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STRUCTURES OF QUASI-GRAPHS AND ω-LIMIT SETS OF QUASI-GRAPH MAPS

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ABSTRACT. An arcwise connected compact metric space X is called a quasi-graph if there is a positive integer N with the following property: for every arcwise connected subset Y of X, the space $\overline{Y} - Y$ has at most N arcwise connected components. If a quasi-graph X contains no Jordan curve, then X is called a quasi-tree. The structures of quasi-graphs and the dynamics of quasi-graph maps are investigated in this paper. More precisely, the structures of quasi-graphs are explicitly described; some criteria for ω-limit points of quasi-graph maps are obtained; for every quasi-graph map f, it is shown that the pseudo-closure of $R(f)$ in the sense of arcwise connectivity is contained in $\omega(f)$; it is shown that $\overline{P(f)} = \overline{R(f)}$ for every quasi-tree map f. Here $P(f)$, $R(f)$ and $\omega(f)$ are the periodic point set, the recurrent point set and the ω-limit set of f, respectively. These extend some well-known results for interval dynamics.

1. INTRODUCTION

In the early 1960s, A. N. Sharkovskii established the famous theorem which describes the coexistence among periods of periodic points of an interval map (see [27]). Since then, the dynamics of continuous interval maps has been intensively studied, and many interesting results have been obtained. One may consult [2,3,25] for a systematic introduction to some topics of this area. A natural question is to what extent can these results be generalized beyond the interval? Some simple examples show that many results on the dynamics of interval maps do not generalize to spaces of topological dimension ≥ 2. It is thus natural to consider the dynamical systems on some 1-dimensional continua (a *continuum* is a compact connected metric space). Great progress has been made in this direction. For the pioneering work of Blokh on dynamics of graph maps, see [5–8], and for some later related works, see [2,9,17,18,20,21]. Very recently, periodic points, recurrent points and transitivity of dendrite maps were studied in [1,11,16,29] (a *dendrite* is a locally connected continuum containing no Jordan curve). However, many remarkable results for interval maps no longer hold for dendrite maps, unless some further restrictions are made on the dendrite. In this paper, we aim to find some natural 1-dimensional spaces to which many results on dynamics of interval maps are extended.

By a *quasi-graph*, we mean a nondegenerate, compact, arcwise connected metric space X such that $\overline{Y} - Y$ has at most N arcwise connected components, for some fixed positive integer N and for every arcwise connected subset Y of X (see also

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Definition 2.1). Recall that a topological space is said to be *nondegenerate* if it contains at least two points. Clearly every graph is a quasi-graph. A *quasi-tree* is a quasi-graph containing no Jordan curve. Trees and the Warsaw circle are quasi-trees. Since the Warsaw circle is a quasi-graph but not a graph, the class of quasi-graphs is indeed strictly larger than that of graphs.

We are mainly concerned with the ω-limit sets for quasi-graph maps. Let us first recall some notation and definitions. Given a metric space X, denote by $C^0(X)$ the set of all continuous maps from X to itself. Let $f \in C^0(X)$. For every $x \in X$, write $O(x, f) = \{f^n(x) : n = 0, 1, 2, \cdots\}$, and call it the *orbit* of x under f. The *ω-limit set of x*, denoted by $\omega(x, f)$, is the collection of all limit points of $O(x, f)$. The set $\omega(f) = \bigcup_{x\in X} \omega(x, f)$ is called the *ω-limit set of f*. A point $x \in X$ is called a *periodic point of f* if $f^n(x) = x$ for some positive integer n. The least positive integer n with $f^n(x) = x$ is called the *period of x*. The point x is called a *recurrent point of f* if $x \in \omega(x, f)$. It is called a *nonwandering point of f* if for every neighborhood V of x there is a positive integer n such that $f^n(V) \cap V \neq \emptyset$. The sets of periodic points, recurrent points and nonwandering points of f are denoted by $P(f)$, $R(f)$ and $\Omega(f)$ respectively. We use $P_n(f)$ to denote the set of all $x \in X$ such that $f^n(x) = x$.

A set $A \subset X$ is called *f-invariant* if $f(A) = A$. The sets $P(f), R(f), \omega(f)$ and $\Omega(f)$ are all f-invariant. Except for $\Omega(f)$, the other three sets are not closed in general. The following containments are well known:

$$P(f) \subset R(f) \subset \omega(f) \subset \Omega(f). \tag{1}$$

Many refinements of (1) are known in the case of 1-dimensional dynamical systems. We list only a few of them which are closely related to the topic of this paper. A. N. Sharkovskii obtained the following theorem in [28], which is a very useful criterion for ω-limit points of interval maps.

Theorem A. *Let f be a continuous map on the interval $[0, 1]$. If every open interval with left (or right) endpoint v contains at least 2 points of some trajectory, then $v \in \omega(f)$. A point $v \in \omega(f)$ if and only if every open interval containing v contains at least 3 points of some trajectory.*

The following theorem is due to Sharkovskii (see [28]).

Theorem B. *Let f be a continuous map on the interval $[0, 1]$. Then $\overline{R(f)} \subset \omega(f)$.*

The following theorem was proven by Sharkovskii in [26] (see also [15, 32]).

Theorem C. *Let f be a continuous map on the interval $[0, 1]$. Then $\overline{P(f)} = \overline{R(f)}$.*

One may refer to [4, 10, 12, 14, 24] for other interesting results about ω-limit sets for interval maps.

We remark that an analog of Theorem A was obtained for tree maps by F. P. Zeng et al. in [33] and for graph maps by N. Chinen in [13]. Theorem B and Theorem C were generalized to graph maps by A. M. Blokh in [6] (see also [18, 20, 31]). Theorem C was generalized to maps on a class of dendrites by J. H. Mai and E. H. Shi in [19], and to maps on the Warsaw circle by J. C. Xiong et al. in [30].

The aim of this paper is to generalize Theorem A, Theorem B and Theorem C to quasi-graph maps. We first study the structures of quasi-graphs in Section 2, and give an explicit description of the structures in Theorem 2.24. Roughly speaking, a quasi-graph is a union of a graph with finitely many inner rays. For example, the Warsaw circle is the union of an arc with a ray. Before going to the dynamics of quasi-graph maps, we study some basic properties of quasi-graph maps in Section 3, which are used in the later sections. In Section 4 and in the setting of quasi-graph maps, we get a criterion (Theorem 4.1) for periodic points and two criteria (Theorem 4.3 and Theorem 4.4) for ω-limit points. These generalize Theorem A and the corresponding theorems in [33] and [13]. In Section 5, we consider the recurrent points of quasi-graph maps and give a criterion for them in Theorem 5.4. To generalize Theorem B to quasi-graphs, we have to introduce a notion of pseudo-closure in the sense of arcwise connectivity (Definition 5.1). Then we show in Corollary 5.5 that the pseudo-closure of $R(f)$ in the sense of arcwise connectivity is contained in $\omega(f)$. This also generalizes the corresponding theorems in [6, 20] for graph maps. On the basis of the above results, we show in the end that $\overline{R(f)} = \overline{P(f)}$ for any quasi-tree map f. This is a generalization of Theorem C and the corresponding theorems in [6, 30, 31].

Throughout the paper, we use the symbols $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$, $\mathbb{Z}_+$, and $\mathbb{N}$ to respectively denote the sets of real numbers, nonnegative real numbers, integers, nonnegative integers, and positive integers. For a set A, denote its cardinality by $\sharp(A)$. All the maps appearing in this paper are assumed to be continuous. Given a metric space X, we use $B(x, r)$ to denote the open ball with center $x \in X$ and radius $r > 0$. For every nonempty subset X_0 of X, $\partial X_0 = \partial_X(X_0)$ denotes the boundary of X_0 in X, $\overset{\circ}{X}_0 = \mathrm{Int}(X_0, X)$ denotes the interior of X_0 in X, and $\overline{X}_0 = \mathrm{Clos}(X_0, X)$ denotes the closure of X_0 in X. Set $B(X_0, r) = \bigcup_{x \in X_0} B(x, r)$.

2. Quasi-graphs

An *arc* is a continuum which is homeomorphic to the closed interval $[0, 1]$. A *graph* is a continuum which is a union of finitely many arcs so that any two of these arcs are either disjoint or intersecting only at one common endpoint. Each of these arcs is called an *edge* of the graph; each endpoint of an edge is called a *vertex*. The set of vertices of a graph G is denoted by $V(G)$, and the set of edges is denoted by $E(G)$. The *valence* of a vertex x is the number of edges incident to x; if the number is n, then one writes $\mathrm{val}(x) = \mathrm{val}(x, G) = n$. A vertex of valence 1 is also called an *endpoint* of G; a vertex x with $val(x) \geqslant 3$ is called a *branch point* of G. The set of endpoints and the set of branch points of G are denoted by $\mathrm{End}(G)$ and $\mathrm{Br}(G)$ respectively. Note that two edges of a graph may meet at a point which is not a branch point of the graph. If A is a single point set, then set $\mathrm{End}(A) = A$. A *tree* is a graph with no subset which is homeomorphic to the unit circle. A *star* is either a tree having only one branch point or an arc. One can refer to [2] for more facts about graphs.

In what follows, we assume that the metric d on a graph G has the following property: $B(x, r)$ is connected for every $r > 0$. Clearly, such a metric d always exists for every graph G.

Definition 2.1. A nondegenerate compact arcwise connected metric space X is called a *quasi-graph* if there exists $N \in \mathbb{N}$ such that $\overline{Y} - Y$ has at most N arcwise connected components, for every arcwise connected subset Y of X. Furthermore, if such N is minimal, i.e., if there exists an arcwise connected subset Y_0 of X such that $\overline{Y}_0 - Y_0$ has exactly N arcwise connected components, then N is called the *separation degree* of X.

Let G be a graph with n edges. Then $\overline{Y} - Y$ has at most $2n$ arcwise connected components for every connected subset Y of G. So a graph G must be a quasi-graph, but not vice versa. A quasi-graph with no subset homeomorphic to the unit circle is called a *quasi-tree*. Obviously, a tree is a quasi-tree.

By Definition 2.1, we deduce the following lemma directly.

Lemma 2.2. *Let X be a quasi-graph. Then every nondegenerate compact arcwise connected subspace of X is also a quasi-graph.*

Example 2.3. Let $W = \{(t, \sin(1/t)) \in \mathbb{R}^2 : 0 < t \leqslant 1\}$. The closure $\overline{W}$ of W in $\mathbb{R}^2$ is called the $\sin(1/t)$ - *continuum*. Clearly, $\overline{W} = E \cup W$, where $E = \{(0, y) : -1 \leq y \leq 1\}$. Take $a \in E$ and take $b \in W$. Suppose that A is an arc in $\mathbb{R}^2$ such that $A \cap \overline{W} = \mathrm{End}(A) = \{a, b\}$. Then $A \cup \overline{W}$ is a quasi-graph. Specifically, the Warsaw circle is a quasi-graph. (The *Warsaw circle* is any continuum Q homeomorphic to $Y \cup Z$ where Y is the $\sin(1/t)$ - continuum and Z is the union of three segments in $\mathbb{R}^2$, one from $(0, -1)$ to $(0, -2)$, one from $(0, -2)$ to $(1, -2)$, and one from $(1, -2)$ to $(1, \sin(1))$; see [23, p. 5].)

Suppose that x and y are points in a quasi-graph X. If $x \neq y$, then let $[x, y]$ denote an arc in X with endpoints x and y; the notation (x, y), $[x, y)$ and $(x, y]$ is analogous to similar notation in the interval case. If $x = y$, then let $[x, y] = \{x\}$. Suppose that A is an arc in X, with two distinct points $u, v \in A$. Let $A[u, v]$ be the unique subarc of A with endpoints u and v. Let $A(u, v] = A[v, u) = A[u, v] - \{u\}$ and let $A(u, v) = A(u, v] - \{v\}$. Let $A[u, v] = \{u\}$ if $u = v$.

Lemma 2.4. *Let $x_1, \cdots, x_n$ be pairwise distinct points in an arcwise connected space X. Then there exists a tree T in X such that $\{x_1, \cdots, x_n\} \subset T$.*

Proof. Since X is arcwise connected, there exists an arc $A_i \subset X$ such that $\mathrm{End}(A_i) = \{x_1, x_i\}$ for each $i = 2, \cdots, n$. Let $T_2 = A_2$. Assume that there is a tree T_i with $\{x_1, \cdots, x_i\} \subset T_i$ for some $i \in \{2, \cdots, n-1\}$; then there exists a point $y_{i+1} \in T_i \cap A_{i+1}$ such that $A_{i+1}[x_{i+1}, y_{i+1}] \cap T_i = \{y_{i+1}\}$. Take $T_{i+1} = T_i \cup A_{i+1}[x_{i+1}, y_{i+1}]$. Then $\{x_1, \cdots, x_{i+1}\} \subset T_{i+1}$. By induction, we see that the tree $T = T_n$ is what we need. □

Lemma 2.5. *Let X be a quasi-graph and let Y be an arcwise connected subset of X. Then for every $\varepsilon > 0$, there exists a tree $T = T_\varepsilon$ in Y such that $Y \subset B(T, \varepsilon)$.*

Proof. For every $\varepsilon > 0$, there exist finitely many points $x_1, \cdots, x_n$ in Y such that $Y \subset B(\{x_1, \cdots, x_n\}, \varepsilon)$ by the compactness of $\overline{Y}$. Then there is a tree T in Y containing $\{x_1, \cdots, x_n\}$ by Lemma 2.4. So $Y \subset B(T, \varepsilon)$. □

Lemma 2.6. *Let T be a tree in a quasi-graph X and let N be the separation degree of X. Then T has at most N endpoints.*

Proof. Let $Y = T - \mathrm{End}(T)$. Then $\sharp(\mathrm{End}(T)) = \sharp(\overline{Y} - Y) \leqslant N$. □

Example 2.7. For each $n \in \mathbb{N}$, we view $\mathbb{R}^n = \mathbb{R}^n \times \{0\}$ as a subspace of $\mathbb{R}^{n+1}$. Define respectively the projections $\pi_n : \mathbb{R}^{n+1} \to \mathbb{R}^n$ and $\pi : \mathbb{R}^{n+1} \to \mathbb{R}$ by $\pi_n(x, t) = x$ and $\pi(x, t) = t$, for all $x \in \mathbb{R}^n$ and for all $t \in \mathbb{R}$. Let X be a quasi-graph in $\mathbb{R}^n$, and let W be a compact connected subset of X which contains more than one point. Assume that W has only finitely many arcwise connected components $W_1, \cdots, W_m$. For each $k \in \{1, \cdots, m\}$ and for each $i \in \mathbb{N}$, by Lemma 2.5, there exists a tree $T_{k,i}$ in W_k such that $W_k \subset B(T_{k,i}, 1/i)$ and $T_{k,i} \subset T_{k,i+1}$. Take $m - 1$ straight line segments $A_{i,1}, \cdots, A_{i,m-1}$ in $\mathbb{R}^n$ with length $2/i$ such that Y_i is arcwise connected, where $Y_i = (\bigcup_{k=1}^{m} T_{k,i}) \cup (\bigcup_{k=1}^{m-1} A_{i,k})$. Let $\psi : \mathbb{R}_+ \to \mathbb{R}^{n+1}$ be a continuous injection such that $\psi(\mathbb{R}_+) \cap X = \{\psi(0)\}$, $\pi(\psi(t)) = 1/t$ for all $t \in [1, \infty)$, and $\pi_n(\psi([i, i+1])) = Y_i$ for all $i \in \mathbb{N}$. Set $X' = X \cup \psi(\mathbb{R}_+)$. Then it is easy to see that X' is a quasi-graph.

Let S be a tree, let $v \in S$ and let $n \in \mathbb{N}$. We call S an *n-star with center v* if there is a continuous injection $\varphi : S \to \mathbb{C}$ such that $\varphi(v) = 0$, and $\varphi(S) = \{re^{2k\pi \mathbf{i}/n} : r \in [0, 1], k = 1, \cdots, n\}$. (Here, $\mathbf{i} = \sqrt{-1}$.)

According to this definition, an arc is a 1-star with either of its endpoints being center, and is also a 2-star with any of its interior points being center. If S is an n-star with center v, then $(S - \mathrm{End}(S)) \cup \{v\}$ is called an *open n-star with center v*.

In the following, we generalize some notions such as valence, endpoint and branch point from graphs to arcwise connected compact metric spaces.

Definition 2.8. Let X be a compact arcwise connected metric space and let $v \in X$. The *valence* of v in X, denoted by $\mathrm{val}(v)$ or by $\mathrm{val}(v, X)$, is the number $\max\{n \in \mathbb{N} :$ there exists an n-star with center v in $X\}$ ($\mathrm{val}(v)$ may be ∞); v is called an *endpoint of* X if $\mathrm{val}(x) = 1$; v is called a *branch point* of X if $\mathrm{val}(v) \geq 3$. We still use the symbols $\mathrm{End}(X)$ and $\mathrm{Br}(X)$ to denote the endpoint set and the branch point set of X respectively.

According to Definition 2.8, the point $(0, 1)$ is the unique endpoint of the Warsaw circle Q (see Example 2.3), and all the other points of Q have valence 2. So Q has no branch point.

Lemma 2.9. *Let X be a quasi-graph and let N be the separation degree of X. Then*

(1) *X has no n-star for every $n > N$.*

(2) *For every arc A in X and every point $v \in A$, there exists a subarc A' of A such that v is an endpoint of A' and $(A' - \{v\}) \cap \mathrm{Br}(X) = \emptyset$.*

(3) *X has at most N endpoints.*

(4) *X has at most $N - 2$ branch points, and $\sum\{\mathrm{val}(v) - 2 : v \in \mathrm{Br}(X)\} \leqslant N - 2$.*

Proof. The conclusions (1) and (3) can easily be deduced from Lemma 2.4 and Lemma 2.6. (2) is a direct corollary of (4). Now we start to prove (4). Let $x_1, \cdots, x_k$ be any k pairwise distinct branch points of X. By Lemma 2.4, we can take a tree Y in X such that $\{x_1, \cdots, x_k\} \subset Y$. By adding some arcs adjacent to x_i if necessary, we may assume further that $\mathrm{val}(x_i, Y) = \mathrm{val}(x_i, X)$ for each i. Since Y is a tree, we have $\sum\{\mathrm{val}(v, Y) - 2 : v \in \mathrm{Br}(Y)\} = \sharp\mathrm{End}(Y) - 2$. This implies $\sum\{\mathrm{val}(x_i, X) - 2 : i = 1, 2, \cdots, k\} \leqslant N - 2$ by Lemma 2.6. By the arbitrariness of k, we get $\sum\{\mathrm{val}(v) - 2 : v \in \mathrm{Br}(X)\} \leqslant N - 2$. Since $\mathrm{val}(v) - 2 \geq 1$, we see that X has at most $N - 2$ branch points. □

Let X be a quasi-graph and let $x \in X$. For every $\varepsilon > 0$, denote by $\mathrm{St}(x, \varepsilon) = \mathrm{St}(x, \varepsilon, X)$ the arcwise connected component of $B(x, \varepsilon)$ containing x. By Lemma 2.9, we immediately have

Lemma 2.10. *Let X be a quasi-graph, let $x \in X$ and let $\mathrm{val}(x) = n$. Then there exists an $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$, $\mathrm{St}(x, \varepsilon)$ is always an open n-star with center x.* □

Lemma 2.11. *Let X be a quasi-graph and let N be the separation degree of X. Then for any two arcwise connected subsets Y and W of X, $Y - W$ has at most N arcwise connected components.*

Proof. Assume to the contrary that $Y - W$ has at least $N + 1$ arcwise connected components $Y_0, Y_1, \cdots, Y_N$. Then $Y \cap W \neq \emptyset$ and $W \cup Y_i$ is arcwise connected from Lemma 2.9 for each $i = 0, 1, \cdots, N$. Take $w \in W$ and take $x_i \in Y_i$. Let A_i be an arc in $W \cup Y_i$ with endpoints w and x_i, and let $W_1 = \bigcup_{i=0}^{N} (A_i - \{x_i\})$. Then W_1 is an arcwise connected subset of X, and $\overline{W}_1 - W_1 = \{x_0, x_1, \cdots, x_N\}$ has $N+1$ arcwise connected components, which contradicts the fact that the separation degree of X is N. Hence $Y - W$ has at most N arcwise connected components. □

Corollary 2.12. *Let X be a quasi-graph and let N be the separation degree of X. Suppose that Y is an arcwise connected subset of X and W is a subset of Y with n arcwise connected components for some $n \in \mathbb{N}$. Then $Y - W$ has at most $N + n - 1$ arcwise connected components.*

Proof. By Lemma 2.9 and the arcwise connectivity of Y, it is easy to see that there are $n - 1$ arcwise connected sets $A_1, \cdots, A_{n-1}$ in Y such that $W \cup (\bigcup_{i=1}^{n-1} A_i)$ is arcwise connected, $W \cap (\bigcup_{i=1}^{n-1} (A_i - \mathrm{End}(A_i))) = \emptyset$, and A_i is either an arc or a single point set for each $i = 1, \cdots, n - 1$. By Lemma 2.11, $Y - (W \cup (\bigcup_{i=1}^{n-1} A_i))$ has at most N arcwise connected components. Hence $Y - W$ has at most $N + n - 1$ arcwise connected components. □

Lemma 2.13. *Let X be a compact metric space. Suppose that X_1 and X_2 are two compact subspaces of X with $X_1 \cup X_2 = X$ and $X_1 \cap X_2 \neq \emptyset$. If X_1 and X_2 are both quasi-graphs and $X_1 \cap X_2$ has only finitely many arcwise connected components, then X is also a quasi-graph.*

Proof. Let N_1 and N_2 be the separation degrees of X_1 and X_2 respectively. Denote by $W_1, \cdots, W_n$ the arcwise connected components of $W = X_1 \cap X_2$ for some $n \in \mathbb{N}$. Let Y be an arcwise connected subset of X. If $Y \cap X_1$ is not arcwise connected, then each arcwise connected component of $Y \cap X_1$ has a nonempty intersection with W. If the number of the arcwise connected components of $Y \cap X_1$ is greater than nN_2, then there exist $k \in \{1, \cdots, n\}$ and $N_2 + 1$ arcwise connected components $Y_0, Y_1, \cdots, Y_{N_2}$ of $Y \cap X_1$ with $Y_i \cap W_k \neq \emptyset$ for each i. By Lemma 2.9, it is easy to see that there are pairwise disjoint arcs $A_0, A_1, \cdots, A_{N_2}$ in $Y \cap X_2$ such that $A_i \cap W$ has just one point and $A_i \cap W = A_i \cap W_k \cap Y_i = \mathrm{End}(A_i) \cap W_k \cap Y_i$ for each $i = 0, 1, \cdots, N_2$. Let $Z_k = W_k \cup (\bigcup_{i=0}^{N_2} (A_i - \mathrm{End}(A_i)))$; then Z_k is an arcwise connected set in X_2. Clearly, the endpoint of A_i not in W_k is an arcwise connected component of $\overline{Z}_k - Z_k$ for each $i = 0, 1, \cdots, N_2$. It follows that $\overline{Z}_k - Z_k$ has at least $N_2 + 1$ arcwise connected components. This contradicts Definition 2.1. So the number of arcwise connected components of $Y \cap X_1$ is not greater than nN_2, which implies that the number of arcwise connected components of $\overline{Y \cap X_1} - (Y \cap X_1)$ is not greater than nN_2N_1. Similarly, the number of arcwise connected components of $\overline{Y \cap X_2} - (Y \cap X_2)$ is not greater than nN_1N_2. Notice that

$$\begin{aligned}\overline{Y} - Y &= (\overline{Y \cap X_1} \cup \overline{Y \cap X_2}) - Y \\ &= (\overline{Y \cap X_1} - Y) \cup (\overline{Y \cap X_2} - Y) \\ &= (\overline{Y \cap X_1} - (Y \cap X_1)) \cup (\overline{Y \cap X_2} - (Y \cap X_2)).\end{aligned}$$

This implies that the number of arcwise connected components of $\overline{Y} - Y$ is not greater than $2nN_1N_2$. Hence, X is a quasi-graph. Then the proof is complete. $\square$

The assumption that $X_1 \cap X_2$ has only finitely many arcwise connected components is crucial for the above lemma. For example, let $X = [0,1] \times \{0\} \subset \mathbb{R}^2$. For each $i = 1, 2, 3, \cdots$, let $A_i \subset \mathbb{R}^2$ be the segment connecting points $(\frac{1}{2^{i-1}}, 0)$ and $(\frac{1}{2^{i-1}}, \frac{1}{2^{i-1}})$, and let $B_i \subset \mathbb{R}^2$ be the segment connecting points $(\frac{1}{2^i}, 0)$ and $(\frac{1}{2^{i-1}}, \frac{1}{2^{i-1}})$. Set $Y = \bigcup_{i=1}^{\infty}(A_i \cup B_i) \cup \{(0,0)\}$. Then X and Y are two arcs under the relative topology, and $X \cap Y$ has infinitely many components. Clearly, $X \cup Y$ is not a quasi-graph.

Definition 2.14. Let X be a compact metric space and let L be an arcwise connected subset of X. Suppose that $\varphi : \mathbb{R}_+ \to X$ is a continuous injection. Set $L = \varphi(\mathbb{R}_+)$. Then L or φ is called a *quasi-arc*, and $\varphi(0)$ is called an *endpoint of* L. We also viewed φ as a continuous bijection $\varphi : \mathbb{R}_+ \to L$. For any $\{s,t\} \subset \mathbb{R}_+$, denote by $[s,t]$ the minimal connected subset of $\mathbb{R}_+$ containing $\{s,t\}$. Let $(s,t] = [t,s) = [s,t] - \{t\}$. Denote $x = \varphi(s)$ and $y = \varphi(t)$; denote $\varphi([s,t])$, $\varphi((s,t])$, $\varphi([s,t))$, $\varphi((s,t))$, $\varphi([t,\infty))$ and $\varphi((t,\infty))$ either by $L[s,t]$, $L(s,t]$, $L[s,t)$, $L(s,t)$, $L[t,\infty)$ and $L(t,\infty)$ respectively, or by $L[x,y]$, $L(x,y]$, $L[x,y)$, $L(x,y)$, $L[y,\infty)$ and $L(y,\infty)$ respectively. Set $\omega(L) = \omega(\varphi) = \cap\{\overline{L[m,\infty)} : m \in \mathbb{N}\}$; $\omega(L)$ (resp. $\omega(\varphi)$) is said to be the ω*-limit set* of L (resp. φ).

From the definition, $x \in \omega(\varphi)$ if and only if there are positive numbers $t_1 < t_2 < t_3 < \cdots$ such that $t_n \to \infty$ and $\varphi(t_n) \to x$ as $n \to \infty$. Since $\overline{L} = \varphi([0,m]) \cup \overline{\varphi([m,\infty))}$ for every $m \in \mathbb{N}$, we have that $\overline{L} = L \cup \omega(L)$. It is easy to see that $\omega(L)$ must be nonempty and connected by the compactness of X. However, there are simple examples showing that the ω-limit set $\omega(L)$ of a quasi-arc L may not be arcwise connected. For example, let $Q = Y \cup Z \subset \mathbb{R}^2$ be the Warsaw circle defined in Example 2.3. Clearly, Y has two arcwise connected components. Then, by Example 2.7, we can construct a quasi-graph X' in $\mathbb{R}^3$ which contains a quasi-arc L with $\omega(L) = Y$.

If $\omega(L)$ contains more than one point, then we call L or φ an *oscillatory quasi-arc*. Obviously, φ is nonoscillatory if and only if $\lim_{t\to\infty}\varphi(t)$ exists. So no quasi-arc in a graph is oscillatory.

Clearly, the circle $\mathbb{S}^1$ is a nonoscillatory quasi-arc. We can choose any point of $\mathbb{S}^1$ as the endpoint. A σ*-graph* is a graph homeomorphic to the subset $X = \{e^{2\pi \mathrm{i}t} | t \in \mathbb{R}\} \cup \{\mathrm{i}t | 1 \leq t \leq 2\}$ of the complex plane. One can easily take a bijective map $\varphi : \mathbb{R}_+ \to X$ such that $\varphi(0) = 2\mathrm{i}, \varphi(1) = \mathrm{i}$ and $\varphi(t) \to \mathrm{i}$ as $t \to \infty$. So the σ*-graph* is a nonoscillatory quasi-arc. We should note that a point of $\omega(L)$ may belong to L, as in the example of the σ-graph. It is easy to see that the Warsaw circle is an oscillatory quasi-arc. Since no continuous injection $\varphi : \mathbb{R}_+ \to [0,1]$ is surjective, an arc cannot be a quasi-arc.

Proposition 2.15. *Let L be a quasi-arc in a compact metric space X, and let $\varphi : \mathbb{R}_+ \to L$ be the corresponding continuous bijection. Suppose that A is an arc in L. Then exactly one of the following holds.*

(a) *There are $s, t \in \mathbb{R}_+$ such that $A = \varphi([s,t])$.*

(b) *$\lim_{t\to\infty}\varphi(t)$ exists; there are real numbers $0 \leq c_0 \leq c_1 < c$ such that $A = \varphi([c,\infty) \cup [c_0,c_1])$ and $\lim_{t\to\infty}\varphi(t) = \varphi(c_0)$ or $\varphi(c_1)$; L is either a circle or a σ-graph; if L is a σ-graph, then $\varphi(0)$ is the unique endpoint of L.*

Proof. Let $W = \varphi^{-1}(A)$. Then W is a closed set in $\mathbb{R}_+$. Since no continuous injection $\phi : [r,\infty) \to A$ is surjective, $W \neq [r,\infty)$ for every $r \in \mathbb{R}_+$. Hence, if W has only one connected component, then W must be a bounded closed interval in $\mathbb{R}_+$. Thus (a) holds.

In the following, we assume that W has at least two components. If each component of W is bounded, then there exists an infinite increasing sequence $r_1 < r_2 < r_3 < \cdots$ such that $\{r_1, r_2, r_3, \cdots\} \subset \mathbb{R}_+ - W$, and $\lim_{n\to\infty} r_n = \infty$. Now, let $W_0 = W \cap [0, r_1]$ and let $W_n = W \cap [r_n, r_{n+1}]$ for each $n \in \mathbb{N}$. Then $\varphi(W_i)$ is a bounded closed set in A for each $i \in \mathbb{Z}_+$. Thus the arc A is the union of countably many pairwise disjoint proper closed subsets $\{\varphi(W_i) : i \in \mathbb{Z}_+\}$. However, it is well known that a compact interval cannot be the disjoint union of its countably many proper closed subsets. So W has an unbounded component, that is, there is $c > 0$ such that $[c,\infty)$ is a connected component of W. Write $W_0 = [c,\infty)$. Since $\varphi(W_0) \subsetneq A$, there is some $v \in A - \varphi(W_0)$ such that $\lim_{t\to\infty}\varphi(t) = v$. Let $c' = \varphi^{-1}(v)$, and let W_1 be the connected component of W containing c'. Then $\varphi(W_0 \cup W_1)$ is a connected closed subset of A.

Assume that $A - \varphi(W_0 \cup W_1) \neq \emptyset$. Then W has more than two connected components, which implies that A is a union of countably many pairwise disjoint proper closed subsets. This leads to a contradiction. Hence W has exactly two connected components W_0 and W_1, and there exist real numbers $0 \le c_0 \le c_1 < c$ such that $W_1 = [c_0, c_1]$ and $c' \in \{c_0, c_1\}$. So we have that $A = \varphi([c,\infty) \cup [c_0, c_1])$ and $\lim_{t\to\infty}\varphi(t) = v = \varphi(c') \in \{\varphi(c_0), \varphi(c_1)\}$.

If $c' = 0$, then $L(= \varphi([0,c]) \cup \varphi([c,\infty)))$ is a circle; if $c' > 0$, then $L(= \varphi([0,c']) \cup \varphi([c',c]) \cup \varphi([c,\infty)))$ is a σ-graph with the unique endpoint $\varphi(0)$. □

Corollary 2.16. *Let L be a quasi-arc in a compact metric space X and let $\varphi : \mathbb{R}_+ \to L$ be the corresponding continuous bijection. Suppose that L' is an arcwise connected subset of L. If L is neither a circle nor a σ-graph, then $\varphi^{-1}(L')$ is an arcwise connected set in $\mathbb{R}_+$.*

Proof. Assume to the contrary that $\varphi^{-1}(L')$ is not arcwise connected; then there is an arc A in L' such that $\varphi^{-1}(A)$ is not arcwise connected. By Proposition 2.15, L is either a circle or a σ-graph, which is a contradiction. So $\varphi^{-1}(L')$ must be arcwise connected. □

Proposition 2.17. *Let X be a compact metric space and let L be a quasi-arc in X. If there are continuous bijections $\varphi : \mathbb{R}_+ \to L$ and $\psi : \mathbb{R}_+ \to L$ such that $\varphi(0) \neq \psi(0)$, then L is a circle.*

Proof. Let $r = \varphi^{-1}(\psi(0))$. Then $\varphi(r) = \psi(0)$, and $r > 0$. Let $a = \psi^{-1}(\varphi(0))$ and let $b = \psi^{-1}(\varphi(2r))$. Then $a > 0$, $b > 0$ and $a \neq b$. Let $A = \psi([a,b])$; then A is an arc in L with endpoints $\psi(a) = \varphi(0)$ and $\psi(b) = \varphi(2r)$. For any $s,t \in \mathbb{R}_+$ with $s \neq t$, set $q = \max\{s,t,2r\}$. Then $\varphi([0,q])$ is an arc in L, and $\varphi([s,t])$ is a subarc of $\varphi([0,q])$. Since the only arc with endpoints $\varphi(0)$ and $\varphi(2r)$ in $\varphi([0,q])$ is $\varphi([0,2r])$, and $\varphi(r) = \psi(0) \notin A$, we have that no subarc of $\varphi([0,q])$ is equal to A. So $\varphi([s,t]) \neq A$. It thus follows from Proposition 2.15 that L is either a circle or a σ-graph. However, if L is a σ-graph, then $\varphi(0)$ and $\psi(0)$ should be the unique endpoint of L by Proposition 2.15. This contradicts the assumption that $\varphi(0) \neq \psi(0)$. Hence L is a circle. □

Proposition 2.18. *Let L be a quasi-arc in a compact metric space X. If L contains a 3-star Y, then L is a σ-graph.*

Proof. Suppose that the center of Y is w, and the endpoints of Y are v_1, v_2 and v_3. Let $\varphi : \mathbb{R}_+ \to L$ be a continuous bijection, and $a_i = \varphi^{-1}(v_i)$ for $i = 1, 2, 3$. Without loss of generality, we may assume that $a_1 < a_2 < a_3$. Let A be the arc in Y with endpoints v_1 and v_3. Similar to the proof of Proposition 2.17, we see that $\varphi([s, t]) \neq A$ for any $s, t \in \mathbb{R}_+$. Then L is a σ-graph by Proposition 2.15. □

It follows from Proposition 2.18 that no quasi-arc in a compact metric space contains an n-star with $n \geq 4$ or contains more than one 3-star.

From Proposition 2.15, we see that a nonoscillatory quasi-arc can only be a half-open arc, a circle, or a σ-graph. Explicitly, we have the following.

Proposition 2.19. *Let L be a nonoscillatory quasi-arc in a compact metric space X. Suppose that $\omega(L) = \{x\}$ for some $x \in X$. If $x \notin L$, then $L \cup \{x\}$ is an arc and L is a half-open arc; if x is an endpoint of L, then L is a circle; if $x \in L$, but does not belong to the endpoint set of L, then L is a σ-graph.* □

Proposition 2.20. *Let L be a quasi-arc in a compact metric space X. Suppose that L is neither a circle nor a σ-graph. If $\varphi : \mathbb{R}_+ \to L$ and $\psi : \mathbb{R}_+ \to L$ are two continuous bijections, then $\psi^{-1}\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ and $\varphi^{-1}\psi : \mathbb{R}_+ \to \mathbb{R}_+$ are two homeomorphisms which are inverses of each other.*

Proof. Since L is neither a circle nor a σ-graph, $L \cup \omega(L)$ is an arc when L is nonoscillatory. In this case, the proposition obviously holds. In the following, we assume that L is oscillatory. Let $h = \psi^{-1}\varphi : \mathbb{R}_+ \to \mathbb{R}_+$; then $h^{-1} = \varphi^{-1}\psi$. By Proposition 2.17, L has only one endpoint. Thus $\varphi(0) = \psi(0)$, or $h(0) = 0$. For every $r > 0$, let $A_r = \varphi([0, r])$. Then A_r is an arc in L with endpoints $\psi(0) = \varphi(0)$ and $\psi(h(r)) = \varphi(r)$. Write $J_r = \psi^{-1}(A_r)$. We have

Claim 1. $J_r = [0, h(r)]$.

In fact, if $J_r \neq [0, h(r)]$, then for any $s, t \in \mathbb{R}_+$, we have $A_r \neq \psi([s, t])$. Thus L is either a circle or a σ-graph by Proposition 2.15, which contradicts the assumption that L is oscillatory. So $J_r = [0, h(r)]$.

Since $\varphi|_{[0,r]} : [0, r] \to A_r$ is a homeomorphism, and $\psi|_{J_r} : J_r \to A_r$ is also a homeomorphism by Claim 1, we have

Claim 2. For every $r > 0$, $h([0, r]) = [0, h(r)]$, and $h|_{[0,r]} : [0, r] \to [0, h(r)]$ is a homeomorphism.

By Claim 2, we get that $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a homeomorphism. □

It follows from Proposition 2.20 that if a quasi-arc L is neither a circle nor a σ-graph, then the continuous bijection $\varphi : \mathbb{R}_+ \to L$ is unique up to a topological conjugacy.

Definition 2.21. Suppose that L and K are two quasi-arcs in a compact metric space X with the corresponding bijections $\varphi : \mathbb{R}_+ \to L$ and $\psi : \mathbb{R}_+ \to K$ respectively. For every $r \in \mathbb{R}_+$, denote $L_r = \varphi([r, \infty))$ and $K_r = \psi([r, \infty))$. We write $L \Rightarrow K$ or $\varphi \Rightarrow \psi$ if there exists $m \in \mathbb{Z}_+$ such that $K_m \subset \omega(\varphi)$, and we write $L \succ K$ or $\varphi \succ \psi$ if $K \subset \omega(L)$.

Obviously we have

Lemma 2.22. *Let L, K, φ, and ψ be as in Definition* 2.21*.*

(1) *If $L \Rightarrow K$, then L is oscillatory.*

(2) *For every $r \in \mathbb{R}_+$, L_r and K_r are also quasi-arcs and $\omega(L_r) = \omega(L)$.*

(3) *$L \Rightarrow K$ if and only if there is $r \in \mathbb{R}_+$ such that $L_r \succ K_r$.*

(4) *$\psi^{-1}(\omega(L) \cap K) = \psi^{-1}(\omega(L))$ is a closed set in $\mathbb{R}_+$.* □

We study quasi-arcs in a quasi-graph in the following.

Lemma 2.23. *Let L and K be quasi-arcs in a quasi-graph X and let $\varphi : \mathbb{R}_+ \to L$ and $\psi : \mathbb{R}_+ \to K$ be the corresponding continuous bijections respectively. If $L \not\Rightarrow K$, then $\psi^{-1}(\omega(L) \cap K)$ is a bounded set in $\mathbb{R}_+$.*

Proof. Assume to the contrary that $L \not\Rightarrow K$ and $\psi^{-1}(\omega(L))$ is an unbounded set in $\mathbb{R}_+$. Then there are $\{r_1, r_2, \cdots\} \subset \mathbb{R}_+ - \psi^{-1}(\omega(L))$ such that $0 < r_1 < r_2 < r_3 < \cdots$, $\lim_{n\to\infty} r_n = \infty$, and $[r_n, r_{n+1}] \cap \psi^{-1}(\omega(L)) \neq \emptyset$ for all $n \in \mathbb{N}$. Let N be the separation degree of quasi-graph X. Since X has at most $N - 2$ branch points, there is $i \in \mathbb{N}$ such that $(K_{r_i} \cup L_{r_i}) \cap \mathrm{Br}(X) = \emptyset$. Let $J = K_{r_i} \cap L_{r_i}$, then J is arcwise connected. By Corollary 2.16, $\varphi^{-1}(J)$ and $\psi^{-1}(J)$ are also arcwise connected. If $\varphi^{-1}(J)$ is bounded, then there exists $s \geq r_i$ such that $L_s \cap K_{r_i} = \emptyset$, which implies that

$$(\overline{L}_s - L_s) \cap K_{r_i} = \omega(L_s) \cap K_{r_i} = \omega(L) \cap K_{r_i}$$

contains infinitely many arcwise connected components. Since $K_{r_i} \cap \mathrm{Br}(X) = \emptyset$, $\overline{L}_s - L_s$ also has infinitely many arcwise connected components. This contradicts the definition of quasi-graph. So $\varphi^{-1}(J)$ is unbounded. Then there are $s > r_i$ and $t > r_{i+N+1}$ such that $L_s = K_t$. Thus $L_s \cap \psi([r_i, t)) = \emptyset$, which implies that

$$(\overline{L}_s - L_s) \cap \psi([r_i, t)) = \omega(L_s) \cap \psi([r_i, t)) = \omega(L) \cap \psi([r_i, t))$$

has at least $N+1$ arcwise connected components. As $\psi([r_i, t)) \cap B_r(X) = \emptyset$, $\overline{L}_s - L_s$ also has at least $N + 1$ arcwise connected components. However, this contradicts the fact that the separation degree of X is N. Hence $\psi^{-1}(\omega(L) \cap K)$ must be a bounded set provided that $L \not\Rightarrow K$. □

Now we prove the main theorem in this section, which explicitly describes the structures of quasi-graphs.

Theorem 2.24. *A continuum X is a quasi-graph if and only if there are a graph G and n pairwise disjoint oscillatory quasi-arcs $L_1, \cdots, L_n$ in X, for some $n \in \mathbb{Z}_+$, such that*

(1) *$X = G \cup (\bigcup_{i=1}^n L_i)$, and $\mathrm{End}(X) \cup (\bigcup\{\mathrm{St}(x, \varepsilon_0) : x \in \mathrm{Br}(X)\}) \subset G$ for some $\varepsilon_0 > 0$,*

(2) *$L_i \cap G = \{a_i\}$ for each $1 \leq i \leq n$, where a_i is the endpoint of L_i,*

(3) *$\omega(L_i) \subset G \cup (\bigcup_{j=1}^{i-1} L_j)$ for each $1 \leq i \leq n$, and*

(4) *if $\omega(L_i) \cap L_j \neq \emptyset$ for some $i, j \in \{1, \cdots, n\}$, then $\omega(L_i) \supset L_j$.*

In addition, if X is a quasi-graph with separation degree N, then the number n appearing above is less than or equal to N.

Proof. Clearly, if X contains a graph G and n pairwise disjoint oscillatory quasi-arcs $L_1, \cdots, L_n$ satisfying the conditions (1) to (4), then X is a quasi-graph. We need only prove the converse direction. Let X be a quasi-graph with separation degree N. By Lemma 2.9 and Lemma 2.4, there exists a tree T_0 in X such that $\mathrm{End}(X) \cup \mathrm{Br}(X) \subset T_0$. By Lemma 2.10, there is an $\varepsilon_0 > 0$ such that $\mathrm{St}(x, \varepsilon_0)$ is an open star for all $x \in \mathrm{Br}(X)$, and $\overline{T}$ is still a tree in X, where $T = T_0 \cup (\bigcup\{\mathrm{St}(x, \varepsilon_0) : x \in \mathrm{Br}(X)\})$. If $X = \overline{T}$, then Theorem 2.24 holds for $G = \overline{T}$ and $n = 0$. In the following, we assume that $X \neq \overline{T}$. By Lemma 2.11, $X - T$ has only finitely many arcwise connected components $L_1, \cdots, L_m$ with some $m \in \{1, \cdots, N\}$. For each $1 \leq i \leq m$, as $\mathrm{Br}(X) \subset T_0 \subset T$, we see that L_i contains no branch points of X. Thus, for any $x, y \in L_i$ with $x \neq y$, there is a unique arc $L_i[x, y]$ in L_i with endpoints x and y. Since X is arcwise connected and $L_1, \cdots, L_m$ are pairwise disjoint, we have $L_i \cap \overline{T} = \mathrm{End}(L_i) \cap (\mathrm{End}(\overline{T}) - \mathrm{End}(T)) \neq \emptyset$. As $\mathrm{End}(X) \subset T_0 \subset T$, L_i contains more than one point. Since $\mathrm{End}(\overline{T}) \cap \mathrm{Br}(X) = \emptyset$, L_i cannot be a circle. If $L_i \cap \overline{T}$ contains two points, denoted by a_i and b_i, then L_i must be an arc with endpoints a_i and b_i, and $\overline{T} \cap L_i - \{a_i, b_i\} = \emptyset$. So we may suppose that, for some $n \in \{0, 1, \cdots, m\}$, $L_i \cap \overline{T}$ contains only one point a_i for each $1 \leq i \leq n$ and $L_i \cap \overline{T}$ contains two points for each $n+1 \leq i \leq m$. Let $G = \overline{T} \cup (\bigcup_{i=n+1}^{m} L_i)$. Then G is a graph in X, and $\mathrm{End}(G) \subset \mathrm{End}(\overline{T})$. If $n = 0$, then $X = G$, and Theorem 2.24 holds. Now assume that $n \in \{1, \cdots, m\}$. Then $X = G \cup (\bigcup_{i=1}^{n} L_i)$ and $L_i \cap G = \{a_i\}$ for each $i \in \{1, \cdots, n\}$, where a_i is an endpoint of G. Thus the conclusions (1) and (2) of Theorem 2.24 hold.

Claim 1. For each $i \in \{1, \cdots, n\}$, there is a continuous bijection $\varphi_i : \mathbb{R}_+ \to L_i$ such that $\varphi_i(0) = a_i$ and $\lim_{t\to\infty} \varphi_i(t)$ does not exist.

As the long line is also an arcwise connected space without branch points (see [22, p. 159]), Claim 1 is not that obvious. So we need to give a proof here.

Proof of Claim 1. Let $\prec_i$ be a linear ordering on L_i defined by $a_i \prec_i x \prec_i y$ if and only if $x \in L_i[a_i, y)$ for any $x, y \in L_i - \{a_i\}$. If L_i has a maximal element, say b_i, then $L_i = [a_i, b_i]$ and b_i is an endpoint of X. This contradicts the fact that $L_i \cap \mathrm{End}(X) = \emptyset$. So L_i has no maximal element with respect to the ordering $\prec_i$. Since X is compact, for every $k \in \mathbb{N}$, there is a finite set V'_k in X such that $X = B(V'_k, 2^{-k-1})$. Thus, there is a finite set V_k in L_i such that $B(V_k, 2^{-k}) \supset L_i$. It follows that there exists a sequence of points $a_i \prec_i x_1 \prec_i x_2 \prec_i x_3 \prec_i \cdots$ in L_i such that $L_i \subset B(L_i[a_i, x_k], 2^{-k})$ for all $k \in \mathbb{N}$. Let $L'_i = \cup_{k=1}^{\infty} L_i[a_i, x_k]$. Evidently, there is a continuous bijection $\varphi_i : [0, \infty) \to L'_i$ with $\varphi_i(0) = a_i$ and $\varphi_i(k) = x_k$ for all $k \in \mathbb{N}$. If $L'_i \neq L_i$, then there exists $y \in L_i - L'_i$. Since L has no maximal element, there is $y' \in L_i$ such that $y \prec_i y'$. Thus we have $L'_i \subset L_i[a_i, y] \subset L_i[a_i, y')$. Let $\delta = d(y', L_i[a_i, y])$. Then $\delta > 0$ by the compactness of $L_i[a_i, y]$. Take $k \in \mathbb{N}$ such that $2^{-k} < \delta$. Then $y' \notin B(L_i[a_i, y], 2^{-k})$, which contradicts the fact that $L_i \subset B(L_i[a_i, x_k], 2^{-k})$ described above. Hence, there must be $L'_i = L_i$. To complete the proof of Claim 1, assume that $\lim_{t\to\infty} \varphi_i(t)$ exists and write $b_i = \lim_{t\to\infty}\varphi_i(t)$. If $b_i \in \mathrm{End}(G)$, then $L_i \cup \{b_i\}$ is an arc with endpoints a_i and b_i, and $L_i \cup \{b_i\}$ is an arcwise connected component of $X - T$, which leads to a contradiction. If $b_i \notin \mathrm{End}(G)$, then b_i is a branch point of X, and $L_i \cap \mathrm{St}(b_i, \varepsilon_0) \neq \emptyset$. However, this contradicts the fact that $L_i \subset X - T \subset X - \mathrm{St}(b_i, \varepsilon_0)$. Hence $\lim_{t\to\infty} \varphi_i(t)$ does not exist. This completes the proof of Claim 1.

From Claim 1, we see that $L_1, \cdots, L_n$ are all oscillatory quasi-arcs. For any $i, j \in \{1, \cdots, n\}$, if $L_i \Rightarrow L_j$, then by (3) of Lemma 2.22, there exists $r \in \mathbb{R}_+$ such that $\varphi_i([r, \infty)) \succ \varphi_j([r, \infty))$. In this case, we replace G, L_i and L_j by $G \cup \varphi_i([0, r]) \cup \varphi_j([0, r])$, $\varphi_i([r, \infty))$ and $\varphi_j([r, \infty))$ respectively. If $L_i \not\Rightarrow L_j$ and $\omega(L_i) \cap L_j \neq \emptyset$, then by Lemma 2.23 there is $r \in \mathbb{R}_+$ such that $\omega(L_i) \cap \varphi_j([r,\infty)) = \emptyset$. In this case, we may replace G and L_j by $G \cup \varphi_j([0,r])$ and $\varphi_j([r,\infty))$ respectively. So, without loss of generality, we always assume that $L_1, \cdots, L_n$ possess the following property.

Property A. For any $i, j \in \{1, \cdots, n\}$, if $\omega(L_i) \cap L_j \neq \emptyset$, then $\omega(L_i) \supset L_j$.

Claim 2. For every $i \in \{1, \cdots, n\}$, $L_i \not\Rightarrow L_i$.

Proof of Claim 2. Let $\varphi_i : \mathbb{R}_+ \to L_i$ be as in Claim 1. For every $r \in \mathbb{R}_+$, denote $L_i[0,r] = \varphi_i([0,r])$ and $L_i[r,\infty) = \varphi_i([r,\infty))$. Note that $G \cup (\bigcup_{j=1}^n L_j[0,r])$ is a compact set in X and is a graph. If there is $i \in \{1, \cdots, n\}$ such that $L_i \Rightarrow L_i$, i.e., $\omega(L_i) \supset L_i[r,\infty)$ for some $r \in \mathbb{R}_+$, then we can choose orderly points $x_0 = \varphi_i(0)$, $x_1 = \varphi_i(r+3), x_2, x_3, \cdots$ in L_i and positive even integers $k_0 = 2 < k_1 < k_2 < k_3 < \cdots$ (the order of choice is $k_0 \to x_0 \to k_1 \to x_1 \to k_2 \to x_2 \to k_3 \to x_3 \to \cdots$) such that

$$x_1 \in L_i[k_0+1, k_1], \; d(x_1, L_i[k_1+1, k_2]) < 1/3,$$

$$x_{m+1} \in L_i[k_m+1, k_{m+1}] \text{ and } d(x_{m+1}, x_m) < \delta_m/3, \text{ for } m = 1, 2, \cdots, \tag{2.1}$$

where

$$\delta_m = d(x_m, G_m) \text{ with } G_m = G \cup (\bigcup_{j=1}^n L_j[0, k_{m-1}]). \tag{2.2}$$

For $m \in \mathbb{N}$, denote $\varepsilon_m = d(x_m, x_{m-1})$. It follows from (2.2) and (2.1) that

$$x_{m-1} \in G_m, \; \varepsilon_{m+1} < \delta_m/3 \le \varepsilon_m/3 \tag{2.3}$$

and

$$B(x_m, \delta_m) \cap G_m = \emptyset. \tag{2.4}$$

Thus the sequence $x_0, x_1, x_2, x_3, \cdots$ converges to a point y in X with

$$d(y, x_m) \le \sum_{\mu=m}^{\infty} d(x_\mu, x_{\mu+1}) = \sum_{\mu=m}^{\infty} \varepsilon_{\mu+1} < \delta_m/2, \text{ for all } m \in \mathbb{N}. \tag{2.5}$$

By (2.5) and (2.4), we have $y \notin G_m$ for all $m \in \mathbb{N}$. So $y \notin \bigcup_{m=1}^\infty G_m = X$, which is a contradiction. Hence, for every $i \in \{1, \cdots, n\}$, we have $L_i \not\Rightarrow L_i$. This completes the proof of Claim 2.

According to Claim 2, we see that in the set of quasi-arcs $\{L_1, \cdots, L_n\}$, the relations $\Rightarrow$ and $\succ$ are both strict partial orderings (see Munkres [22, p. 69]). Thus, by adjusting the order of $L_1, \cdots, L_n$, we may assume that the following property holds.

Property B. When $1 \le i \le j \le n$, $L_i \not\Rightarrow L_j$.

By Property B and Property A, we see that conditions (3) and (4) of the theorem hold. This completes the proof of Theorem 2.24. □

Definition 2.25. Let L and K be quasi-arcs in a compact metric space X, neither of which is a circle or a σ-graph, and let $\varphi : \mathbb{R}_+ \to L$ and $\psi : \mathbb{R}_+ \to K$ be the corresponding continuous bijections. L and K are said to be *eventually same* if there exists $\{s, t\} \in \mathbb{R}_+$ such that $\varphi([s,\infty)) = \psi([t,\infty))$. K is said to be *eventually homeomorphic to* $\mathbb{R}_+$ if there exists $t \in \mathbb{R}_+$ such that $\psi|_{[t,\infty)} : [t,\infty) \to \psi([t,\infty))$ is a homeomorphism.

The following two propositions can be easily deduced from Theorem 2.24.

Proposition 2.26. *Let X be a quasi-graph. Suppose that G and $L_1, \cdots, L_n$ are as in Theorem 2.24. Then for every oscillatory quasi-arc K in X, there always exists a unique $i \in \{1, \cdots, n\}$ such that K and L_i are eventually same.* □

Proposition 2.27. *Every oscillatory quasi-arc K in a quasi-graph X is eventually homeomorphic to $\mathbb{R}_+$.* □

Remark 2.28. Proposition 2.27 does not hold for general compact metric spaces. For example, let $X = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-torus, let $\pi : \mathbb{R}^2 \to X$ be the natural projection, and let $J = \{(t, rt) \in \mathbb{R}^2 : t \in \mathbb{R}_+\}$ where r is an irrational number. Then J is a ray starting from the origin in $\mathbb{R}^2$. Let $K = \pi(J)$. Then K is a one-way infinite helix in X and is a quasi-arc. It is obvious that $\omega(K) = X \supset K$ and K is not eventually homeomorphic to $\mathbb{R}_+$.

Definition 2.29. A nonoscillatory quasi-arc in a compact metric space X is called a *0-order oscillatory quasi-arc*. An oscillatory quasi-arc L is called a *k-order oscillatory quasi-arc* for some $k \in \mathbb{N}$ if $\omega(L)$ contains at least one $(k-1)$-order oscillatory quasi-arc, and $\omega(K)$ contains no $(k-1)$-order oscillatory quasi-arc for every quasi-arc K in $\omega(L)$.

From this definition, it is easy to see that for any two integers $k > i \geq 0$ and any k-order oscillatory quasi-arc L, $\omega(L)$ always contains at least one i-order oscillatory quasi-arc and cannot contain an oscillatory quasi-arc with order $\geq k$.

By Theorem 2.24 and Proposition 2.26, we have

Proposition 2.30. *Let X be a quasi-graph with separation degree N. Suppose that G and $L_1, \cdots, L_n$ are as in Theorem 2.24. Then no oscillatory quasi-arc in X is of $(n+1)$-order. In particular, every oscillatory quasi-arc in X has order $\leq N-1$.* □

Let X be a quasi-graph. From Proposition 2.26, it is easy to see that the number n appearing in Theorem 2.24 is just the maximal number of pairwise disjoint oscillatory quasi-arcs in X. We call this number the *oscillation degree* of X and denote it by $n(X)$.

Proposition 2.31. *Suppose that X is a quasi-graph with oscillation degree $n(X) = n$. Then every compact connected set W in X has at most $n+1$ arcwise connected components.*

Proof. Let $G, L_1, \cdots, L_n$ and the continuous bijections $\varphi_i : \mathbb{R}_+ \to L_i$, $i = 1, \cdots, n$, be as in Theorem 2.24. Note that $0 \leq n \leq N$, where N is the separation degree of X.

Claim 1. If there exist $i \in \{0, 1, \cdots, n\}$ and $\{r_{i+1}, \cdots, r_n\} \subset \mathbb{R}_+$ such that $W \subset X - \bigcup_{j=i+1}^{n} L_j(r_j, \infty)$, then W has at most $i+1$ arcwise connected components.

Proof of Claim 1. Let $X_i = X - \bigcup_{j=i+1}^{n} L_j(r_j, \infty)$. According to Theorem 2.24, we see that X_i is a quasi-graph in X and the maximal number of pairwise disjoint oscillatory quasi-arcs in X_i is i. Since X_0 is a graph, every connected set in X_0 is also arcwise connected. Then Claim 1 holds for $i = 0$.

In the following, we assume that $i > 0$ and $W \cap L_i[t, \infty) \neq \emptyset$ for all $t \in \mathbb{R}_+$. It follows from (3) and (4) of Theorem 2.24 and the connectivity of W that there exists $r_i \in \mathbb{R}_+$ such that $L_i[r_i, \infty) \subset W$. Note that $\omega(L_i)$ is a connected closed set contained in $W \cap (G \cup (\bigcup_{j=1}^{i-1} L_j))$. Let $W' = W - L_i(r_i, \infty)$. By (3) of Theorem 2.24, we see that W' is a closed set. Let W_1 and W_2 be the connected components of W' which contain $\varphi_i(r_i)$ and $\omega(L_i)$ respectively. Since $d(L_i[t_1, t_2], W') > 0$ for any $t_2 > t_1 > r_i$, we have $W' = W_1 \cup W_2$; that is, W' has no other connected components except W_1 and W_2. If $W_1 = W_2$, i.e., W' is connected, then let $W'' = W'$. If $W_1 \neq W_2$, then take an arc A in X such that $W' \cup A$ is connected and $W' \cap A = \text{End(A)}$; in this case, let $W'' = W' \cup A$. Obviously, W'' is a compact connected set in $X_{i-1} \equiv X - \bigcup_{j=i}^{n} L_j\,(r_i, \infty)$. By induction, we can assume that W'' has at most i arcwise connected components. Thus, there are two cases: if $W_1 = W_2$, then W' and W also have at most i arcwise connected components; if $W_1 \neq W_2$, then W' and W have at most $i+1$ arcwise connected components. The proof of Claim 1 is complete.

Proposition 2.31 follows from Claim 1 immediately. □

Theorem 2.32. *A nondegenerate compact arcwise connected metric space X is a quasi-graph if and only if $\overline{Y} - Y$ has only finitely many arcwise connected components for every arcwise connected set Y in X.*

Proof. By Definition 2.1, the necessity is obvious. Now we prove the sufficiency. Suppose that $\overline{Y} - Y$ has only finitely many arcwise connected components for every arcwise connected set Y in X.

Claim 1. If X has only finitely many branch points, and $\sum\{\text{val}(w) : w \in \text{Br}(X)\} < \infty$, then X is a quasi-graph.

Proof of Claim 1. Suppose that X has finitely many branch points. Then the endpoint set of X is also finite. Let $\{v_1, \cdots, v_n\}$ be the branch point set of X. For each $i = 1, \cdots, n$, take a star T_i with center v_i such that $\text{val}(v_i, T_i) = \text{val}(v_i, X)$. Then we can take a graph G in X such that $(\bigcup_{i=1}^{n} T_i) \cup \text{End}(X) \subset G$. It follows that for every point $x \in X \setminus G$ there are a unique $v_x \in \text{End}(G)$ and a unique arc $[v_x, x]$ such that $[v_x, x] \cap G = \{v_x\}$. Clearly, for any two distinct points $x, y \in X \setminus G$, exactly one of the following three cases occurs: $[v_x, x] \subset [v_y, y]$; $[v_y, y] \subset [v_x, x]$; $[v_x, x] \cap [v_y, y] = \emptyset$, by the definition of G. Thus $(X \setminus G) \cup \{v_x : x \in X \setminus G\}$ can be partitioned into finitely many quasi-arcs (must be oscillatory) with endpoints in $\text{End}(G)$. So X is a quasi-graph.

Claim 2. X has only finitely many branch points, and $\sum\{\text{val}(w) : w \in \text{Br}(X)\} < \infty$.

Proof of Claim 2. If Claim 2 does not hold, then exactly one of the following two cases will occur.

Case 1. There exists $w \in X$ such that $\text{val}(w) = \infty$. In this case, there are infinitely many arcs $A_1, A_2, \cdots$ such that they share a common endpoint w: $A_i \cap A_j = \{w\}$ for any $1 \leq i < j < \infty$, and $A_{i+1} \subset B(w, d(v_i, w)/2)$ for all $i \in \mathbb{N}$, where v_i is the other endpoint of A_i besides w. Let $S = \bigcup_{i=1}^{\infty}(A_i - \{v_i\})$; then $\overline{S} - S = \{v_i : i \in \mathbb{N}\}$. Thus $\overline{S} - S$ has infinitely many connected components. This contradicts the initial assumption. So, Case 1 does not occur.

Case 2. X has infinitely many branch points and each branch point of X has finite valence. In this case, we claim that

(∗) there is a continuous injection $\varphi : \mathbb{R}_+ \to X$ such that $\{\varphi(i) : i \in \mathbb{N}\} \subset \text{Br}(X)$.

We postpone the proof of $(*)$ to the next paragraph. Now suppose $(*)$ holds. Denote $L = \varphi(\mathbb{R}_+)$, and $v_i = \varphi(i)$ for all $i \in \mathbb{Z}_+$. For each $i \in \mathbb{N}$, take an arc A_i with an endpoint v_i such that $A_i \cap (L \cup \mathrm{Br}(X)) = \{v_i\}$, and $A_i \cap A_j = \emptyset$ for any $1 \leq i < j < \infty$. Let w_i be the other endpoint of A_i besides v_i and let $Y = L \cup (\bigcup_{i=1}^{\infty}(A_i - \{w_i\}))$. Then $\{w_i : i \in \mathbb{N}\} \subset \overline{Y} - Y$. By the previous assumption, $\overline{Y} - Y$ has only finitely many arcwise connected components. Thus, there exist an arcwise connected component W of $\overline{Y} - Y$ and positive integers $i_1 < i_2 < i_3 < \cdots$ such that $\{w_{i_n} : n \in \mathbb{N}\} \subset W$. Let $W_1 = W - \{w_{i_n} : n \in \mathbb{N}\}$. Since $w_{i_n} \notin \mathrm{Br}(X)$ for every $n \in \mathbb{N}$, W_1 is arcwise connected. (Otherwise, take an arc A in W such that the two endpoints of A belong to different arcwise connected components of W_1. Then there exists $k \in \mathbb{N}$ such that $w_{i_k} \in A - \mathrm{End}(A)$, which implies that $\mathrm{val}(w_{i_k}) \geq 3$. This contradicts the fact that $w_{i_k} \notin \mathrm{Br}(X)$.) Let $Y_1 = Y \cup W_1 \cup \{w_{i_1}\}$. Then Y_1 is arcwise connected, and $\overline{Y_1} - Y_1 = \overline{Y} - Y - W_1 - \{w_{i_1}\}$. So, $\{w_{i_n}\}$ is an arcwise connected component of $\overline{Y_1} - Y_1$ for each $n \geq 2$. This contradicts the assumption at the beginning of the proof. So, Case 2 does not occur.

Now we start to prove $(*)$. This obviously holds if there exists an arc A in X such that A contains infinitely many branch points. So, we may assume that there is no arc in X containing infinitely many branch points. Let $\{x_i\}_{i=1}^{\infty}$ be a sequence of branch points and let $v_0 = x_0$. Fix an arc $[v_0, x_i]$ for each i. Since $\mathrm{val}(v_0) < \infty$ and no arc contains infinitely many branch points, there is a star Y_0 with center v_0 such that $\mathrm{val}(v_0, Y_0) = \mathrm{val}(v_0, X)$ and $Y_0 \cap \mathrm{Br}(X) = \{v_0\}$. This implies that there is an infinite set $B_0 \subset \{x_i\}_{i=1}^{\infty}$ such that $\bigcap_{x \in B_0}[v_0, x]$ contains an arc $[v_0, v_1]$ for some $v_1 \neq v_0$. Clearly, we may take v_1 to be a branch point. Similarly to the case of v_0, we can get a branch point $v_2 \neq v_1$ and an infinite set $B_1 \subset B_0$ such that $[v_0, v_1] \subset [v_0, v_2] \subset \bigcap_{x \in B_1}[v_0, x]$. Going on in this way, we get a sequence of branch points $\{v_i\}_{i=0}^{\infty}$ and a strictly increasing arc sequence $[v_0, v_1] \subset [v_0, v_2] \subset \cdots$. This obviously shows the existence of a continuous injection $\varphi : \mathbb{R}_+ \to X$ with $\varphi(\mathbb{R}_+) = \bigcup_{i=1}^{\infty}[v_0, v_i]$ and $v_i = \varphi(i)$ for all $i \in \mathbb{Z}_+$.

Hence, Claim 2 holds.

It follows from Claim 2 and Claim 1 that X is a quasi-graph. □

3. Some basic properties of quasi-graph maps

A continuous map from a quasi-graph to itself is called a *quasi-graph map*. In this section, we are mainly interested in quasi-graph maps.

Lemma 3.1. *Let X be a quasi-graph. Suppose that G is a graph in X and $f \in C^0(X)$. Then $f(G)$ contains no oscillatory quasi-arcs.*

Proof. Let L be an oscillatory quasi-arc in X, and let $\varphi : \mathbb{R}_+ \to L$ be a continuous bijection. Take $x, y \in \omega(L)$ with $x \neq y$ and positive numbers $s_1 < t_1 < s_2 < t_2 < s_3 < t_3 < \cdots$ such that $\lim_{n\to\infty} s_n = \infty$, $\lim_{n\to\infty}\varphi(s_n) = x$, $\lim_{n\to\infty}\varphi(t_n) = y$, and $\varphi([s_1, \infty)) \cap \mathrm{Br}(X) = \emptyset$. Let $\varepsilon = d(x, y)/3$. Assume that $f(G) \supset L$. For every $n \in \mathbb{N}$, take a point $x_n \in f^{-1}(\varphi(s_n))$ and take a point $y_n \in f^{-1}(\varphi(t_n))$. Since G is compact and locally connected, for every $\delta > 0$, there exist integers $j > k > 0$ such that

(i) $\varphi(s_k) \in B(x, \varepsilon)$, $\varphi(t_k) \in B(y, \varepsilon)$;

(ii) there exists an arc $A = A[x_k, x_j]$ in G such that $A \subset B(x_k, \delta)$.

So, there is a point $w_k \in A$ such that $f(w_k) = \varphi(t_k)$. Noting that $d(w_k, x_k) < \delta$ and $d(f(w_k), f(x_k)) = d(\varphi(t_k), \varphi(s_k)) > \varepsilon$, we know that f is not uniformly continuous. This is a contradiction. Hence $f(G) \not\supseteq L$. The proof is complete. □

By Lemma 3.1 and Theorem 2.24, we have

Corollary 3.2. *Let X be a quasi-graph. Suppose that G is a graph in X and $f \in C^0(X)$. If $f(G)$ contains more than one point, then $f(G)$ is also a graph in X.* □

Corollary 3.3. *Suppose that L and K are both oscillatory quasi-arcs in a quasi-graph X, $\varphi : \mathbb{R}_+ \to L$ and $\psi : \mathbb{R}_+ \to K$ are the corresponding continuous bijections, and $f \in C^0(X)$. If $f(L) \supset K$, then*

(1) for every $r > 0$, there is $t = t(r) \geq 0$ such that $f(L[r, \infty)) \supset K[t, \infty)$, and

(2) $f(\omega(L)) \supset \omega(K)$.

Proof. (1) For every $r > 0$, since $L[0, r]$ is an arc in X, $f(L[0, r])$ contains no oscillatory quasi-arcs by Lemma 3.1. Hence, by Theorem 2.24, there is $t = t(r) \geq 0$ such that $f(L[r, \infty)) \supset K[t, \infty)$.

(2) For every $\varepsilon > 0$, take $r = r(\varepsilon) > 0$ such that $L[r, \infty) \subset B(\omega(L), \varepsilon)$. Let $t = t(r) \geq 0$ be as above. Then

$$f(\overline{B(\omega(L), \varepsilon)}) \supset \overline{K[t, \infty)} \supset \omega(K).$$

It follows that $f(\omega(L)) \supset \omega(K)$ by the arbitrariness of ε. □

Proposition 3.4. *Let X be a quasi-graph. Suppose that L is a k-order oscillatory quasi-arc for some $k \geq 0$ and $f \in C^0(X)$. Then $f(L)$ contains no oscillatory quasi-arcs with order $> k$.*

Proof. When $k = 0$, L is a nonoscillatory quasi-arc. So $\overline{L}$ is a graph ($\overline{L}$ can only be an arc, a circle, or a σ-graph). By Lemma 3.1, $f(\overline{L})$ (and thus $f(L)$) contains no oscillatory quasi-arcs. Hence, the proposition holds for $k = 0$.

In the following, we assume $k > 0$. By Definition 2.29, $\omega(L)$ contains oscillatory quasi-arcs with order from 0 to $k - 1$, but contains no oscillatory quasi-arcs with order $> k - 1$. Let K be a j-order oscillatory quasi-arc contained in $f(L)$. If $j > k$, then $\omega(K)$ contains a k-order oscillatory quasi-arc. From Corollary 3.3, we have that $f(\omega(L)) \supset \omega(K)$. By Proposition 2.31, $\omega(L)$ has only finitely many arcwise connected components. So, $\omega(L)$ is a union of finitely many oscillatory quasi-arcs with order $\leq k - 1$ and finitely many graphs in X. By induction, we can assume that the f-image of an oscillatory quasi-arc with order $\leq k - 1$ contains no oscillatory quasi-arc with order $> k - 1$. This together with Lemma 3.1 implies that $f(\omega(L))$ contains no oscillatory quasi-arcs with order greater than $k - 1$. We get a contradiction. So, there must be $j \leq k$. The proof is complete. □

Proposition 3.5. *Let X be a quasi-graph. Suppose that L and K are k-order oscillatory quasi-arcs in X for some $k \geq 1$, $\varphi : \mathbb{R}_+ \to L$ and $\psi : \mathbb{R}_+ \to K$ are two fixed continuous bijections, and $f \in C^0(X)$. If $f(L) \supset K$, then there are r, $s \in \mathbb{R}_+$ such that $f(L[r, \infty)) = K[s, \infty)$, $\psi^{-1} f \varphi | [r, \infty) : [r, \infty) \to [s, \infty)$ is continuous, and $\lim_{t \to \infty} \psi^{-1} f \varphi(t) = \infty$. In addition, $f(\omega(L)) = \omega(K)$.*

Proof. It follows from Proposition 2.27, Theorem 2.24 and Proposition 2.26 that there exists $s \in \mathbb{R}_+$ such that $\psi | [s, \infty) : [s, \infty) \to K[s, \infty)$ is a homeomorphism and $K[s, \infty) \cap (\omega(K) \cup \mathrm{Br}(X)) = \emptyset$. Let $S = \varphi^{-1} f^{-1} \psi(s)$. Then S is a nonempty closed set in $\mathbb{R}_+$. Assume that S is unbounded; then by Lemma 3.1 there are positive integers $t_1 < s_1 < t_2 < s_2 < t_3 < s_3 < \cdots$ such that $\lim_{i\to\infty} t_i = \infty$, $\lim_{i\to\infty} \varphi(t_i) = x$ for some $x \in \omega(L)$, $f\varphi(t_i) \in K(s,\infty)$, and $s_i \in S$ for all $i \in \mathbb{N}$. Thus $\lim_{i\to\infty} f\varphi(t_i) = f(x) \in \omega(K)$, and $f(L[t_i, s_i]) \supset K[\psi(s), f\varphi(t_i)]$, which imply $f(L[t_i,\infty)) \supset K[s,\infty)$, for all $i \in \mathbb{N}$. For every $\varepsilon > 0$, there is $j = j(\varepsilon) \in \mathbb{N}$ such that $L[t_j,\infty) \subset B(\omega(L),\varepsilon)$. So we have $f(\overline{B(\omega(L),\varepsilon)}) \supset f(L[t_i,\infty)) \supset K[s,\infty)$. It follows that $f(\omega(L)) \supset K[s,\infty)$. (Otherwise, there exist $y \in K[s,\infty)$ and $\delta > 0$ such that $d(y, f(\omega(L))) > \delta$. By the uniform continuity of f, there is an $\varepsilon > 0$ such that $f(B(\omega(L),\varepsilon)) \subset B(f(\omega(L)),\delta)$. It follows that $y \notin f(B(\omega(L),\varepsilon))$, which leads to a contradiction.) On the other hand, $\omega(L)$ contains no oscillatory quasi-arcs with order $> k-1$, and $K[s,\infty)$ is a k-order oscillatory quasi-arc. Thus we have $f(\omega(L)) \not\supseteq K[s,\infty)$ by Proposition 2.31, Lemma 3.1 and Proposition 3.4. This is a contradiction. So S is a nonempty bounded closed set in $\mathbb{R}_+$. Set $r = \max(S)$; then $f(L[r,\infty)) = K[s,\infty)$. Since $\psi^{-1}|K[s,\infty) : K[s,\infty) \to [s,\infty)$ is a homeomorphism, $\psi^{-1}f\varphi|[r,\infty) : [r,\infty) \to [s,\infty)$ is continuous. For every $s' \in [s,\infty)$, let $S' = \varphi^{-1}f^{-1}\psi([s,s'])$. Similarly to the above, it can be shown that S' is a nonempty bounded closed set in $\mathbb{R}_+$. Hence, we have $\lim_{t\to\infty} \psi^{-1}f\varphi(t) = \infty$.

In addition, for every $x \in \omega(L)$, take $\{t_1, t_2, \cdots\} \subset [r,\infty)$ such that $\lim_{i\to\infty} t_i = \infty$ and $\lim_{i\to\infty}\varphi(t_i) = x$. Let $s_i = \psi^{-1}f\varphi(t_i)$. Then $\lim_{i\to\infty} s_i = \infty$ and $\lim_{i\to\infty}\psi(s_i) = \lim_{i\to\infty} f\varphi(t_i) = f(x)$, which implies that $f(x) \in \omega(K)$, and $f(\omega(L)) \subset \omega(K)$. Combining with (2) of Corollary 3.3, we have $f(\omega(L)) = \omega(K)$. This completes the proof of Proposition 3.5. □

4. Periodic points and ω-limit points of quasi-graph maps

In this section, we study the dynamical systems generated by the iteration of quasi-graph maps and give some criteria to determine when a point is a periodic point or a ω-limit point for a quasi-graph map.

Theorem 4.1. *Let X be a quasi-graph, let $f \in C^0(X)$, and let $v \in X$. If there are an arc A with an endpoint v and an arcwise connected set W such that $\bigcup_{i=1}^\infty f^i(W) \supset A - \{v\}$, $\bigcup_{i=1}^n f^i(W) \not\supseteq A - \{v\}$ for all $n \in \mathbb{N}$, and $f^\mu(W) \cap f^\lambda(W) \neq \emptyset$ for some integers $\mu > \lambda \geq 0$, then $v \in P(f)$.*

Proof. For every subset Z of X, let $c(Z)$ denote the number of arcwise connected components of Z. Let $Y = \bigcup_{i=1}^\infty f^i(W)$, let $Y_n = \bigcup_{i=1}^n f^i(W)$, and let $X_n = Y - Y_n$ for all $n \in \mathbb{N}$. Then we have

$$\mu \geq \max\{c(Y_1), \cdots, c(Y_\mu)\} \geq c(Y_{\mu+1}) \geq c(Y_{\mu+2}) \geq \cdots \geq c(Y).$$

It follows from Corollary 2.12 that $c(X_n) < c(Y)\cdot(N + c(Y_n)) \leq \mu(N+\mu)$, where N is the separation degree of X. Since there are at most N pairwise disjoint oscillatory quasi-arcs and $\mathrm{Br}(X)$ contains at most $N-2$ points, there exists an integer $\mu' \geq \mu$ such that, for all $n \geq \mu'$, the largest number of pairwise disjoint oscillatory quasi-arcs in Y_n is equal to that in $Y_{\mu'}$, $c(Y_n) = c(Y_{\mu'})$ and $X_{\mu'} \cap \mathrm{Br}(X) = \emptyset$. This implies $c(X_{\mu'}) \geq c(X_{\mu'+1}) \geq c(X_{\mu'+2}) \geq \cdots$. Hence, there is an integer $\mu'' \geq \mu'$ such that $c(X_n) = c(X_{\mu''})$ for all $n \geq \mu''$. Let $k_0 = c(Y)$. Take $k_0 - 1$ arcs $A_1, \cdots, A_{k_0-1}$ in X such that $Y \cup (\bigcup_{i=1}^{k_0-1} A_i)$ is arcwise connected and $Y \cap (\bigcup_{i=1}^{k_0-1}(A_i - \mathrm{End}(A_i))) = \emptyset$.

Noting that $c(Y_{\mu''} \cup (\bigcup_{i=1}^{k_0-1} A_i)) \le c(Y_{\mu''}) + k_0 - 1$, we get from Corollary 2.12 that

$$\begin{aligned} c(X_{\mu''}) &= c(Y - Y_{\mu''}) \\ &= c((Y \cup (\bigcup_{i=1}^{k_0-1} A_i)) - (Y_{\mu''} \cup (\bigcup_{i=1}^{k_0-1} A_i))) \\ &\le N + c(Y_{\mu''}) + k_0 - 2 \\ &\le N + 2\mu - 2. \end{aligned}$$

Set $k = c(X_{\mu''})$. For $n \ge \mu''$, let $L_{n1}, \cdots, L_{nk}$ be the arcwise connected components of X_n. Since $X_{n+1} \subset X_n$, we may assume that $L_{(n+1)i} \subset L_{ni}$ for each $i = 1, \cdots, k$. In addition, we may as well assume that $L_{\mu''1} \cap A \neq \emptyset$ and $(\bigcup_{i=2}^{k} L_{\mu''i}) \cap (A - \{v\}) = \emptyset$.

Claim 1. For each $i = 1, \cdots, k$, there are an integer $m = m_i \ge \mu''$, a continuous injection $\varphi_i : \mathbb{R}_+ \to L_{mi}$ and real numbers $0 = r_{mi} \le r_{(m+1)i} \le r_{(m+2)i} \le \cdots$ such that $\lim_{n\to\infty} r_{ni} = \infty$, and $\varphi_i((r_{ni}, \infty)) \subset L_{ni} \subset \varphi_i([r_{ni}, \infty))$ for all $n \ge m$.

Proof of Claim 1. Note that L_{ni} is an arcwise connected set containing no branch points of X for all $n \ge \mu''$. If $L_{\mu''i}$ contains an oscillatory quasi-arc, then set $m = m_i = \mu''$ and set $r_{mi} = 0$. By Theorem 2.24, we see that there is a continuous injection $\varphi_i : \mathbb{R}_+ \to L_{mi}$ such that $\varphi_i((0, \infty)) \subset L_{mi} \subset \varphi_i([0, \infty))$. For $n \ge m$, since the largest number of pairwise disjoint oscillatory quasi-arcs in X_n is equal to that in X_m, L_{ni} also contains an oscillatory quasi-arc. Hence, there is $r_{ni} \in \mathbb{R}^+$ such that $\varphi_i((r_{ni}, \infty)) \subset L_{ni} \subset \varphi_i([r_{ni}, \infty))$. As $L_{(n+1)i} \subset L_{ni}$, we know that $r_{(n+1)i} \ge r_{ni}$. Since $\bigcap_{n=m}^{\infty} L_{ni} = \emptyset$, $\lim_{n\to\infty} r_{ni} = \infty$.

If $L_{\mu''i}$ contains no oscillatory quasi-arcs, then $\overline{L_{\mu''i}}$ is an arc and $L_{\mu''i} \supset \overline{L_{\mu''i}} - \mathrm{End}(\overline{L_{\mu''i}})$. As $\bigcap_{n=\mu''}^{\infty} L_{ni} = \emptyset$, it is easy to see that there exists $m \ge \mu''$ such that, for every $n \ge m$, $L_{mi} - L_{ni}$ contains at most one connected component, and there exist a continuous injection $\varphi_i : \mathbb{R}_+ \to L_{mi}$ and real numbers $0 = r_{mi} \le r_{(m+1)i} \le r_{(m+2)i} \le \cdots$ satisfying the conditions of the claim.

Thus the proof of Claim 1 is complete.

Let $q = \max\{m_1, \cdots, m_k\}$. By Claim 1, for every $n \ge q$ and every $\lambda \in \{1, \cdots, k\}$, $L_{n\lambda}$ is either a quasi-arc or a quasi-arc without its endpoint. For any $i, j \in \{1, \cdots, k\}$, define $L_{qi} \Rightarrow L_{qj}$ if there is $n' \ge q$ such that $f(L_{ni}) \supset L_{n'j}$ for every $n \ge q$. It is easy to see that $\Rightarrow$ so defined is a transitive relation on $\{L_{q1}, \cdots, L_{qk}\}$. Suppose that L_{qi} is a_i-orderly oscillatory. Then $a_1 = 0$. We may as well assume that $a_k \ge a_{k-1} \ge \cdots \ge a_2 \ge a_1$. By Propositions 3.4 and 3.5, we have

Claim 2. If $L_{qi} \Rightarrow L_{qj}$, then $a_i \ge a_j$.

Claim 3. If $L_{qi} \Rightarrow L_{qj}$ and $L_{qi} \Rightarrow L_{qj'}$ with $j \neq j'$, then $a_i > \max\{a_j, a_{j'}\}$.

Noting that $f(X_n) \supset X_{n+1}$ for all $n \ge q$, we have

Claim 4. For every $j \in \{1, \cdots, k\}$, there exists at least one $i \in \{1, \cdots, k\}$ such that $L_{qi} \Rightarrow L_{qj}$.

When $L_{qi} \Rightarrow L_{qj}$, we say that L_{qi} *can flow to* L_{qj} *under* f, or L_{qj} *can be filled with* L_{qi}.

Claim 2, Claim 3 and Claim 4 show that if some L_{qi} flows to two different L_{qj} and $L_{qj'}$, then it can only flow to the one with lower oscillatory order, and it can only be filled with some $L_{qi'}$ with equal or greater oscillatory order. In the end, there must be some $L_{qi''}$ which cannot be filled with any $L_{qi'''}$. This leads to a contradiction. Hence, we have

Claim 5. If $L_{qi} \Rightarrow L_{qj}$ and $L_{qi} \Rightarrow L_{qj'}$, then $j = j'$ and $a_i = a_j$.

By Claim 4 and Claim 5, we have

Claim 6. There is a unique bijection $\zeta : \{1, \cdots, k\} \to \{1, \cdots, k\}$ such that $L_{qi} \Rightarrow L_{qj}$ if and only if $j = \zeta(i)$, for all $i \in \{1, \cdots, k\}$.

It is known by Claim 6 that there exists $\lambda \in \{1, \cdots, k\}$ such that $\zeta^\lambda(1) = 1$. Thus we have

Claim 7. For every $n \geq q$, there is $n' \geq q$ such that $f^\lambda(L_{n1}) \supset L_{n'1}$.

From Claim 7, we have $v \in \overline{L_{n'1}} \subset f^\lambda(\overline{L_{n1}})$. Since $\operatorname{diam}(\overline{L_{n1}}) \to 0$ and $\operatorname{diam}(f^\lambda(\overline{L_{n1}})) \to 0$ as $n \to \infty$, we have $f^\lambda(v) = v$. The proof of Theorem 4.1 is complete. □

Corollary 4.2. *Let X be a quasi-graph. Suppose that W is an arcwise connected closed set in X, $v \in W$, $f \in C^0(X)$, and $Y = \bigcup_{n=1}^\infty f^n(W)$. If $Y \cap \operatorname{St}(v, \varepsilon) \neq \emptyset$ for every $\varepsilon > 0$, and $f^\mu(W) \cap f^\lambda(W) \neq \emptyset$ for some integers $\mu > \lambda \geq 0$, then $v \in Y$.*

Proof. By the conditions of the corollary, we see that Y has at most μ arcwise connected components, and there exists an arc A in X with v being one of its endpoints and with $A - \{v\} \subset Y$. For every $n \in \mathbb{N}$, write $Y_n = \bigcup_{i=1}^n f^i(W)$. If $A - \{v\} \not\subseteq Y_n$ for every $n \in \mathbb{N}$, then v is a periodic point of order m for some $m \in \mathbb{N}$ by Theorem 4.1, which implies that $v = f^m(v) \in f^m(W) \subset Y$. If there is some $n \in \mathbb{N}$ such that $A - \{v\} \subset Y_n$, then we also have $v \in Y_n \subset Y$ by the closedness of Y_n. □

Theorem 4.3. *Let X be a quasi-graph, let $f \in C^0(X)$, and let $v \in X$. If there is an arc A in X with v being one of its endpoints, such that for every $x \in A - \{v\}$ there exist $y_x \in A[x, v)$ and $n_x \in \mathbb{N}$ such that $f^{n_x}(y_x) \in A[x, v)$, then $v \in \omega(f)$. Furthermore, if $v \in \omega(f) \setminus P(f)$, then there exist $w \in A - \{v\}$ and positive integers $m_1 < m_2 < m_3 < \cdots$ such that $\{f^{m_i}(w) : i \in \mathbb{N}\} \subset A - \{v\}$ and $\lim_{i \to \infty} f^{m_i}(w) = v$.*

Proof. If $v \in P(f)$, then $v \in \omega(f)$. So we may assume that $v \notin P(f)$ in what follows. For any $x, y \in A$, if $y \in A(x, v)$, then we write $x < y < v$. Thus we get naturally a linear ordering $<$ on A. Assume that $\operatorname{Br}(X) \cap A - \{v\} = \emptyset$. (Otherwise, take a subarc of A instead of A.)

Claim 1. For any $w_0 \in A - \{v\}$, there exist $\{x_0, y_0, w_1\} \subset A - \{v\}$ and $m_1 \in \mathbb{N}$ such that $w_0 \leq x_0 < y_0 < w_1$, $d(w_1, v) < d(w_0, v)/2$, and $f^{m_1}(A[x_0, y_0]) = A[w_1, v]$.

Proof of Claim 1. Set $W = A[w_0, v]$, $Y = \bigcup_{i=1}^\infty f^i(W)$, $Y_n = \bigcup_{i=1}^n f^i(W)$, and $\mu = \min\{n_x : x \in W\}$. Then $f^\mu(W) \cap W \neq \emptyset$, and thus Y has at most μ arcwise connected components. So there exists $u \in A(w_0, v)$ such that $A[u, v) \subset Y$ by the conditions of the theorem. Then it follows from Theorem 4.1 and the assumption $v \notin P(f)$ that there exists $m \in \mathbb{N}$ such that $A[u, v) \subset Y_m$. Since Y_m is closed, $A[u, v] \subset Y_m$. If for every $i \in \{1, \cdots, m\}$ and every $x \in W \cap f^{-i}(v)$ there is $\varepsilon_{ix} > 0$ such that $f^i(B(x, \varepsilon_{ix}) \cap W) \cap A[u, v] = \{v\}$, then there is $u' \in A[u, v)$ such that

$A[u', v) \cap Y_m = \emptyset$, which is a contradiction. Hence there exists $m_1 \in \mathbb{N}_m$ and $z \in W \cap f^{-m_1}(v)$ such that

$$f^{m_1}(B(z, \varepsilon) \cap W) \cap A[u, v) \neq \emptyset \quad \text{for every } \varepsilon > 0. \tag{4.1}$$

Take $\varepsilon_0 > 0$ such that $\varepsilon_0 + \text{diam}(f^{m_1}(B(z, \varepsilon_0))) < d(\{z, w_0\}, v)/2$. By (4.1), we know that there exist $\{x_0, y_0\} \subset B(z, \varepsilon_0) \cap W$ and $w_1 \in A[u, v)$ such that $\{f^{m_1}(x_0), f^{m_1}(y_0)\} = \{w_1, v\}$ and $f^{m_1}(A[x_0, y_0]) = A[w_1, v]$. The proof of Claim 1 is complete.

According to Claim 1, there exist $\{w_i, x_i, y_i : i \in \mathbb{Z}_+\} \subset A - \{v\}$ and integers $0 = m_0 < m_1 < m_2 < m_3 < \cdots$ such that $w_0 \le x_0 < y_0 < w_1 \le x_1 < y_1 < w_2 \le x_2 < y_2 < w_3 \le \cdots$, $\lim_{i\to\infty} d(w_i, v) = 0$ and

$$f^{m_{i+1}-m_i}(A[x_i, y_i]) = A[w_{i+1}, v] \quad \text{for all } i \in \mathbb{Z}_+.$$

Take an arbitrary point w from $\bigcap_{i=0}^{\infty} f^{-m_i}(A[x_i, y_i])$. Then w together with m_1, m_2, $m_3 \cdots$ meets the requirements of the theorem. Thus the proof of Theorem 4.3 is complete. □

Theorem 4.4. *Let X be a quasi-graph, let $f \in C^0(X)$, and let $v \in X$ with* $\text{val}(v) = k$*. If for every $n \in \mathbb{N}$, there exist $y_n \in \text{St}(v, 1/n)$ and positive integers $m_{1n} < m_{2n} < \cdots < m_{kn}$ such that $\{f^{m_{in}}(y_n) : i = 1, \cdots, k\} \subset \text{St}(v, 1/n)$, then $v \in \omega(f)$.*

Proof. Take $c > 0$ such that $\overline{\text{St}(v, c)} \cap \text{Br}(X) - \{v\} = \emptyset$. Let $m_{0n} = 0$. Write $y_{in} = f^{m_{in}}(y_n)$. We may as well assume that $\{y_{in} : i = 0, \cdots, k,$ and $n \in \mathbb{N}\} \subset \text{St}(v, c)$. Suppose that $w_1, \cdots, w_k$ are k endpoints of $\overline{\text{St}(v, c)}$. Let $\mathbb{M} = \{n \in \mathbb{N} :$ there are $\lambda \in \{1, \cdots, k\}$ and $i \neq j \in \{0, \cdots, k\}$ with $\{y_{in}, y_{jn}\} \subset [w_\lambda, v)\}$. If $\mathbb{M}$ is an infinite set, then Theorem 4.4 holds by Theorem 4.3. So, we assume that $\mathbb{M}$ is a finite set in what follows. For convenience, we may further assume that $\mathbb{M} = \emptyset$. Thus, for every $n \in \mathbb{N}$, we have $v \in \{y_{jn} : j = 0, \cdots, k\}$ and $[w_i, v) \cap \{y_{jn} : j = 0, \cdots, k\}$ contains just one point for every $i \in \{1, \cdots, k\}$.

Let $\mathbb{M}_1 = \{n \in \mathbb{N} : v \neq y_{kn}\}$. If $\mathbb{M}_1$ is an infinite set, then $v \in R(f) \subset \omega(f)$ by the fact that $\{y_{kn} : n \in \mathbb{M}_1\} \subset O(v, f)$, and the theorem holds. So we may assume that $\mathbb{M}_1$ is a finite set. For convenience, we may further assume that $\mathbb{M}_1 = \emptyset$, that is, $y_{kn} = v$ for every $n \in \mathbb{N}$. In addition, we may also assume that $y_{(i-1)n} \in [w_i, v)$ for all $i \in \{1, \cdots, k\}$ and for all $n \in \mathbb{N}$.

If $v \in R(f)$, then the theorem holds. Now assume that $v \notin R(f)$. Then $d(v, O(f(v), f)) > 0$. We may as well assume that $c < d(v, O(f(v), f))$. Let $\mu_{in} = m_{kn} - m_{in}$. Then for every $n \in \mathbb{N}$ and every $i \in \{0, \cdots, k-1\}$, there is $j_{in} \in [0, \cdots, k-1]$ such that $f^{\mu_{in}}([y_{in}, v]) \supset [w_{j_{in}}, v]$. It follows that, for every $n \in \mathbb{N}$, there exist $i_n \in \{0, \cdots, k-1\}$ and $\mu_n \in \mathbb{N}$ such that $f^{\mu_n}([y_{i_n n}, v]) \supset [w_{i_n}, v]$. Thus, we get

Claim 1. There is $q \in \{0, \cdots, k-1\}$ such that for any $u_0 \in [w_q, v)$ there exist $\{x_0, y_0, u_1\} \subset [w_q, v)$ and $\beta_1 \in \mathbb{N}$ satisfying $x_0 \in [u_0, y_0)$, $y_0 \in (x_0, v)$, $d(u_1, v) < d(u_0, v)/2$, and $f^{\beta_1}([x_0, y_0]) = [u_1, v]$.

Similarly to the proof of Theorem 4.3, it can be deduced from Claim 1 that $v \in \omega(f)$. Thus Theorem 4.4 is proved. □

According to Theorem 4.3 and Theorem 4.4, we have

Corollary 4.5. *Let X be a quasi-graph. Suppose that $f \in C^0(X)$ and $v \in X - \bigcup\{\,\omega(L) : L$ is an oscillatory quasi-arc in $X\}$ with $\mathrm{val}(v) = k$. Then the following three items are equivalent.*

(i) *$v \in \omega(f)$.*

(ii) *At least one of the following two items holds:*

(a) *$v \in P(f)$.*

(b) *For every $\varepsilon > 0$, there exist an arc A in $B(v, \varepsilon)$ with v being an endpoint and a point $x \in A - \{v\}$ with $O(f(x), f) \cap A - \{v\} \neq \emptyset$.*

(iii) *For every $\varepsilon > 0$, there exist an $x \in B(v, \varepsilon)$ and positive integers $m_1 < m_2 < \cdots < m_k$ such that $\{f^{m_i}(x) : i = 1, \cdots, k\} \subset B(v, \varepsilon)$.* □

By Corollary 4.5, we have

Corollary 4.6. *Let X be a quasi-graph, let $f \in C^0(X)$, and let $L_1, \cdots, L_n$ be as in Theorem 2.24. Suppose that each oscillatory quasi-arc L_i is of k_i-order, $k_m = k_{m+1} = \cdots = k_n$ for some $m \in \{1, \cdots, n\}$, and $v \in \bigcup_{i=m}^{n} L_i$. Then $\mathrm{val}(v) = 2$ and the conditions (i), (ii) and (iii) in Corollary 4.5 are equivalent.* □

Theorem 4.7. *Suppose that G is a graph, $f \in C^0(G)$, and $v \in G$ with $\mathrm{val}(v) = k$. Then the conditions (i), (ii) and (iii) in Corollary 4.5 are equivalent.* □

We remark that Chinen proved in [13] that conditions (iii) and (i) of Theorem 4.7 are equivalent, and part (b) of condition (ii) implies (i).

For a quasi-graph map $f : X \to X$, each of Theorem 4.3 and Theorem 4.4 gives a sufficient condition for a point v to be in $\omega(f)$. However, these conditions are not necessary in general as the following example shows.

Example 4.8. Suppose that $\mathbb{S}^1 = \{e^{\mathbf{i}\theta} : \theta \in \mathbb{R}\}$ is the unit circle in the complex plane $\mathbb{C}$. Let $H = \{(e^{2\pi \mathbf{i} t},\ 1/t) : t \in [1, \infty)\}$ be a helical curve in $\mathbb{C} \times \mathbb{R}$. Then $\overline{H} = H \cup \mathbb{S}^1$. Let A be an arc in $\mathbb{C} \times \mathbb{R}$ such that $A \cap \overline{H} = \mathrm{End}(A) = \{(1+0\mathbf{i}, 1), (1+0\mathbf{i}, 0)\}$. Then $X = A \cup \overline{H}$ is a quasi-graph. Let $p : \mathbb{C} \times \mathbb{R} \to \mathbb{R}$, $(s, t) \mapsto t$, be the natural projection. Obviously, there exists a homeomorphism $h : X \to X$ satisfying the following two conditions:

(i) $\mathrm{Fix}(h) = A$.

(ii) For every $x \in H - A$, $p(h(x)) < p(x)$.

It is easy to show that $\omega(h) \subset \mathbb{S}^1 \cup A$, $\omega(h) - A \neq \emptyset$, and for every point $v \in \omega(h) - A (\subset \mathbb{S}^1 - A)$ there always exists $\varepsilon_v > 0$ such that $\{n \in \mathbb{Z} : h^n(y) \in \mathrm{St}(v, \varepsilon_v)\} = \{0\}$ for every $y \in \mathrm{St}(v, \varepsilon_v)$.

The following corollary can be deduced immediately from Theorem 4.7, which was given by Blokh in [6] (see also [20]).

Corollary 4.9. *Let G be a graph and let $f \in C^0(G)$. Then the set of accumulation points of $\Omega(f)$ is contained in the ω-limit set of f, and hence $\omega(f)$ is a closed set in G.* □

From Corollary 4.9, we know that $\overline{P(f)} \subset \overline{R(f)} \subset \overline{\omega(f)} = \omega(f)$ for every graph map f. However, it does not hold for general quasi-graph maps as the following example shows.

Example 4.10. Let $J = [-1, 1]$. Suppose that $A_0 = \{0\} \times J$, $A_n = \{(t, \cos(\pi/t)) \in \mathbb{R}^2 : 1/(n+1) \leq t \leq 1/n\}$ for all $n \in \mathbb{N}$, and $Y = \bigcup_{i=0}^{\infty} A_i$. Then Y is homeomorphic to the $\sin(1/t)$-continuum. Take an arc A in $\mathbb{R}^2$ such that $A \cap Y = \mathrm{End}(A) = \{(0,-1),(1,-1)\}$. Let $X = Y \cup A$. Then X is a Warsaw circle. Set $a_0 = -1, a_1 = 0, a_2 = 1/4, a_3 = 1/2$, and $a_4 = 1$. For each $i \in \mathbb{Z}_+$, let $\varphi_i : J \to J$ be the map such that $\varphi_i(a_0) = -1, \varphi_i(a_4) = 1, \varphi_i(a_3) = 0, \varphi_0(a_1) = \varphi_0(a_2) = 1$, $\varphi_i(a_1) = \varphi_i(a_2) = 1 - 2^{-i}$, and $\varphi_i|[a_{j-1}, a_j]$ is linear for each $j = 1, \cdots, 4$ (see Figure 1).

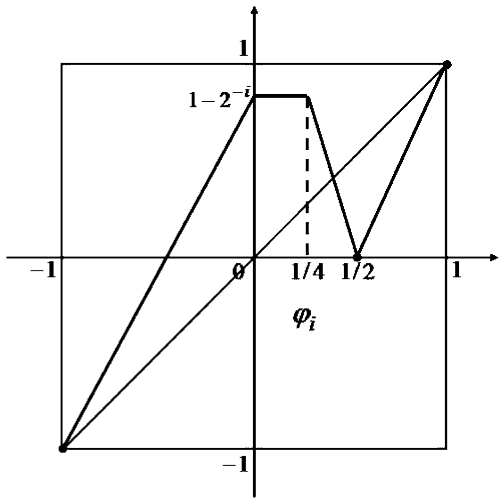

FIGURE 1

For every $i \in \mathbb{Z}_+$, define a homeomorphism $h_i : A_i \to J$ by

$$h_i(r,s) = s, \qquad \text{for all } (r,s) \in A_i.$$

Define a map $f : X \to X$ by $f(x) = x$ for every $x \in A$ and $f|A_i = h_i^{-1}\varphi_i h_i$ for every $i \in \mathbb{Z}_+$. Then f is continuous. It is easy to verify that $h_i^{-1}(0) \in P_{i+1}(f) \cap A_i$ for all $i \in \mathbb{N}$, $h_0^{-1}(0) = \lim_{i\to\infty} h_i^{-1}(0) \in \overline{P(f)} \cap A_0$, and $h_0^{-1}(0) \notin \omega(f)$. Hence, the closures of $R(f)$ and $P(f)$ are not contained in $\omega(f)$, the set of accumulation points of $\Omega(f)$ is not contained in the ω-limit set of f, and $\omega(f)$ is not a closed set.

5. Recurrent points of quasi-graph maps

As a slight modification of the notion of limit points, we introduce the following.

Definition 5.1. Let X be a metric space. A point v in X is said to be a *connectivity limit point* (resp. *arcwise connectivity limit point*) of a point sequence $x_1, x_2, \cdots$ in X if for every $\varepsilon > 0$ and every $m \in \mathbb{N}$ there exists a connected set (resp. arcwise connected set) W_ε such that $v \in W_\varepsilon \subset B(v,\varepsilon)$ and $W_\varepsilon \cap \{x_m, x_{m+1}, \cdots\} \neq \emptyset$. Similarly, a point v in X is said to be a *connectivity limit point* (resp. *arcwise connectivity limit point*) of a subset Y of X if for every $\varepsilon > 0$ there exists a connected set (resp. arcwise connected set) W_ε such that $v \in W_\varepsilon \subset B(v,\varepsilon)$ and $Y \cap W_\varepsilon - \{v\} \neq \emptyset$.

Obviously, if X is a locally connected (resp. locally arcwise connected) space, then there is no difference between the notion of connectivity limit point (resp. arcwise connectivity limit point) and the notion of limit point. Specifically, this is the case when X is a graph.

A point sequence $x_1, x_2, \cdots$ is said to be *connectedly (resp. arcwise connectedly) convergent* if there exists a unique limit point of the sequence in the sense of connectivity (resp. arcwise connectivity).

A connectivity (resp. arcwise connectivity) limit point of a set Y is also called a *connectivity (resp. arcwise connectivity) accumulation point* of Y. Denote by $\mathrm{Clp}(Y)$ (resp. $\mathrm{Aclp}(Y)$) the set of all accumulation points of Y in the sense of connectivity (resp. arcwise connectivity). Write

$$\mathrm{Cpclos}(Y) = Y \cup \mathrm{Clp}(Y) \quad \text{and} \quad \mathrm{Acpclos}(Y) = Y \cup \mathrm{Aclp}(Y).$$

The set $\mathrm{Cpclos}(Y)$ (resp. $\mathrm{Acpclos}(Y)$) is called the *pseudo-closure in the sense of connectivity (resp. arcwise connectivity)* of Y. If $\mathrm{Acpclos}(Y) = Y$, then Y is called a *pseudo-closed set in the sense of arcwise connectivity.* Obviously, for any $Y \subset X$, $\mathrm{Acpclos}(Y)$ is always pseudo-closed in the sense of arcwise connectivity.

In Example 4.8, observe that for every $v \in \omega(h) - A$, there is an $x \in X$ such that the sequence $x, h(x), h^2(x), \cdots$ is contained in the oscillatory quasi-arc H, and v is a limit point of this sequence. But there is no $y \in X$ such that v is a connectivity limit point of the sequence $y, h(y), h^2(y), \cdots$. So, for a general quasi-graph map f, "(a) or (b)" in Corollary 4.5 (ii) is only a sufficient condition but not a necessary condition for $v \in \omega(f)$.

Definition 5.2. Let X be a quasi-graph. Suppose that L is an oscillatory quasi-arc in X and $\varphi : \mathbb{R}_+ \to L$ is the corresponding continuous bijection. A point sequence $x_1, x_2, x_3, \cdots$ in X is said to be cofinal with L if there exist $m \in \mathbb{N}$ and positive numbers $t_0 < t_1 < t_2 < \cdots$ such that $x_{m+i} = \varphi(t_i)$ for all $i \in \mathbb{Z}_+$ and $\lim_{i\to\infty} t_i = \infty$.

Obviously, we have

Lemma 5.3. *Let L be an oscillatory quasi-arc in a quasi-graph X and let Y be an arcwise connected set in X. If Y has an infinite point sequence cofinal with L, then $Y \supset L[x, \infty)$ for some $x \in L$.* □

Theorem 5.4. *Let X be a quasi-graph and let $f \in C^0(X)$. Then for every $v \in X$ the following three items are equivalent:*

(i) *$v \in R(f)$.*

(ii) *There is $\varepsilon_0 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0]$, $B(v, \varepsilon) \cap O(f(v), f) = \mathrm{St}(v, \varepsilon) \cap O(f(v), f) \neq \emptyset$.*

(iii) *v is an arcwise connected limit point of the point sequence $v, f(v), f^2(v), \cdots$.*

Proof. (ii)⇒(iii)⇒(i) is obvious.

Suppose that the graph G, the quasi-arcs $L_1, \cdots, L_n$, and the corresponding bijections $\varphi_i : \mathbb{R}_+ \to L_i, i = 1, \cdots, n$, are as in Theorem 2.24 and its proof. If (i)⇒(ii) does not hold, then there exist $v \in R(f) - P(f)$, $\mu_0 \in \{1, \cdots, n\}$, and integers $m_1 < m_2 < m_3 < \cdots$ greater than n such that $f^{m_1}(v), f^{m_2}(v), f^{m_3}(v), \cdots$ is cofinal with L_{μ_0} and $\lim_{i\to\infty} f^{m_i}(v) = v$. For every $i \in \mathbb{Z}_+$, denote $v_i = f^i(v)$. For $j = 1, \cdots, n$, by Lemma 3.1, there exists $\mu_j \in \{1, \cdots, n\}$ such that the sequence $v_{m_1-j}, v_{m_2-j}, v_{m_3-j}, \cdots$ has a convergent subsequence cofinal with L_{μ_j}. For convenience, we may as well assume that the sequence $v_{m_1-j}, v_{m_2-j}, v_{m_3-j}, \cdots$ itself is cofinal with L_{μ_j} and converges to a point v_{-j}. Obviously, $f^j(v_{-j}) = v$ and $v_{-j} \in \omega(f)$. Let k_j be the oscillatory order of quasi-arc L_{μ_j} for $j = 1, \cdots, n$, and let k be the oscillatory order of quasi-arc L_{μ_0}. It follows from Proposition 3.4 that $k_j \geq k$. If $k_j > k$, then use v_{-j} in place of v. Thus, we may as well assume that

$k_j = k$ for $j = 1, \cdots, n$. So, for $0 \le i < j \le n$, we get from Proposition 3.5 and Corollary 3.2 that there exists a graph G_{ij} in X such that

$$f^{j-i}(L_{\mu_j}) \subset L_{\mu_i} \cup G_{ij}. \tag{5.1}$$

Since $\{\mu_0, \mu_1, \cdots, \mu_n\} \subset \{1, \cdots, n\}$, there are $0 \le i_0 < j_0 \le n$ with $\mu_{j_0} = \mu_{i_0}$. Let $q = j_0 - i_0$. By (5.1), we have $\mu_q = \mu_0$ and

$$f^q(L_{\mu_0}) \subset L_{\mu_0} \cup G_{0q}. \tag{5.2}$$

Because $f^q(L_{\mu_0})$ contains a sequence $v_{m_1}, v_{m_2}, v_{m_3}, \cdots$ cofinal with L_{μ_0}, there is $w \in L_{\mu_0}$ such that $f^q(L_{\mu_0}) \supset L_{\mu_0}[w, \infty)$. Thus $f^q(\omega(L_{\mu_0})) = \omega(L_{\mu_0}[w, \infty)) = \omega(L_{\mu_0})$ by Proposition 3.5, which together with $v \in \omega(L_{\mu_0})$ implies that $O(v, f^q) \subset \omega(L_{\mu_0})$ and $O(v, f) \subset Y$, where $Y = \bigcup_{i=0}^{q-1} f^i(\omega(L_{\mu_0}))$. On the other hand, by Definition 2.29 and Proposition 3.4, we know that Y contains no quasi-arc with order $> k-1$. So Y cannot contain the sequence $v_{m_1}, v_{m_2}, v_{m_3}, \cdots$ cofinal with the k-order quasi-arc L_{μ_0}, which contradicts the initial assumption. Hence (i) implies (ii). This completes the proof of Theorem 5.4. □

Theorem 5.4 shows that if v is a recurrent point of f on a quasi-graph X, then the orbit $O(v, f)$ can only pass through the "arcwise connected component neighborhood" $\mathrm{St}(v, \varepsilon)$ of v (not an oscillatory quasi-arc of X) to return to a sufficiently small neighborhood of v.

By Theorem 5.4 and Theorem 4.3, we immediately get

Corollary 5.5. *Let X be a quasi-graph, let $f \in C^0(X)$, and let $v \in X$. If for every $\varepsilon > 0$, $\mathrm{St}(v, \varepsilon) \cap R(f) \neq \emptyset$, then $v \in \omega(f)$, that is, the pseudo-closure in the sense of arcwise connectivity of $R(f)$ is contained in $\omega(f)$.*

Let X be a quasi-graph. Denote $\omega(X) = \cup\{\omega(L) : L$ is an oscillatory quasi-arc in $X\}$. From Corollary 5.5, we get the following corollary whose special case was given by Blokh in [6] (see also [20]).

Corollary 5.6. *Let X be a quasi-graph and let $f \in C^0(X)$. Then $\overline{R(f)} - \omega(X) \subset \omega(f)$. Specifically, if f is a graph map, then $\overline{R(f)} \subset \omega(f)$.* □

It is well known that $\overline{P(f)} = \overline{R(f)}$ if f is an interval map (see [15, 26, 32]), a tree map (see [6, 31]), or a Warsaw circle map (see [30]). In the following, we will prove that $\overline{P(f)} = \overline{R(f)}$ when f is a quasi-tree map. First, we have

Proposition 5.7. *Let X be a metric space, let $f \in C^0(X)$, and let $v \in R(f)$. Suppose that there is no Jordan curve containing v in X, there exists an arc A with v being an endpoint and with $O(v, f) \cap A \cap B(v, \varepsilon) - \{v\} \neq \emptyset$ for every $\varepsilon > 0$, and there is no 3-star in X with center in $A - \{v\}$. Then $v \in \overline{P(f)}$.*

Proof. Let w be the endpoint of A other than v. For every $i \in \mathbb{Z}_+$, denote $v_i = f^i(v)$. If $v \notin \overline{P(f)}$, then there exists $\varepsilon \in (0, d(w, v)/2]$ such that $B(v, 2\varepsilon) \cap P(f) = \emptyset$, and there exist integers $n > k > 0$ such that $v_k \in B(v, \varepsilon) \cap A - \{v\}$ and $v_n \in A(v, v_k) \subset B(v, \varepsilon)$. Let $Y = (\bigcup_{i=0}^{\infty} f^{ki}(A[v, v_k])) \cup (\bigcup_{i=0}^{\infty} f^{i(n-k)}([v_k, v_n])) \cup A$. Then Y is arcwise connected. For every $x \in A(v, v_k]$, let Y_{x1} and Y_{x2} be the arcwise connected components of Y containing v and w respectively. Since X has no Jordan curve containing v and has no 3-star with center in $A - \{v\}$, we have that $Y_{x1} \neq Y_{x2}$ and they are the only two arcwise connected components of $Y - \{x\}$. Let Y_{v_2} be the arcwise connected component of $Y - \{v\}$ containing w.

Claim 1. For every $i \in \mathbb{N}$, we have $v_{ik+k} \in Y_{v_k 2}$.

In fact, if Claim 1 does not hold, then there exists $i \in \mathbb{N}$ such that $v_{ik} \in Y_{v_k 2} \cup \{v_k\}$ and $v_{ik+k} \in Y_{v_k 1}$. For $j = 1, 2$, let

$$X_j = \{x \in A[v, v_k] : f^{ik}(x) \in Y_{xj}\}.$$

Then $v \in X_2$, $v_k \in X_1$, and X_1 and X_2 are both open subsets of $A[v, v_k]$ (with respect to the topology of $A[v, v_k]$). Thus $A[v, v_k] - X_1 - X_2 \neq \emptyset$ and $f^{ik}(y) = y$ for every $y \in A[v, v_k] - X_1 - X_2$. This contradicts the assumption that $A[v, v_k] \cap P(f) \subset B(v, 2\varepsilon) \cap P(f) = \emptyset$. So Claim 1 holds.

By Claim 1, we get

Claim 2. For every $i \in \mathbb{N}$, $v_{ik} \in Y_{v2}$.

Similarly to the proof of Claim 1, we get from Claim 2 that

Claim 3. For every $i \in \mathbb{N}$ and every $x \in A[v, v_k]$, $f^{ik}(x) \in Y_{x2}$.

Similarly to Claim 3, we have

Claim 4. For every $j \in \mathbb{N}$ and every $x \in A[v_k, v_n]$, $f^{j(n-k)}(x) \in Y_{x1}$.

Applying Claim 3 and Claim 4 to $x \in [v_k, v_n]$, $i = n - k$ and $j = k$, we obtain simultaneously that $f^{k(n-k)}(x) \in Y_{x2}$ and $f^{k(n-k)}(x) \in Y_{x1}$. This is a contradiction. Hence we must have $v \in \overline{P(f)}$. The proof is complete. □

By Proposition 5.7 and Theorem 5.4, we immediately get

Proposition 5.8. *Let X be a quasi-graph. Suppose that $G_0 = \bigcup\{C : C$ is a Jordan curve in $X\}$ and $f \in C^0(X)$. Then $R(f) - G_0 \subset \overline{P(f)}$, and hence $\overline{R(f)} - G_0 = \overline{P(f)} - G_0$. Specifically, if the oscillatory quasi-arcs $L_1, \cdots, L_n$ in X are as in Theorem 2.24, then $R(f) \cap (\bigcup_{i=1}^n L_i) \subset \overline{P(f)}$, and hence $\overline{R(f)} \cap (\bigcup_{i=1}^n L_i) = \overline{P(f)} \cap (\bigcup_{i=1}^n L_i)$.* □

Theorem 5.9. *If f is a quasi-tree map, then $\overline{R(f)} = \overline{P(f)}$.* □

Noting that the Warsaw circle is a quasi-tree, we immediately get the following corollary, which was first given by Xiong et al. in [30].

Corollary 5.10. *Let f be a continuous self-map on the Warsaw circle. Then $\overline{R(f)} = \overline{P(f)}$.* □

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