$

It is well defined by Claim 3 and is a total order by the equality $G = \widetilde{P} \cup \widetilde{H} \cup \widetilde{P}^{-1}$. It is obvious that $\leq$ is G-invariant, i.e. for any $g, x, y \in G$, $gx\widetilde{H} < gy\widetilde{H}$ whenever $x\widetilde{H} < y\widetilde{H}$. Thus the quotient $G/\widetilde{H}$ is left-orderable.

□

6.3. **Existence of pointwise fixed arcs.** Now we are ready to prove Proposition 1.4 for any higher rank lattices.

Proof of Proposition 1.4. Now take a finite set $\{g_1, \cdots, g_n\}$ of generators of Γ. For each $i \in \{1, \cdots, n\}$, by Lemma 2.1, we can take $x_i \in X \setminus \{z\}$ fixed by g_i. Let t be the point such that $[z,t] = \cap_{i=1}^{n} [z, x_i]$. Define a positive cone P of Γ by

$$P = \left\{ g \in \Gamma : g^{-1}(t) \in [z,t] \right\}.$$

Then P leads to a preorder $\preceq$ on Γ. Let $H = P \cap P^{-1}$ and $\widetilde{H} = \bigcup_{p \in P} \bigcap_{q^{-1} \preceq p^{-1}} q^{-1} H q$. Note that H is just the stabilizer of t in Γ.

Claim. There is some point $s \in (z,t]$ fixed by Γ.

We discuss into two cases for the proof of the Claim.

Case 1. There is a sequence (f_i) in P^{-1} such that $f_i(t) \to z$ as $i \to \infty$. Then P gives rise to a semilinear left preorder $\preceq$ on Γ: for $g, h \in \Gamma$,

$$g \preceq h \text{ if and only if } [z, g(t)] \subset [z, h(t)].$$

By Proposition 6.4, Γ admits a left-orderable quotient $\Gamma/\widetilde{H}$. By Lemma 2.7, either $\widetilde{H}$ is contained in the center of G or $\Gamma/\widetilde{H}$ is a finite group, where G is the ambient Lie group of Γ. In the former case, $\Gamma/\widetilde{H}$ is also a higher rank lattice, which contradicts Deroin-Hurtado's theorem in [7]. In the latter case, $\Gamma/\widetilde{H}$ is a finite group. Then the orderability implies that it is trivial and hence $\widetilde{H} = \Gamma$. Then we have $H = \Gamma$, by Proposition 6.4 (2). Thus t is fixed by Γ and take $s = t$.

Case 2. There is no sequence (f_i) in P^{-1} with $f_i(t) \to z$ as $i \to \infty$.
Fix a canonical ordering $<$ on $[z,t]$ with $z < t$. Let $s = \inf_{g^{-1} \in P^{-1}} g(t)$. Then $s \in (z,t]$ and we claim that s is fixed by Γ. Indeed, let $g^{-1} \in P^{-1}$ be given. Suppose that $s = \lim\limits_{n \to \infty} \gamma_n^{-1}(t)$ for some sequence (γ_n^{-1}) in P^{-1}. Then $g^{-1}(s) \geq s$ with respect to the canonical ordering $<$ on $[z,t]$. If $g^{-1}(s) \neq s$ then $g^{-1}\gamma_n^{-1}(t) > \gamma_n^{-1}(t)$, for all sufficiently large n. Then $g\gamma_n^{-1}(t) < \gamma_n^{-1}(t) \leq t$ and hence $g\gamma_n^{-1} \in P^{-1}$. Thus

$$s \leq \lim_{n \to \infty} g\gamma_n^{-1}(t) \leq \lim_{n \to \infty} \gamma_n^{-1}(t) = s.$$

So $g(s) = s$ and hence $g^{-1}(s) = s$. This contradiction shows that s is fixed by each element in P. Since for each i, either $g_i \in P$ or $g_i^{-1} \in P$, s is fixed by Γ.

Thus the Claim holds. Now, by Theorem 2.8, Γ fixes the $[z,s]$ pointwise. □

7. Examples and questions

As supplements to the main theorems, we give some examples in this section.

Example 7.1. For each $i \neq 0 \in \mathbb{Z}$, let $\theta_i = \mathrm{sgn}(i)(1 - \frac{1}{2|i|})\pi$; and let I_i be the arc in the complex plane defined by $I_i = \{re^{\mathrm{i}\theta_i} : 0 \leq r \leq \frac{1}{|i|}\}$. Let $X = \cup_{i \in \mathbb{Z}\setminus\{0\}} I_i$ and take the subspace topology of $\mathbb{C}$. Then X is a dendrite of infinite order. If G is a countably infinite group, then it can act on the end point set of X transitively and freely. We then extend this action to X by letting g map I_i affinely to $I_{g(i)}$ for each $g \in G$ and $i \neq 0 \in \mathbb{Z}$. Clearly, the extended action is almost free. So, the condition "with no infinite order points" in Theorem 1.1 cannot be removed.

The following example indicates that the exact analogy to the Zimmer's rigidity for higher rank lattice actions on the circle does not hold for higher rank lattice actions on dendrites of finite order. Although we construct the action for $SL_n(\mathbb{Z})$, it holds for any residually finite groups.

Example 7.2. Let $\Gamma = SL_n(\mathbb{Z})$ with $n \geq 3$. Fix a prime number p. For each positive integer α, let $\phi_\alpha : SL_n(\mathbb{Z}) \to SL_n(\mathbb{Z}/p^\alpha\mathbb{Z})$ denote the canonical homomorphism and let $\Gamma_\alpha = \ker(\phi_\alpha)$. Then Γ_α is a finite index subgroup of Γ, the so called *principal congruence subgroup* of Γ. It is known that for each α the index

$$[\Gamma_\alpha : \Gamma_{\alpha+1}] = p^{n^2-1}.$$

Set $\Gamma_0 = \Gamma$. The sequence $(\Gamma_\alpha)_{\alpha=0}^\infty$ forms a *group chain* of Γ:

$$\Gamma_0 > \Gamma_1 > \Gamma_2 > \cdots.$$

Thus we get a sequence of finite sets $\{\Gamma/\Gamma_\alpha : \alpha = 0,1,2,\cdots\}$ on which G acts by left translations.

Now we associate to each α a finite combinatorial tree Y_α whose end point set is Γ/Γ_α, and a finite action of G on Y_α whose restriction to Γ/Γ_α coincides with the left translation action. Precisely, let $Y_0 = \{V_0\} = \Gamma/\Gamma_0$ and let Γ act on it trivially. Assume that for each $0 \leq \beta \leq \alpha$, Y_β and the finite action of G on it is defined. We let $Y_{\alpha+1}$ be the union of Y_α and the set of edges:

$$\{(\gamma\Gamma_\alpha, \gamma\gamma_1\Gamma_{\alpha+1}), (\gamma\Gamma_\alpha, \gamma\gamma_2\Gamma_{\alpha+1}), \cdots, (\gamma\Gamma_\alpha, \gamma\gamma_{p^{n^2-1}}\Gamma_{\alpha+1}) : \gamma\Gamma_\alpha \in \Gamma/\Gamma_\alpha\},$$

where $\gamma_1, \cdots, \gamma_{p^{n^2-1}}$ is a set of coset representatives of $\Gamma_{\alpha+1}$ in Γ_α. Then we extend the action of G from Y_α to $Y_{\alpha+1}$ by letting $g.\gamma\Gamma_{\alpha+1} = (g\gamma)\Gamma_{\alpha+1}$ for any $g, \gamma \in \Gamma$.

For each α, let T_α be the geometric realization of Y_α, which is a finite topological tree. Then G induces a finite action on each T_α in a canonical way. Let $\psi_\alpha : T_{\alpha+1} \to T_\alpha$ be a continuous surjective map defined by

(1) $\psi_\alpha|_{T_\alpha} = \mathrm{id}_{T_\alpha}$;
(2) for each arc $[u,v]$ with $u \in V_\alpha, v \in V_{\alpha+1}$, ψ_α maps the whole arc $[u,v]$ to u.

From the definition, we see that $\psi_\alpha(gx) = g\psi_\alpha(x)$ for $x \in T_\alpha$ and $g \in \Gamma$. Thus we get an inverse system $\{(T_\alpha, \Gamma) : \alpha = 0,1,2,\cdots\}$ with bonding maps ψ_α. Since each ψ_α is monotone and onto, $\varprojlim(T_\alpha, \Gamma)$ is a dendrite by [22, Theorem 10.36], which is of finite order by the construction. Clearly, the inverse limit $(\varprojlim(T_\alpha, \Gamma), \Gamma)$ is almost finite.

The following example shows that there does exist a non-almost-finite action by $SL_n(\mathbb{Z})$ with $n \geq 3$ on a dendrite of finite order.

Example 7.3. Let (X,Γ) be the system constructed in Example 7.2. Since dendrites are planar continua, we may assume that X is contained in $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$. Fix an end point e of X and label its orbit Γe as $\{e_i : i = 1,2,\cdots\}$. Suppose the coordinate of e_i is $(x_i, y_i, 0)$. Let $I_i = \{(x_i, y_i, t) : 0 \leq t \leq 1/i\}$ for each i, and let $Z = X \cup (\cup_{i=1}^{\infty} I_i)$. Then Z is a dendrite of finite order contained in $\mathbb{R}^3$. Extend the action of Γ from X to Z by letting g map I_i to I_j affinely if $e_j = ge_i$, for each $g \in \Gamma$. Then we get an action (Z,Γ), which is not almost finite.

Finally, we recall the following open questions.

Question 1. Let Γ be a countable group having Kazhdan's Property (T).

(1) Does every action of Γ on a dendrite admit a finite orbit?
(2) Is every action of Γ on $\mathbb{S}^1$ finite (i.e., every orbit is finite)?

If the above questions both have positive answers, then we can also establish Theorem 1.3 for countable groups having Kazhdan's Property (T). The item (1) above is proposed in [9, 10]. Duchesne and Monod also show that the item (1) does not hold for some Polish groups having strong Kazhdan's property (T) but it still open for countable groups. The item (2) is answered positively by Navas in [24] for every $C^{1+\tau}$ action with $\tau > \frac{1}{2}$ but is still open for continuous actions.

REFERENCES

[1] E. Abdalaoui and I. Naghmouchi, *Group action with finite orbits on local dendrites*. Dyn. Syst. 36 (2021), 714-730.

[2] K. A. Bencsáth, M. C. Bonanome, M. H. Dean and M. Zyman, *Lectures on finitely generated solvable groups*. Springer Briefs Math., Springer, New York, 2013.

[3] B. Bowditch, *Hausdorff dimension and dendritic limit sets*. Math. Ann. 332 (2005), 667-676.

[4] A. Brown, *Recent progress in the Zimmer program.* Current developments in mathematics 2018, 1-56.

[5] M. Burger and N. Monod, *Bounded cohomology of lattices in higher rank Lie groups*, J. Eur. Math. Soc. 1 (1999), 199-235.

[6] I. M. Chiswell and P. H. Kropholler, *Soluble right orderable groups are locally indicable*. Canad. Math. Bull. Vol. **36** (1), (1993), 22-29.

[7] B. Deroin and S. Hurtado, *Non left-orderability of lattices in higher rank semi-simple Lie groups*. arXiv:2008.10687.

[8] C. Drutu and M. Kapovich, *Geometric Group Theory*. Colloquium Publications, 2018.

[9] B. Duchesne and N. Monod, *Structural properties of dendrite groups*. Trans. Amer. Math. Soc. 371 (2019), 1925-1949.

[10] B. Duchesne and N. Monod, *Group actions on dendrites and curves*. Ann. Inst. Fourier (Grenoble) 68 (2018), 2277-2309.

[11] D. Fisher, *Recent developments in the Zimmer program.* Notices Amer. Math. Soc. 67 (2020), no. 4, 492-499.

[12] É. Ghys. *Actions de réseaux sur le cercle*. Invent. Math. 137 (1999), 199-231.

[13] É. Ghys. *Groups acting on the circle*. L'Enseign. Math. 47 (2001), 329-407.

[14] E. Glasner and M. Megrelishvili, *Group actions on treelike compact spaces*. Sci. China Math. 62 (2019), 2447-2462.

[15] A. Glass, *Partially ordered groups*. World Scientific, 1999.

[16] D. Kerr and H. Li, *Ergodic Theory: Independence and Dichotomies*. Springer Monographs in Mathematics, Springer, Berlin, 2016.

[17] V. Kopytov, *Semilinearly ordered groups*. Selected surveys presented at two conferences on Advances in algebra and model theory, Pages 149-170, 1997.

[18] A. Malyutin, *On groups acting on dendrons.* Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), 415(Geometriya i Topologiya. 12):62-74, 2013. transl. J. Math. Sci. (N.Y.) 212 (2016), 558-565.

[19] G. Margulis, *Discrete subgroups of semisimple Lie groups*, volume 17 of Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer, 1991.

[20] J. Mennicke, *A remark on the congruence subgroup problem*, Math. Scand. 86 (2000), no. 2, 206-222.

[21] Y. Minsky, *On rigidity, limit sets, and end invariants of hyperbolic 3-manifolds*. J. Amer. Math. Soc., 7 (1994), 1994.

[22] S. Nadler, Jr. *Continuum theory*, volume 158 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 1992. An introduction.

[23] A. Navas, *Groups of circle diffeomorphisms*, Translation of the 2007 Spanish edition, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 2011.

[24] A. Navas, *Actions de groupes de Kazhdan sur le cercle*. Annales Sci. de l'École Normale Supérieure **35** (2002), 749-758.

[25] E. Shi, *Free subgroups of dendrite homeomorphism group.* Topology Appl. 159 (2012), 2662-2668.

[26] E. Shi, H. Xu and X. Ye, *An alternative for minimal group actions on totally regular curves*. Journal of Differential Equations **341** (2022), 402-421.

[27] E. Shi and X. Ye, *Equicontinuity of minimal sets for amenable group actions on dendrites.* Dynamics: topology and numbers, 175-180, Contemp. Math., 744, Amer. Math. Soc., [Providence], RI, 2020.

[28] A. Sikora. *Topology on the spaces of left-orders of groups* . Bull. London Math. Soc. **36** (2004), 519-526.

[29] J. Tits, *Systémes générateurs de groupes de congruence*, C. R. Acad. Sci. Paris Sér. A 283 (1976), no. 9, 693-695.

[30] G. Whyburn, *Analytic Topology*, American Mathematical Society Colloquium Publications, v. 28, 1942.

[31] D. Witte-Morris, *Arithmetic groups of higher* $\mathbb{Q}$*-rank cannot act on 1-manifolds*, Proc. Amer. Math. Soc. 122 (1994), 333-340.

[32] R. Zimmer, *Ergodic theory and semisimple groups*. Monographs in mathematics, Birkhäuser, 1984.

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