

## Topological transitivity and wandering intervals for group actions on the line $\mathbb{R}$

Enhui Shi and Lizhen Zhou

**Abstract.** For every group  $G$ , we show that either  $G$  has a topologically transitive action on the line  $\mathbb{R}$  by orientation-preserving homeomorphisms, or every orientation-preserving action of  $G$  on  $\mathbb{R}$  has a wandering interval. According to this result, all groups are divided into two types: transitive type and wandering type, and the types of several groups are determined. We also show that every finitely generated orderable group of wandering type is indicable. As a corollary, we show that if a higher rank lattice  $\Gamma$  is orderable, then  $\Gamma$  is of transitive type.

**Mathematics Subject Classification (2010).** Primary 37B05; Secondary 20F16.

**Keywords.** Topological transitivity, orderable group, indicable group, solvable group, Baumslag–Solitar group.

### 1. Introduction

The theory of topological dynamical system study the orbit structure of group actions on topological spaces. An important question in this area is classifying “irreducible” actions up to topological conjugations. The Poincaré’s classification theorem for minimal orientation-preserving circle homeomorphisms is the first celebrated result toward this question; this theorem had been extended to orientation-preserving minimal group actions on the circle by Ghys (see [9]). Like minimality, the notion of topological transitivity is also a kind of irreducibility for group actions, which has been intensively studied since the foundation of dynamical system. Many important systems are topologically transitive, such as the Anosov automorphisms of torus, the irrational rotations on the circle, and the full shifts on symbol spaces. Some well-known definitions of chaos such as Devaney’s chaos and Auslander–Yorke’s chaos (Ruelle–Taken’s chaos) are based on topological transitivity (see [1, 6]). One may consult [18] for a classification theorem about topologically transitive  $\mathbb{Z}^n$  actions on  $\mathbb{R}$ .

Apart from the classification problem of topological transitive actions, the following question is more basic: given a group  $G$  and a topological space  $X$ , does  $G$  have a topologically transitive action on  $X$ ? Certainly, the answer to this question depends on the algebraic structure of  $G$  and the topology of  $X$ . In this paper, we are mainly concerned with the case that  $G$  is a discrete group and  $X$  is the real line  $\mathbb{R}$  (the simplest noncompact manifold). In fact, there is a large literature covering group actions on the line. We recommend the references [9, 13] for more details in this area. In addition, group actions on the real line are closely related to the study of orderable groups. It is well known that a countable group  $G$  is orderable (that is,  $G$  admits a left-invariant total order relation) if and only if it admits a faithful orientation-preserving action on the real line (see [14, Proposition 2.1]). Many important groups coming from geometry and topology are known to be orderable or nonorderable (see e.g. [3, 5, 19, 22]). Moreover, an orderable group may possess some interesting algebraic properties (see e.g. [12, 15, 21]).

In the next section, we recall some notions in dynamical system and group theory, and give some examples of transitive group actions on the line. Section 3 contains the statements of the main results in the paper. In Section 4, we show a dichotomy theorem which motivates us to divide all groups into two types: transitive type and wandering type. Applying the dichotomy theorem, the types of several groups are determined in Section 5 (with emphasis on solvable groups). A necessary condition is obtained for a finitely generated orderable group to be of wandering type in Section 6. The last section contains some properties in group theory which are used in the previous sections.

## 2. Preliminaries

Let  $X$  be a topological space and let  $\text{Homeo}(X)$  be the homeomorphism group of  $X$ . Suppose  $G$  is a group. A group homomorphism  $\phi: G \rightarrow \text{Homeo}(X)$  is called an *action* of  $G$  on  $X$ ; the action is said to be *faithful* if  $\phi$  is injective. If  $G$  is a subgroup of  $\text{Homeo}(X)$ , then the action of  $G$  on  $X$  always refers to the inclusion homomorphism  $\iota: G \hookrightarrow \text{Homeo}(X)$ . For brevity, we usually use  $gx$  or  $g(x)$  instead of  $\phi(g)(x)$  for  $g \in G$  and  $x \in X$ . The *orbit* of  $x \in X$  under the action of  $G$  is the set  $Gx \equiv \{gx: g \in G\}$ ;  $x$  is called a *fixed point* of  $\phi$  or of  $G$ , if  $gx = x$  for all  $g \in G$ . We use  $\text{Fix}(G)$  to denote the fixed point set of  $G$ ; use  $\text{Fix}(g)$  to denote the fixed point set of the cyclic group  $\langle g \rangle$  generated by  $g \in G$ . A subset  $Y$  of  $X$  is called  *$G$ -invariant*, if  $g(Y) \subset Y$  for all  $g \in G$ .

Let  $\phi$  be an action of group  $G$  on a topological space  $X$ . The action  $\phi$  (or  $G$ ) is said to be *topologically transitive*, if for every nonempty open subsets  $U$  and  $V$  of  $X$ , there is some  $g \in G$  such that  $g(U) \cap V \neq \emptyset$ . It is well known that, when  $G$  is countable and  $X$  is a Polish space,  $G$  is topologically transitive if and only if

there is a point  $x \in X$  such that the orbit  $Gx$  is dense in  $X$ . Furthermore,  $G$  is said to be *minimal* if for every  $x \in X$  the orbit  $Gx$  is dense in  $X$ ; this is equivalent to saying that there is no proper  $G$ -invariant nonempty closed subset of  $X$ . A closed subset  $Y$  of  $X$  is said to be *minimal*, if  $Y$  is  $G$ -invariant and the restriction action  $G|_Y$  of  $G$  to  $Y$  is minimal. An argument using Zorn's lemma shows the existence of minimal sets when  $X$  is a compact metric space, but this is not true in general when  $X$  is not compact.

Let  $\mathbb{R}$  be the real line and let  $\text{Homeo}_+(\mathbb{R})$  be the orientation-preserving homeomorphism group of  $\mathbb{R}$ . A group homomorphism  $\phi$  from  $G$  to  $\text{Homeo}_+(\mathbb{R})$  is called an *orientation-preserving action* of  $G$  on  $\mathbb{R}$ . An open interval  $(a, b) \subset \mathbb{R}$  is said to be a *wandering interval* of  $\phi$  or of  $G$  if, for every  $g \in G$ , either the restriction  $g|_{(a,b)} = \text{Id}_{(a,b)}$  or  $g((a, b)) \cap (a, b) = \emptyset$ .

For any  $\alpha \in \mathbb{R}$ , define  $L_\alpha, M_\alpha: \mathbb{R} \rightarrow \mathbb{R}$  by letting  $L_\alpha(x) = x + \alpha$  and  $M_\alpha(x) = \alpha x$  for every  $x \in \mathbb{R}$ . We use  $\mathbb{Z}^n$  to denote the free abelian group of rank  $n$ . Now we give some examples to illustrate the above notions, which will be used in Section 5.

**Example 2.1.** Every open interval  $(a, b)$  with  $b - a < 1$  is a wandering interval for the  $\mathbb{Z}$  action generated by  $L_1$ .

**Example 2.2.** Let  $\alpha$  be an irrational number, then the  $\mathbb{Z}^2$  action generated by  $L_1$  and  $L_\alpha$  on  $\mathbb{R}$  is minimal.

**Example 2.3.** Let  $n$  be a positive integer. Let  $T = L_1$  and let  $S = M_n$ . Then  $S^{-m}TS^m(x) = x + n^{-m}$  for all non negative integers  $m$  and all  $x \in \mathbb{R}$ , which clearly implies the minimality of the action of the group  $G$  generated by  $S$  and  $T$ .

**Example 2.4.** Let  $f = L_1$  and let  $k \geq 2$  be a positive integer. Define a homeomorphism  $g$  on  $\mathbb{R}$  by setting

$$g(x) = (x - n)^{2^{(-1)^n k - n}} + n$$

for all integers  $n$  and all  $x \in [n, n + 1)$ . Then, for  $x \in [n, n + 1)$ , we have

$$\begin{aligned} fgf^{-1}(x) &= (x - 1 - (n - 1))^{2^{(-1)^{n-1} k - (n-1)}} + (n - 1) + 1 \\ &= (x - n)^{2^{(-1)^{n-1} k - (n-1)}} + n, \end{aligned}$$

$$g^{-k}(x) = (x - n)^{2^{(-1)^{n+1} k - n k}} + n = (x - n)^{2^{(-1)^{n-1} k - (n-1)}} + n.$$

So  $fgf^{-1} = g^{-k}$ . Since  $f^m g^l f^n(\frac{1}{2}) = (\frac{1}{2})^{2^{(-1)^n k - n l}} + n + m$  for all integers  $m, l, n$ , the set  $\{f^m g^l f^n(\frac{1}{2}): m, l, n \in \mathbb{Z}\}$  is dense in  $\mathbb{R}$ . This implies that the action by the group  $\langle f, g \rangle$  generated by  $f$  and  $g$  is topologically transitive.

### 3. Main results

In this section, we will list the results obtained in this paper. We first give the following dichotomy theorem.

**Theorem 3.1.** *Let  $G$  be a group. Then either  $G$  has a topologically transitive action on the line  $\mathbb{R}$  by orientation-preserving homeomorphisms, or every orientation-preserving action of  $G$  on  $\mathbb{R}$  has a wandering interval.*

We should note that the “dichotomy phenomenon” in Theorem 3.1 is far from being true for group actions on spaces of dimension  $\geq 2$ . For example, if  $D$  is the closed unit disk in the plane and  $S^2$  is the unit sphere in  $\mathbb{R}^3$ , then any one point union of  $D$  and  $S^2$  admits no topologically transitive homeomorphism but admits a homeomorphism with no wandering open set.

Theorem 3.1 motivates us to give the following definition.

**Definition 3.2.** A group  $G$  is of *transitive type* if it has a topologically transitive action on the line  $\mathbb{R}$  by orientation-preserving homeomorphisms; it is of *wandering type* if every orientation-preserving action of  $G$  on  $\mathbb{R}$  has a wandering interval.

Recall that a group  $G$  is *poly-cyclic* (resp. *super-poly-cyclic*) if it admits a decreasing sequence of subgroups  $G = N_0 \triangleright N_1 \triangleright \cdots \triangleright N_k = \{e\}$  for some positive integer  $k$  such that  $N_{i+1}$  is normal in  $N_i$  (resp.  $N_{i+1}$  is normal in  $G$ ) and  $N_i/N_{i+1}$  is cyclic for each  $i \leq k-1$ ; it is called *poly-infinite-cyclic* (resp. *super-poly-infinite-cyclic*) if  $N_i/N_{i+1}$  is infinitely cyclic for each  $i \leq k-1$ . It is well known that all poly-cyclic groups are solvable and all finitely generated torsion free nilpotent groups are super-poly-infinite-cyclic. One may see the appendix for more properties about poly-cyclic groups.

Suppose  $G = N_0 \triangleright N_1 \triangleright \cdots \triangleright N_k = \{e\}$  is super-poly-infinite-cyclic. Take  $f_i \in N_i \setminus N_{i+1}$  such that  $N_i/N_{i+1} = \langle f_i N_{i+1} \rangle$  for each  $i$ . Then  $f_i f_{i+1} N_{i+2} f_i^{-1} = f_{i+1}^{n_i} N_{i+2}$  where  $n_i = \pm 1$ . We call the  $(k-1)$ -tuple  $(n_0, n_1, \dots, n_{k-2})$  the *name* of  $G$ . Clearly, the name of  $G$  is independent of the choice of  $f_i$ .

Baumslag–Solitar groups are examples of two-generator one-relator groups that play an important role in combinatorial group theory and geometric group theory. For each integer  $n$ , the solvable *Baumslag–Solitar group*  $B(1, n)$  is the group  $\langle a, b : ba = a^n b \rangle$ ;  $B(1, 1)$  is the free abelian group of rank 2;  $B(1, -1)$  is the fundamental group of the Klein Bottle, which is a classical example being of orderable but non bi-orderable (see [13, Exercise 2.2.68]). One may consult [7] for the study of quasi-isometry property of these groups.

**Theorem 3.3.** *The following groups are of transitive type: the nonabelian free group  $\mathbb{Z} * \mathbb{Z}$ ; any super-poly-infinite-cyclic group  $G = N_0 \triangleright N_1 \triangleright \cdots \triangleright N_k = \{e\}$  having the name  $(n_0, n_1, \dots, n_{k-2})$  with some  $n_i = 1$ ; any poly-infinite-cyclic, non super-poly-infinite-cyclic group  $G$ ; the Baumslag–Solitar group  $B(1, n)$  with  $n \neq 0$  and  $n \neq -1$ .*

*The following groups are of wandering type: finite groups; the infinite cyclic group  $\mathbb{Z}$ ;  $\mathrm{SL}(2, \mathbb{Z})$ ; finite index subgroups of  $\mathrm{SL}(n, \mathbb{Z})$  with  $n \geq 3$ ; any super-poly-infinite-cyclic group  $G = N_0 \triangleright N_1 \triangleright \cdots \triangleright N_k = \{e\}$  having the name  $(-1, -1, \dots, -1)$ ; the Baumslag–Solitar group  $B(1, -1)$ .*

**Remark 3.4.** The wandering type of finite index subgroups of  $\mathrm{SL}(n, \mathbb{Z})$  with  $n \geq 3$  in the above theorem is almost a restatement of the celebrated and difficult result due to Witte-Morris in [22].

Recall that a group is *indicable* if it has a homomorphism onto the infinite cyclic group. One may consult [2, 10, 21] for the discussions about indicability of orderable groups.

**Theorem 3.5.** *If  $G$  is a finitely generated nontrivial orderable group of wandering type, then  $G$  is indicable.*

A higher rank lattice is a lattice of a simple Lie group with finite center and with real rank  $\geq 2$ . The 1-dimensional Zimmer’s rigidity conjecture says that every continuous action of a higher rank lattice on the circle  $\mathbb{S}^1$  must factor through a finite group action. Though the conjecture is still open now, Ghys (see [8]) and Burger and Monod (see [4]) proved independently the existence of periodic points for such actions. This implies that 1-dimensional Zimmer’s rigidity conjecture is equivalent to that no higher rank lattice is orderable. We get immediately the following corollary by Theorem 3.5 and Theorem 7.5 in the appendix.

**Corollary 3.6.** *Suppose  $G$  is a higher rank lattice. If  $G$  is orderable, then it is of transitive type.*

#### 4. The dichotomy theorem

**Lemma 4.1.** *Let  $G$  be a group. Suppose  $G$  has no topologically transitive action on the line  $\mathbb{R}$  by orientation-preserving homeomorphisms. Then, for every action  $\phi: G \rightarrow \mathrm{Homeo}_+(\mathbb{R})$  and for every  $x \in \mathbb{R}$ ,  $\overline{Gx}$  is countable; in particular,  $\overline{Gx}$  is nowhere dense.*

*Proof.* Assume to the contrary that there is some action  $\phi_0: G \rightarrow \text{Homeo}_+(\mathbb{R})$  and some  $x_0 \in \mathbb{R}$  such that  $\overline{Gx_0}$  is uncountable. Then, by collapsing the maximal open intervals of  $\mathbb{R} \setminus \overline{Gx_0}$ , we get an induced topologically transitive action of  $G$  on either  $[0, 1]$ , or  $[0, 1)$ , or  $(0, 1]$ , or  $(0, 1)$ . By removing the endpoints of the phase space of the induced action if necessary, we get a topologically transitive action of  $G$  on  $\mathbb{R}$  by orientation-preserving homeomorphisms. This contradicts the assumption.  $\square$

*Proof of Theorem 3.1.* Assume to the contrary that the following two items hold simultaneously:

- (a)  $G$  has no topologically transitive action on  $\mathbb{R}$  by orientation-preserving homeomorphisms;
- (b) there is an action  $\phi: G \rightarrow \text{Homeo}_+(\mathbb{R})$  such that  $\phi$  has no wandering interval.

From Assumption (b) and the definition of wandering interval, there is some  $x_1 \in \mathbb{R}$  and some  $g_1 \in G$  such that  $x_1 < \phi(g_1)(x_1)$  (otherwise,  $\phi(g) = \text{Id}_{\mathbb{R}}$  for all  $g \in G$ ; then every open interval in  $\mathbb{R}$  is wandering). Without loss of generality, we suppose that  $\{x_1, \phi(g_1)(x_1)\} \subset (0, 1)$ . Set  $U_0 = (0, 1)$ . For the simplicity of notations, we use  $g(x)$  instead of  $\phi(g)(x)$  in what follows.

Now we define inductively a sequence of open intervals  $U_i$  and  $g_i \in G$ ,  $i = 1, 2, \dots$ , such that

- (1) for each  $i \geq 1$ ,  $U_{i-1} \supset U_i$ ;
- (2) for every  $g \in G$  and every  $i \geq 1$ , either  $g(U_i) = U_i$ , or  $g(U_i) \cap U_i = \emptyset$ ;
- (3) for every  $g \in G$  and every  $i \geq 1$ ,  $\text{diam}(g(U_i) \cap [0, 1]) < \frac{1}{i}$ ;
- (4) for each  $i \geq 1$ ,  $\overline{U_i} \cap g_i(\overline{U_i}) = \emptyset$  and  $\overline{U_i} \cup g_i(\overline{U_i}) \subset U_{i-1}$ .

For  $i = 1$ , take a sufficiently small interval  $V_1 \subset (0, 1)$  such that  $x_1 \in V_1$ ,

$$\overline{V_1} \cup g_1(\overline{V_1}) \subset U_0, \text{ and } \overline{V_1} \cap g_1(\overline{V_1}) = \emptyset. \quad (4.1)$$

Take a sufficiently large positive integer  $i_1 > 1$  such that

$$\frac{1}{i_1} < \frac{1}{2} \text{diam}(V_1). \quad (4.2)$$

Let  $A_1 = \{k \frac{1}{i_1} : k = 0, 1, \dots, i_1\}$  and let  $B_1 = \overline{GA_1}$ . It follows from Lemma 4.1 that  $B_1$  is a nowhere dense  $G$ -invariant closed subset of  $\mathbb{R}$ . From (4.2) and the definition of  $A_1$ , there exists a maximal open interval  $U_1$  of  $\mathbb{R} \setminus B_1$  such that  $U_1 \subset V_1$ . Then (1)–(4) hold for  $U_1$  and  $g_1$  by (4.1), (4.2), and the definition of  $U_1$ .

Suppose that for  $1 \leq i \leq k$  we have defined  $U_i$  and  $g_i$  which satisfy (1)–(4). Then define  $U_{k+1}$  and  $g_{k+1}$  as follows. From Assumption (b),  $U_k$  is nonwandering, which together with (2) implies that there is some point  $x_{k+1} \in U_k$  and some  $g_{k+1} \in G$  such that  $g_{k+1}(x_{k+1}) \in U_k$  and  $g_{k+1}(x_{k+1}) > x_{k+1}$ . Take a sufficiently small open interval  $V_{k+1}$  such that  $x_{k+1} \in V_{k+1}$ ,

$$\overline{V_{k+1}} \cup g_{k+1}(\overline{V_{k+1}}) \subset U_k \quad \text{and} \quad \overline{V_{k+1}} \cap g_{k+1}(\overline{V_{k+1}}) = \emptyset. \quad (4.3)$$

Take a sufficiently large positive integer  $i_{k+1} > k + 1$  such that

$$\frac{1}{i_{k+1}} < \frac{1}{2} \text{diam}(V_{k+1}). \quad (4.4)$$

Let  $A_{k+1} = \{k \frac{1}{i_{k+1}} : k = 0, 1, \dots, i_{k+1}\}$  and let  $B_{k+1} = \overline{GA_{k+1}}$ . Similar to the case of  $i = 1$ , we get a maximal open interval  $U_{k+1}$  of  $\mathbb{R} \setminus B_{k+1}$  which satisfy conditions (1)–(4).

Now we define a sequence of subsets  $G_i$  of  $G$  for  $i = 1, 2, \dots$  as follows. Let  $G_1 = \{e, g_1\}$ . Assume  $G_i$  have been defined for  $1 \leq i \leq k$ . Then let  $G_{k+1} = G_k \cup \{gg_{k+1} : g \in G_k\}$ . For each  $k = 1, 2, \dots$ , set  $\Lambda_k = \bigcup_{g \in G_k} g(\overline{U_k})$ , and set  $\Lambda = \bigcap_{k=1}^{\infty} \Lambda_k$ . It follows from (3) and (4) that  $\Lambda$  is homeomorphic to the Cantor set, and, for any point  $x \in \Lambda$ ,  $\overline{Gx} \supset \Lambda$  (one may see Figure 1 for the illustration of the ideas of the construction). This implies that  $G$  has a topologically transitive orientation-preserving action on  $\mathbb{R}$  by Lemma 4.1, which contradicts the assumption (a).  $\square$

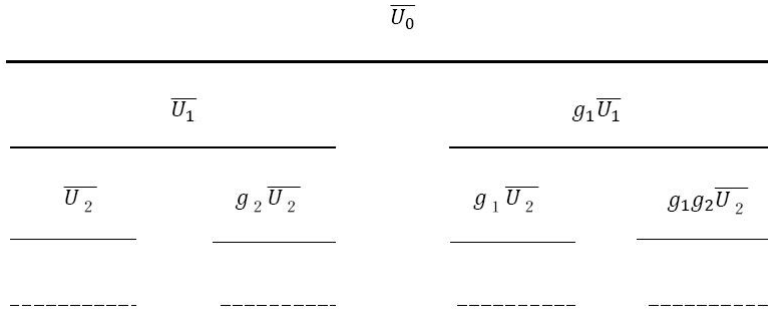


Figure 1

## 5. Types of some groups

In this section, we start to prove Theorem 3.3. From Definition 3.2, we immediately have that all finite groups and the infinite cyclic group  $\mathbb{Z}$  are of wandering type.

**Proposition 5.1.** *Suppose  $G = N_0 \triangleright N_1 \triangleright \cdots \triangleright N_k = \{e\}$  is super-poly-infinite-cyclic and has the name  $(-1, -1, \dots, -1)$ . Then  $G$  is of wandering type.*

*Proof.* By the hypothesis, we can take a sequence  $g_i \in N_i \setminus N_{i+1}$  and take  $g_k = e$  such that  $N_i = \langle g_i, \dots, g_{k-1} \rangle$  and

$$g_i g_{i+1} N_{i+2} = g_{i+1}^{-1} g_i N_{i+2} \quad \text{for each } i: 0 \leq i \leq k-2. \quad (5.1)$$

Let  $\phi: G \rightarrow \text{Homeo}_+(\mathbb{R})$  be any orientation-preserving action of  $G$  on  $\mathbb{R}$  and let  $f_i = \phi(g_i)$  for each  $i$ . If  $f_i = \text{Id}_{\mathbb{R}}$  for each  $i$ , then every open interval in  $\mathbb{R}$  is wandering by Definition 3.2. So, we may as well suppose that

$$\mathbb{R} \setminus \text{Fix}(N_{k-1}) \neq \emptyset. \quad (5.2)$$

**Claim A.** *There exists a sequence of open intervals  $J_i$ ,  $0 < i \leq k$ , such that  $J_1 \supset \cdots \supset J_k$ ,  $J_i \subset \mathbb{R} \setminus \text{Fix}(f_i)$  for  $i < k$ ,  $f_{i-1}(J_i) \cap J_i = \emptyset$  and  $f_j(J_i) = J_i$  for  $j \geq i$ .*

We prove this claim by induction. Take a maximal open interval  $J_{k-1}$  in  $\mathbb{R} \setminus \text{Fix}(N_{k-1})$  by (5.2), and take an open interval  $J_k \subset J_{k-1}$  such that  $f_{k-1}(J_k) \cap J_k = \emptyset$ . Assume that, for some  $l \geq 0$ , we have obtained open intervals  $J_{l+1} \supset \cdots \supset J_k$  such that  $J_i \subset \mathbb{R} \setminus \text{Fix}(f_i)$  for each  $i: k > i > l$ ,  $f_{i-1}(J_i) \cap J_i = \emptyset$  for  $i > l$ , and  $f_j(J_i) = J_i$  for  $j \geq i$ . Let  $J_i = (\alpha_i, \beta_i)$ ,  $k \geq i \geq l+1$ . Let  $J_l = (\alpha_l, \beta_l)$  be the maximal open interval of  $\mathbb{R} \setminus \text{Fix}(f_l)$ , which contains  $J_{l+1}$ . Since either  $\alpha_l = \lim_{m \rightarrow +\infty} f^m(\alpha_{l+1})$  or  $\alpha_l = \lim_{m \rightarrow -\infty} f^m(\alpha_{l+1})$ , and  $\alpha_{l+1} \in \text{Fix}(N_{l+1})$ , we have  $\alpha_l \in \text{Fix}(N_l)$ . Similarly,  $\beta_l \in \text{Fix}(N_l)$ . So,  $f_j(J_l) = J_l$  for  $j \geq l$ . In addition, we have  $f_{l-1}(J_l) \cap J_l = \emptyset$ . Otherwise,  $f_{l-1}(J_l) = J_l$  since  $f_{l-1}(\text{Fix}(N_l)) = \text{Fix}(N_l)$  by (5.1). Without loss of generality, we suppose  $f_l(x) > x$  for every  $x \in J_l$ . Let  $w = \alpha_{l+1}$ . Then  $w \in \text{Fix}(N_{l+1})$ , and  $\alpha_l = \lim_{m \rightarrow +\infty} f_{l-1} f_l^{-m}(w) = \lim_{m \rightarrow +\infty} f_l^m(f_{l-1}(w)) = \beta_l$  by (5.1), which is a contradiction. Thus we complete the proof of Claim A.

**Claim B.**  *$J_k$  is a wandering interval of  $\phi$ . In fact, for any  $g \in G$ ,  $\phi(g)$  can be expressed as  $\phi(g) = f_0^{n_0} f_1^{n_1} \cdots f_{k-1}^{n_{k-1}}$  for some integers  $n_0, n_1, \dots, n_{k-1}$ . If  $\phi(g)(J_k) \cap J_k \neq \emptyset$ , then it follows from Claim A that  $n_0 = n_1 = \cdots = n_{k-1} = 0$ . Thus  $\phi(g) = \text{Id}_{\mathbb{R}}$ .*

From Claim B, we get that  $G$  is of wandering type. □

**Lemma 5.2.** *Let  $H$  be a normal subgroup of  $G$  such that  $G/H$  is infinite cyclic. Then  $G$  is of transitive type provided that  $H$  is of transitive type.*

*Proof.* Suppose  $G/H = \langle aH \rangle$  for some  $a \in G \setminus H$ . Let  $a$  act on the line by the unit translation  $\phi(a): x \mapsto x + 1$ . By the assumption,  $H$  has an orientation-preserving topologically transitive action on  $(0, 1)$ , which extends to an action  $\phi$



on the interval  $[0, 1]$  by fixing the endpoints. Then extend this  $H$  action to  $G$  action on the line by setting, for each  $j \in \mathbb{Z}$ ,

$$\phi(b)(x) = \phi(a^{-j}ba^j)(x - j) + j \quad (5.3)$$

for all  $x \in [j, j + 1]$  and  $b \in H$ . Then define  $\phi$  on  $G$  by setting  $\phi(a^i b) = (\phi(a))^i \phi(b)$  for all  $i \in \mathbb{Z}$  and  $b \in H$ . It is direct to check that  $\phi(g_1 g_2) = \phi(g_1) \phi(g_2)$  for arbitrary two elements  $g_1 = a^{l_1} b_1$  and  $g_2 = a^{l_2} b_2$  in  $G$  where  $b_1, b_2 \in H$ . In fact, for all  $x \in [i, i + 1]$ , we have

$$\begin{aligned} \phi(g_1 g_2)(x) &= \phi(a^{l_1} b_1 a^{l_2} b_2)(x) \\ &= \phi(a^{l_1 + l_2} a^{-l_2} b_1 a^{l_2} b_2)(x) \\ &= \phi(a^{l_1 + l_2}) \phi(a^{-l_2} b_1 a^{l_2} b_2)(x) \\ &= \phi(a^{-i} a^{-l_2} b_1 a^{l_2} b_2 a^i)(x - i) + i + l_1 + l_2, \end{aligned}$$

while

$$\begin{aligned} \phi(g_1) \phi(g_2)(x) &= \phi(a^{l_1} b_1) \phi(a^{l_2} b_2)(x) \\ &= \phi(a^{l_1} b_1) (\phi(a^{-i} b_2 a^i)(x - i) + i + l_2) \\ &= \phi(a^{-i - l_2} b_1 a^{i + l_2}) \phi(a^{-i} b_2 a^i)(x - i) + i + l_1 + l_2 \\ &= \phi(a^{-i} a^{-l_2} b_1 a^{l_2} b_2 a^i)(x - i) + i + l_1 + l_2, \end{aligned}$$

as required. Thus  $\phi$  is an orientation-preserving action of  $G$  on the line and the topological transitivity of  $\phi$  is clear.  $\square$

**Proposition 5.3.** *Suppose  $G = N_0 \triangleright N_1 \triangleright \cdots \triangleright N_k = \{e\}$  is super-poly-infinite-cyclic with  $k \geq 2$  and has the name  $(n_0, n_1, \dots, n_{k-2})$  with some  $n_i = 1$ . Then  $G$  is of transitive type.*

*Proof.* If  $n_{k-2} = 1$ , then  $N_{k-2}$  is isomorphic to  $\mathbb{Z}^2$ , which is of transitive type by Example 2.2. By repeated applications of Lemma 5.2, we get that  $G$  is of transitive type. If  $n_i = 1$  for some  $i < k - 2$ , then  $G/N_{i+2}$  is of transitive type by the previous argument. Since  $G/N_{i+2}$  is a factor of  $G$ ,  $G$  is of transitive type.  $\square$

**Proposition 5.4.** *Suppose  $G$  is poly-infinite-cyclic. If  $G$  is not super-poly-cyclic, then  $G$  is of transitive type.*

*Proof.* By Proposition 7.4 in the appendix, there is a decreasing sequence of normal subgroups of  $G$ :  $G = N_0 \triangleright N_1 \triangleright \cdots \triangleright N_k = \{e\}$  for some  $k > 0$ , such that each  $N_i/N_{i+1}$  is isomorphic to  $\mathbb{Z}^{d_i}$  for some  $d_i \geq 1$ . If every  $N_i/N_{i+1}$  is isomorphic to  $\mathbb{Z}$ , then  $G$  is super-poly-cyclic, which contradicts the assumption. So, there is some  $i'$  such that  $N_{i'}/N_{i'+1}$  is isomorphic to  $\mathbb{Z}^{d_{i'}}$  with  $d_{i'} \geq 2$ . Let  $\tilde{G} = G/N_{i'+1}$ . Since  $\tilde{N}_{i'} \equiv N_{i'}/N_{i'+1}$  is of transitive type by Example 2.2,  $\tilde{G}$  is of transitive type by repeated applications of Lemma 5.2 (note that  $\tilde{G}/\tilde{N}_{i'}$  is poly-infinite-cyclic). Then  $G$  is of transitive type since  $\tilde{G}$  is a factor of  $G$ .  $\square$

**Proposition 5.5.** *The Baumslag–Solitar group  $B(1, n)$  is of transitive type if and only if  $n \neq 0$  and  $n \neq -1$ .*

*Proof.* ( $\Rightarrow$ ) Suppose  $B(1, n) = \langle a, b : ba = a^n b \rangle$ . If  $n = 0$ , then  $B(1, n)$  is an infinite cyclic group; in this case, every orientation-preserving action of  $B(1, n)$  on  $\mathbb{R}$  has a wandering interval. Suppose  $n = -1$ . Let  $\phi: B(1, -1) \rightarrow \text{Homeo}_+(\mathbb{R})$  be any orientation-preserving action of  $B(1, -1)$  on  $\mathbb{R}$ . Let  $g = \phi(a)$  and  $f = \phi(b)$ ; then  $fg = g^{-1}f$ . We wish to show that  $\phi$  has a wandering interval. If  $\text{Fix}(g) = \mathbb{R}$ , then  $\phi$  factors through a cyclic group action on  $\mathbb{R}$ , which ensures the existence of wandering intervals. So, we may suppose that  $\text{Fix}(g) \neq \mathbb{R}$ .

**Claim A.**  $f(\text{Fix}(g)) = \text{Fix}(g)$ . In fact, let  $x \in \text{Fix}(g)$ . Then  $g^{-1}f(x) = fg(x) = f(x)$ . So,  $f(x) \in \text{Fix}(g^{-1}) = \text{Fix}(g)$ . On the other hand,  $g^{-1}f^{-1}(x) = f^{-1}g(x) = f^{-1}(x)$ , which means  $f^{-1}(x) \in \text{Fix}(g^{-1}) = \text{Fix}(g)$ .

From Claim A, we see that  $f$  permutes the maximal open intervals in the set  $\mathbb{R} \setminus \text{Fix}(g)$ . Fix a maximal open interval  $(u, v)$  in  $\mathbb{R} \setminus \text{Fix}(g)$  ( $u$  may be  $-\infty$ , and  $v$  may be  $+\infty$ ).

**Claim B.**  $f((u, v)) \cap (u, v) = \emptyset$ . Otherwise,  $f((u, v)) = (u, v)$ . Without loss of generality, we suppose  $g(x) > x$  for every  $x \in (u, v)$ . Fix any point  $w \in (u, v)$ , then  $v = \lim_{i \rightarrow +\infty} fg^i(w) = \lim_{i \rightarrow +\infty} g^{-i}(f(w)) = u$  by the relation  $fg = g^{-1}f$ . This is a contradiction.

From Claim A and Claim B, we immediately get:

**Claim C.** Suppose  $f^{m_1}g^{n_1}f^{m_2}g^{n_2}\dots f^{m_l}g^{n_l}((u, v)) = (u, v)$  for some integers  $l, m_i$ , and  $n_i$  ( $1 \leq i \leq l$ ). Then  $m_1 + m_2 + \dots + m_l = 0$ .

Take an open interval  $J \subset (u, v)$  such that  $J \cap g(J) = \emptyset$ .

**Claim D.** If  $h(J) \cap J \neq \emptyset$  for some  $h \in \phi(B(1, -1))$ , then  $h = \text{Id}_{\mathbb{R}}$ . In fact, suppose  $h = f^{m_1}g^{n_1}f^{m_2}g^{n_2}\dots f^{m_l}g^{n_l}$  such that  $|m_1| + |n_1| + |m_2| + |n_2| + \dots + |m_l| + |n_l|$  attains minimum among all expressions of  $h$  by  $f$  and  $g$ . This implies that all  $m_i$  with  $m_i \neq 0$  have the same signs by the relation  $f^{-1}gf = g^{-1}$ . However, this forces all  $m_i = 0$  by Claim C (noting that  $h((u, v)) = (u, v)$ ). Then  $h = \text{Id}_{\mathbb{R}}$ , since  $J \cap g(J) = \emptyset$ .

It follows from Claim D that  $J$  is a wandering interval for  $\phi$ .

( $\Leftarrow$ ) Since  $B(1, 1)$  is isomorphic to  $\mathbb{Z}^2$ , it is of transitive type by Example 2.2. If  $n > 1$ , then  $B(1, n)$  is of transitive type by Example 2.3 (note that  $T$  and  $S$  in Example 2.3 satisfy the relation  $ST = T^nS$ ). In Example 2.4, we see that  $f$  and  $g$  satisfy the relation  $fg = g^{-k}f$  with  $k \geq 2$ , which implies that  $B(1, n)$  is of transitive type when  $n \leq -2$ .  $\square$

From Example 2.2, we see that the free nonabelian group  $\mathbb{Z} * \mathbb{Z}$  has a topologically transitive action on  $\mathbb{R}$  by orientation-preserving homeomorphisms, since  $\mathbb{Z}^2$  is a factor of  $\mathbb{Z} * \mathbb{Z}$ . In fact, we can further require the action to be faithful as the following lemma shows.

**Proposition 5.6.** *The free nonabelian group  $\mathbb{Z} * \mathbb{Z}$  has a faithful topologically transitive action on  $\mathbb{R}$  by orientation-preserving homeomorphisms.*

*Proof.* Let  $f, g \in \text{Homeo}_+(\mathbb{R})$  be defined by  $f(x) = x + 1$  and  $g(x) = x^3$  for every  $x \in \mathbb{R}$ . Then for any nonempty open intervals  $U$  and  $V$  in  $\mathbb{R}$ , we have  $\text{diam}(g^n(U)) > 1$  for some integer  $n$ , and then there is some integer  $m$  such that  $f^m(g^n(U)) \cap V \neq \emptyset$ . Thus the action by the group  $H$  generated by  $f$  and  $g$  is topologically transitive. By the main result in [20], we see that  $H$  is isomorphic to  $\mathbb{Z} * \mathbb{Z}$ .  $\square$

The following theorem is due to D. Witte-Morris (see [22]).

**Theorem 5.7** (Witte-Morris). *The group  $\text{SL}(2, \mathbb{Z})$  and all finite index subgroups of  $\text{SL}(n, \mathbb{Z})$  with  $n \geq 3$  are non-orderable.*

**Corollary 5.8.** *The group  $\text{SL}(2, \mathbb{Z})$  and all finite index subgroups of  $\text{SL}(n, \mathbb{Z})$  with  $n \geq 3$  are of wandering type.*

*Proof.* Since  $\text{SL}(2, \mathbb{Z})$  is generated by elements with finite orders, any orientation preserving action of  $\text{SL}(2, \mathbb{Z})$  on  $\mathbb{R}$  must be trivial. So,  $\text{SL}(2, \mathbb{Z})$  is of wandering type. Suppose  $n \geq 3$  and  $H$  is a subgroup of  $\text{SL}(n, \mathbb{Z})$  with finite index. Assume  $H$  is of transitive type and let  $\phi: H \rightarrow \text{Homeo}_+(\mathbb{R})$  be a topologically transitive action of  $H$  on  $\mathbb{R}$ . By Selberg's Lemma (see Theorem 7.6 in the appendix), there is a torsion free normal subgroup  $F$  of  $H$  which has finite index in  $H$ . It follows from Theorem 7.5 (see the appendix) and the topological transitivity of  $\phi$  that  $\text{Ker}(\phi)$  is finite, which implies that  $\text{Ker}(\phi) \cap F$  is trivial. So the restriction  $\phi|_F: F \rightarrow \text{Homeo}_+(\mathbb{R})$  is injective. Thus  $F$  is orderable, which contradicts Theorem 5.7.  $\square$

**Remark 5.9.** Since the free non-abelian group  $\mathbb{Z} * \mathbb{Z}$  is a finite index subgroup of  $\text{SL}(2, \mathbb{Z})$  and  $\mathbb{Z} * \mathbb{Z}$  is of transitive type, Corollary 5.8 does not hold for finite index subgroups of  $\text{SL}(2, \mathbb{Z})$ .

Then Theorem 3.3 follows from all the propositions in this section.

## 6. Indicability

To prove Theorem 3.5, we need several well-known results about group actions on  $\mathbb{R}$ . The following lemma can be shown by the dynamical realization method (see [13, Theorem 2.2.19] and its remark).

**Lemma 6.1.** *Every countable nontrivial orderable group has a faithful orientation-preserving action on the line  $\mathbb{R}$  without fixed points.*

The following lemma is the combination of [13, Proposition 2.1.12] and the remarks after it (see also [11]).

**Lemma 6.2.** *If  $G$  is a finitely generated group acting on the line  $\mathbb{R}$  by orientation-preserving homeomorphisms, then  $G$  admits a nonempty minimal closed subset  $\Lambda$  of  $\mathbb{R}$ , and  $\Lambda$  has four possibilities:*

- (a)  $\Lambda$  is a point (in this case,  $\Lambda$  is a fixed point of  $G$ );
- (b)  $\Lambda$  is an infinite sequence  $(a_n)_{n \in \mathbb{Z}}$  satisfying  $a_n < a_{n+1}$  for all  $n$  and without accumulation points in  $\mathbb{R}$ ;
- (c)  $\Lambda$  is locally a Cantor set;
- (d)  $\Lambda = \mathbb{R}$ .

*Proof of Theorem 3.5.* Suppose  $G$  is a finitely generated nontrivial orderable group of wandering type. From Lemma 6.1, we can fix a faithful action  $\phi: G \rightarrow \text{Homeo}_+(\mathbb{R})$  without fixed points. By Lemma 6.2, there is a minimal set  $\Lambda \subset \mathbb{R}$ . Since  $\phi$  has no fixed points,  $\Lambda$  cannot be a single point. By Definition 3.2 and Lemma 4.1, we see that  $\Lambda$  is countable, which together with Lemma 6.2 implies that  $\Lambda$  is an infinite sequence  $(a_n)_{n \in \mathbb{Z}}$  satisfying  $a_n < a_{n+1}$  for all  $n$  and without accumulation points in  $\mathbb{R}$ . Set  $H = \{g \in G: g((a_0, a_1)) = (a_0, a_1)\}$  and fix an  $f \in G$  with  $f(a_0) = a_1$ . By the structure of  $\Lambda$ , we have  $H = \{g \in G: g(a_n) = a_n \text{ for all } n\}$  and  $f(a_n) = a_{n+1}$  for all  $n$ . Thus  $H$  is normal in  $G$ , and  $G/H = \{f^n H, n \in \mathbb{Z}\}$  which is an infinite cyclic group. This completes the proof.  $\square$

## 7. Appendix

In this section, we first supply some basic results about poly-cyclic groups, which have been used in the previous sections. One may consult [16] for more details.

**Proposition 7.1.** *Let  $G$  be a poly-cyclic group and let  $H$  be a subgroup of  $G$ . Then  $H$  is poly-cyclic.*

**Proposition 7.2** ([15]). *Let  $G$  be a poly-cyclic group. Then  $G$  is poly-infinite-cyclic if and only if  $G$  is orderable.*

Since every subgroup of an orderable group is orderable, we immediately get the following corollary by propositions 7.1 and 7.2.

**Corollary 7.3.** *Let  $G$  be a poly-infinite-cyclic group. Then every non-trivial subgroup of  $G$  is poly-infinite-cyclic.*

Suppose  $G$  is a poly-cyclic group. Then  $G = N_0 \triangleright N_1 \triangleright \cdots \triangleright N_k = \{e\}$  with  $N_i/N_{i+1}$  cyclic for each  $i$ . The cyclic groups  $N_i/N_{i+1}$  are called the *cyclic factors*. The *Hirsch number* of  $G$  is the number of infinite cyclic factors among these  $N_i/N_{i+1}$ . It is an invariant of polycyclic groups.

**Proposition 7.4.** *Let  $G$  be a poly-infinite-cyclic group. Then there is a decreasing sequence of normal subgroups of  $G$ :  $G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_l = \{e\}$  for some  $l > 0$ , such that  $G_i/G_{i+1}$  is a free abelian group of finite rank for each  $i$ .*

*Proof.* Suppose  $G = N_0 \triangleright N_1 \triangleright \cdots \triangleright N_k = \{e\}$  for some  $k > 0$ , where  $N_{i+1}$  is normal in  $N_i$  and  $N_i/N_{i+1}$  is infinitely cyclic for each  $i \geq 0$ . Since  $G/N_1$  is abelian, the commutator group  $[G, G] \subset N_1$ . So,  $G/[G, G]$  is isomorphic to  $\mathbb{Z}^d \times F$  for some  $d > 1$ , where  $F$  is a finite abelian group. Let  $\pi: G \rightarrow G/[G, G]$  be the quotient homomorphism and let  $G_1 = \pi^{-1}(F)$ . Then  $G_1$  is a character subgroup of  $G$  and  $G/G_1$  is isomorphic to  $\mathbb{Z}^d$ . Similarly, we can get a character subgroup  $G_2$  of  $G_1$  (which is also a character subgroup of  $G$ ) such that  $G_1/G_2$  is a free abelian group of finite rank, since  $G_1$  is still poly-infinite-cyclic by Corollary 7.3. Going on in this way, we get a sequence of character subgroups of  $G$ :  $G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_i \triangleright \cdots$  such that  $G_i/G_{i+1}$  is a free abelian group of finite rank for each  $i$ . Since the Hirsch number of  $G$  is finite, there exists a positive integer  $l$  such that  $G_l = \{e\}$ . Thus we complete the proof.  $\square$

The following theorem is due to Margulis and Kazhdan (see e.g. [23, Theorem 8.1.2]).

**Theorem 7.5.** *Let  $\Gamma$  be a higher rank lattice and let  $H$  be a normal subgroup of  $\Gamma$ . Then either  $H$  is finite or  $\Gamma/H$  is finite.*

The following theorem is known as Selberg's Lemma (see [17]).

**Theorem 7.6.** *Let  $G$  be a finitely generated subgroup of  $GL(n, \mathbb{C})$ . Then  $G$  contains a torsion free normal subgroup  $H$  with finite index in  $G$ .*

**Acknowledgements.** The work of this article is supported by NSFC (No. 11771318, No. 11790274). We would like to thank Prof. Binyong Sun for providing us the idea of the proof of Proposition 7.4.

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Received April 21, 2017

Enhui Shi, School of Mathematical Sciences, Soochow University, Suzhou 215006,  
P. R. China

e-mail: [ehshi@suda.edu.cn](mailto:ehshi@suda.edu.cn)

Lizhen Zhou, School of Mathematical Sciences, Soochow University, Suzhou 215006,  
P. R. China

e-mail: [zhoulizhen@suda.edu.cn](mailto:zhoulizhen@suda.edu.cn)