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Invariant Radon measures and minimal sets for subgroups of $\mathrm{Homeo}_+(\mathbb{R})$

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ABSTRACT

Let G be a subgroup of $\mathrm{Homeo}_+(\mathbb{R})$ without crossed elements. We show the equivalence among three items: (1) existence of G-invariant Radon measures on $\mathbb{R}$; (2) existence of minimal closed subsets of $\mathbb{R}$; (3) nonexistence of infinite towers covering the whole line. For a nilpotent subgroup G of $\mathrm{Homeo}_+(\mathbb{R})$, we show that G always has an invariant Radon measure and a minimal closed set if every element of G is $C^{1+\alpha}(\alpha > 0)$; a counterexample of C^1 commutative subgroup of $\mathrm{Homeo}_+(\mathbb{R})$ is constructed.

1. Introduction

In the theory of dynamical systems, the following two facts are well known:

(1) if G is a group consisting of homeomorphisms on a compact metric space X, then G has a minimal set K in X, that is K is minimal among all nonempty G-invariant closed subsets with respect to the inclusion relation on sets;

(2) if G is an amenable group consisting of homeomorphisms on a compact metric space X, then G has an invariant Borel probability measure on X.

In general, these two results do not hold if X is not compact. However, if the topology of X is very constrained and the acting group G possesses some specified structures, then the existence of invariant Radon measures (Borel measures which are finite on every compact set) or minimal sets can still be true, even if X is noncompact.

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When X is the real line $\mathbb{R}$ and Γ is a finitely generated virtually nilpotent group, Plante obtained the following theorem in [7].

Theorem 1.1. *If Γ is a finitely generated virtually nilpotent subgroup of Homeo$_+(\mathbb{R})$, then Γ preserves a Radon measure on the line.*

Here, Homeo$_+(\mathbb{R})$ means the orientation preserving homeomorphism group on $\mathbb{R}$. More generally, for every finitely generated subgroup of Homeo$_+(\mathbb{R})$ without crossed elements, there always exists an invariant Radon measure (see Prop. 2.2.45 in [5]). Note that the condition having crossed elements implies the existence of free sub-semigroup (see Lemma 2.2.44 in [5]). In particular, a nilpotent group has no crossed elements, since it contains no free sub-semigroup. In [8] V. Solodov obtained an alternative that for a finitely generated subgroup of Homeo$_+(\mathbb{R})$, either it preserves a Radon measure or it contains a free sub-semigroup on two generators.

Considering the existence of minimal set, the following theorem is well known which can be seen in A. Navas' book (see Prop. 2.1.12 in [5]).

Theorem 1.2. *Every finitely generated subgroup of Homeo$_+(\mathbb{R})$ admits a nonempty minimal invariant closed set.*

In [1], L. Beklaryan obtained the following relation between the existences of invariant Radon measure and minimal set.

Theorem 1.3 *(Theorem B [1]). If G is a subgroup of Homeo$(\mathbb{R})$, there exists a G-invariant Radon measure if and only if there exists a nonempty minimal set and the quotient group G/H_G does not contain a free sub-semigroup on two generators. Here H_G is a subgroup of G canonically defined in [1].*

In this paper, we are interested in non-finitely generated subgroups of Homeo$_+(\mathbb{R})$ and get the following theorem.

Theorem 1.4. *Let G be a subgroup of* Homeo$_+(\mathbb{R})$ *without crossed elements. Then the following items are equivalent:*

(1) *there exists a G-invariant Radon measure;*
(2) *there exists a nonempty closed minimal set;*
(3) *there does not exist any infinite tower $\{(I_i, f_i)\}_{i=1}^{\infty}$ such that $\bigcup_{i=1}^{\infty} I_i = \mathbb{R}$.*

For a nilpotent group G of Homeo$_+(\mathbb{R})$, if it does not preserve any Radon measures, we can construct a better infinite tower. This together with a beautiful generalization of Kopell's Lemma due to A. Navas implies the following theorem.

Theorem 1.5. *For every $\alpha > 0$, every nilpotent $C^{1+\alpha}$ subgroup of Homeo$_+(\mathbb{R})$ has an invariant Radon measure and has a nonempty minimal closed invariant set.*

We should note that there is no requirement of finite generation or even countability for the group appearing in Theorem 1.4 and Theorem 1.5. This is the key point that differs from Theorem 1.1 and Theorem 1.2. In [4], N. Guelman and C. Rivas obtained the following criterion for the existence of invariant Radon measure the similar ideas of which will be used frequently in this paper.

Proposition 1.6. *Let G be a subgroup of Homeo$_+(\mathbb{R})$ locally of subexponential growth. If there is $g \in G$ having no fixed points, then G preserves a Radon measure on the line.*

As a supplement of Theorem 1.5, we construct in Section 5 a C^1 commutative subgroup of Homeo$_+(\mathbb{R})$, which has neither invariant Radon measure nor minimal closed set. Finally, we remark that in [6], Navas showed that there exists a subgroup of $\mathrm{Diff}^1_+([0,1])$ having intermediate growth but does not exist a subgroup of $\mathrm{Diff}^{1+\alpha}_+([0,1])$ having intermediate growth, for any $\alpha > 0$.

2. Notions and auxiliary lemmas

In this section, we give some definitions and lemmas which will be used in the proof of the main theorems.

Let G be a subgroup of Homeo$_+(\mathbb{R})$. For $x \in \mathbb{R}$, we denote the *orbit* of x by $Gx \equiv \{g(x) : g \in G\}$. For $g \in G$, we denote by $\mathrm{Fix}(g)$ the set of fixed points of g and denote by $\mathrm{Fix}(G)$ the set of global fixed points of G, i.e. $\mathrm{Fix}(G) = \{x \in \mathbb{R} : \forall g \in G, g(x) = x\}$.

Definition 2.1. Tow elements $f, g \in$ Homeo$_+(\mathbb{R})$ are called *crossed* if there exists an interval (a, b) such that one of f, g, saying f, $\mathrm{Fix}(f) \cap [a, b] = \{a, b\}$ while g sends either a or b into (a, b). Here we allow the cases $a = -\infty$ or $b = +\infty$.

We recall the following facts which will be used frequently.

Fact 2.2. *Let G be a subgroup of Homeo$_+(\mathbb{R})$ and $F = \{f \in G : Fix(f) \neq \emptyset\}$. Then for any $f \in F$ and $g \in G$, $gfg^{-1} \in F$.*

Proof. For any $x \in \mathrm{Fix}(f)$, $(gfg^{-1})(g(x)) = g(f(x)) = g(x)$. Thus $g(x) \in \mathrm{Fix}(gfg^{-1})$ and hence $gfg^{-1} \in F$. □

Fact 2.3. *Let f and g be in Homeo$_+(\mathbb{R})$ which are not crossed and $Fix(f) \cap [\alpha, \beta] = \{\alpha, \beta\}$. If $g(\alpha) > \alpha$ or $g(\beta) < \beta$, then $g((\alpha, \beta)) \cap (\alpha, \beta) = \emptyset$.*

Fact 2.4. *Let G be a subgroup of Homeo$_+(\mathbb{R})$. For any $x \in \mathbb{R}$, set*

$$\alpha := \inf\{Gx\}, \quad \beta := \sup\{Gx\}.$$

Then either $\alpha = -\infty$ (resp. $\beta = +\infty$) or $\alpha \in \mathrm{Fix(G)}$ (resp. $\beta \in \mathrm{Fix(G)}$).

Proof. We may assume that $\alpha \neq -\infty$. Then for any $g \in G$,

$$g(\alpha) \geq \alpha, \quad \text{and} \quad g^{-1}(\alpha) \geq \alpha \Longrightarrow g(\alpha) \leq \alpha.$$

Hence $g(\alpha) = \alpha$. It is similar for β. □

Definition 2.5. If $\{I_i\}_{i=1}^{\infty}$ is a sequence of closed intervals such that $I_1 \subsetneq I_2 \subsetneq \dots$, and $\{f_i\}_{i=1}^{\infty}$ is a sequence of orientation preserving homeomorphisms on $\mathbb{R}$ such that $\mathrm{Fix}(f_i) \cap I_i = \mathrm{End}(I_i)$ for each i, where $\mathrm{End}(I_i)$ denotes the endpoint set of interval I_i, then we call the sequence of pairs $\{(I_i, f_i)\}_{i=1}^{\infty}$ an *infinite tower.*

Lemma 2.6. *Let H be a subgroup of Homeo$_+(\mathbb{R})$ without crossed elements. Suppose H has no infinite tower $\{(I_i, f_i)\}_{i=1}^{\infty}$ such that $\bigcup_{i=1}^{\infty} I_i = \mathbb{R}$. If for every $f \in H$, $Fix(f) \neq \emptyset$, then $Fix(H) \neq \emptyset$.*

Proof. Assume to the contrary that $\mathrm{Fix}(H) = \emptyset$. Choose an $f_1 \neq \mathrm{id}$ in H. Then $\emptyset \neq \mathrm{Fix}(f_1) \subsetneq \mathbb{R}$. Take a connected component (α_1, β_1) of $\mathbb{R} \setminus \mathrm{Fix}(f_1)$.

We claim that $-\infty < \alpha_1 < \beta_1 < +\infty$. In fact, since $\mathrm{Fix}(f_1) \neq \emptyset$, at least one of α_1, β_1 is finite. We may assume that $\alpha_1 \in \mathbb{R}$. Since $\mathrm{Fix}(H) = \emptyset$, by Fact 2.4, there exists $f_2 \in H \setminus \{f_1\}$ such that $f_2(\alpha_1) > \max\{\alpha_1, 2\}$. Since f_1 and f_2 are not crossed, $f_2((\alpha_1, \beta_1)) \cap (\alpha_1, \beta_1) = \emptyset$ by Fact 2.3. Therefore, $\beta_1 \leq f_2(\alpha_1) < +\infty$.

Set $\alpha_2 = \inf\{f_2^i(\alpha_1) : i \in \mathbb{Z}\}$ and $\beta_2 = \sup\{f_2^i(\alpha_1) : i \in \mathbb{Z}\}$. Then either $\alpha_2 \neq -\infty$ or $\beta_2 \neq +\infty$ by the assumption that $\mathrm{Fix}(f) \neq \emptyset$ for every $f \in H$. Similar to the argument of the previous claim, we have $\alpha_2 \in \mathbb{R}$ and $\beta_2 \in \mathbb{R}$. Then $\alpha_2 < \alpha_1 < \beta_1 < \beta_2$ and $\mathrm{Fix}(f_2) \cap [\alpha_2, \beta_2] = \{\alpha_2, \beta_2\}$ and $\beta_2 > 2$.

Similar to the above arguments, we get $\alpha_3, \beta_3 \in \mathbb{R}$ and $f_3 \in H$ such that $\alpha_3 < \alpha_2 < \beta_2 < \beta_3$, and $\mathrm{Fix}(f_3) \cap [\alpha_3, \beta_3] = \{\alpha_3, \beta_3\}$, and $\alpha_3 < -3$.

Continuing this process, we obtain a nested closed intervals $[\alpha_1, \beta_1] \subsetneq [\alpha_2, \beta_2] \subsetneq \cdots$ and a sequence $f_1, f_2, \cdots \in H$ such that

$$\mathrm{Fix}(f_i) \cap [\alpha_i, \beta_i] = \{\alpha_i, \beta_i\}, i = 1, 2, ...,$$

and $\alpha_{2i-1} < -(2i-1)$ and $\beta_{2i} > 2i$ for each $i > 0$. Set $I_i = [\alpha_i, \beta_i]$. Then $\{(I_i, f_i)\}_{i=1}^{\infty}$ is an infinite tower such that $\bigcup_{i=1}^{\infty} I_i = \mathbb{R}$, which contradicts the hypothesis. □

Lemma 2.7. *Let F be a subset of $\mathrm{Homeo}_+(\mathbb{R})$ and let $H = \langle F \rangle$ be the group generated by F. Suppose H has no crossed elements. If for every $f \in F$, $\mathrm{Fix}(f) \neq \emptyset$, then $\mathrm{Fix}(g) \neq \emptyset$ for every $g \in H$.*

Proof. Since H is generated by F, we need only to prove that for any $g_1 \neq g_2 \in H$, if $\mathrm{Fix}(g_1) \neq \emptyset$ and $\mathrm{Fix}(g_2) \neq \emptyset$, then $\mathrm{Fix}(g_1 g_2) \neq \emptyset$. Otherwise, $\mathrm{Fix}(g_2) \subset \mathbb{R} \setminus \mathrm{Fix}(g_1)$ and $\mathrm{Fix}(g_1) \subset \mathbb{R} \setminus \mathrm{Fix}(g_2)$. This clearly implies the existence of crossed elements in H, which is a contradiction. □

Recall that a subgroup H of $\mathrm{Homeo}_+(\mathbb{R})$ is said to act on $\mathbb{R}$ *freely*, if every non-identity element of H has no fixed points.

Lemma 2.8 *(Hölder [5] Proposition 2.2.29). Every group acting freely by homeomorphisms of the real line is isomorphic to a subgroup of $(\mathbb{R}, +)$.*

Lemma 2.9. *Let G be a subgroup of $\mathrm{Homeo}_+(\mathbb{R})$ and let $\Gamma = \{f \in G : \mathrm{Fix}(f) \neq \emptyset\}$. Suppose Γ is a normal subgroup of G. If $\mathrm{Fix}(\Gamma)$ is uncountable, then there exist a G-invariant Radon measure on $\mathbb{R}$ and a nonempty minimal closed subset of $\mathbb{R}$.*

Proof. If $G = \Gamma$, then each point x in $\mathrm{Fix}(\Gamma)$ is minimal and the Dirac measure δ_x is a G-invariant Radon measure on $\mathbb{R}$. So, we may suppose that Γ is a proper subgroup of G.

Let φ be the map on $\mathbb{R}$ defined by collapsing the closure of each component of $\mathbb{R} \setminus \mathrm{Fix}(\Gamma)$ into a point. Then the space $\varphi(\mathbb{R})$ is homeomorphic to an interval K (with or without endpoints). Since Γ is normal in G, $g(\mathrm{Fix}(\Gamma)) = \mathrm{Fix}(\Gamma)$ for every $g \in G$. Thus G/Γ naturally acts on K by letting $g\Gamma.\varphi(x) = \varphi(g(x))$.

If p is an end point of K, then p is G/Γ-invariant and hence $\varphi^{-1}(p)$ is a connected G-invariant closed set J in $\mathbb{R}$. Let q be a boundary point of J. Then q is G-fixed, which contradicts the assumption that Γ is properly contained in G. So, $\varphi(\mathbb{R}) \cong \mathbb{R}$.

We claim that the action of G/Γ on $\varphi(\mathbb{R})$ is free. Otherwise, there is some $g \in G \setminus \Gamma$ and $y \in \varphi(\mathbb{R})$ such that $g\Gamma(y) = y$. Then $\varphi^{-1}(y)$ is a g-invariant closed interval, and hence each point of the boundary of $\varphi^{-1}(y)$ is g-fixed. This is a contradiction.

By the claim and Hölder's Lemma 2.8, this G/Γ action on $\varphi(\mathbb{R})$ is conjugate to translations on the line. We may as well assume that G/Γ are translations on $\mathbb{R}$. Then the Lebesgue measure λ on $\mathbb{R}$ is a G/Γ-invariant Radon measure.

Since φ is increasing and continuous, it is well known that there is a unique continuous Radon measure ℓ on $\mathbb{R}$ such that

$$\ell([a,b]) = \varphi(b) - \varphi(a) = \lambda(\varphi([a,b])).$$

The G-invariance of ℓ can be seen from

$$\ell(g[a,b]) = \lambda(g\Gamma\varphi([a,b])) = \lambda(\varphi([a,b])) = \ell([a,b]).$$

Then ℓ is the required Radon measure on $\mathbb{R}$.

To prove the existence of minimal sets, we discuss in two cases.

Case 1. The G/Γ-action on $\mathbb{R}$ is minimal.

Set $K = \mathbb{R} \setminus \bigcup_{x\in\mathbb{R}} \operatorname{int}(\varphi^{-1}(x))$. Firstly, K is nonempty, since φ is monotonic and $\varphi(\mathbb{R}) \cong \mathbb{R}$. Furthermore, for any $x \in K$, $\varphi^{-1}(\varphi(x))$ has at most two points. We claim that K is a minimal closed subset for G. For any $x, y \in K$, by the minimality of the G/Γ-action, there exists a sequence $(g_n)_{n=1}^{\infty}$ in G such that

$$g_n\Gamma \cdot \varphi(x) = \varphi(g_n x) \to \varphi(y), \text{ as } n \to \infty.$$

If $\varphi^{-1}(\varphi(y)) = \{y\}$, then there is a subsequence of $(g_{n_k})_{k=1}^{\infty}$ such that $g_{n_k}(x) \to y$ as $k \to \infty$. If $\varphi^{-1}(\varphi(y)) = \{y, y'\}$, then we may assume that $y < y'$ and that the choice of (g_n) satisfying that $\varphi(g_n x)$ tends to $\varphi(y)$ from left. Then there is also a subsequence of $(g_{n_k})_{k=1}^{\infty}$ such that $g_{n_k}(x) \to y$ as $k \to \infty$. Therefore, K is a nonempty minimal closed subset for G.

Case 2. The G/Γ-action on $\mathbb{R}$ is not minimal.

Noting that the action of G/Γ on $\mathbb{R}$ consists of translations, $\Lambda \equiv \{g\Gamma(0) : g \in G\}$ is discrete and minimal in this case. Take $x \in \varphi^{-1}(\Lambda) \cap \operatorname{Fix}(\Gamma)$ and let $E = \overline{Gx}$. For any $y \in E$, there is an $\epsilon > 0$ such that if $d(gx, y) < \epsilon$ then $\varphi(gx) = \varphi(y)$ by the discreteness of Λ. Supposed that there exist $g, g' \in G$ such that $\varphi(gx) = \varphi(g'x)$. Then $g\Gamma \cdot \varphi(x) = g'\Gamma \cdot \varphi(x)$. Hence $g\Gamma = g'\Gamma$, by the freeness of the G/Γ-action. Thus $g(x) = g'(x)$, since $x \in \operatorname{Fix}(\Gamma)$. Therefore, $Gx \cap \varphi^{-1}(z)$ has at most one point, for every $z \in \mathbb{R}$. Thus we have $gx = g'x$ for any $g, g' \in G$ with $d(gx, y) < \epsilon$ and $d(g'x, y) < \epsilon$. This forces $y = g(x)$ for some $g \in G$. Thus $E = Gx$ is only a single orbit, which is clearly a nonempty minimal closed subset. □

3. Proof of Theorem 1.4

In this section, we will prove Theorem 1.4. We prove the theorem by showing (1) $\Longleftrightarrow$ (3) and (2) $\Longleftrightarrow$ (3).

Claim *((1) $\Longrightarrow$ (3)). For any subgroup G of* $\mathrm{Homeo}_+(\mathbb{R})$ *without crossed elements, if there exists a G-invariant Radon measure, then there does not exist an infinite tower covering the line.*

Let μ be a G invariant Radon measure on $\mathbb{R}$. If there exists an infinite tower $\{(I_i, f_i)\}_{i=1}^{\infty}$ such that $\bigcup_{n=1}^{\infty} I_n = \mathbb{R}$, then there is $N \in \mathbb{N}^+$ such that $\mu(\operatorname{int}(I_N)) > 0$. Let $B = \operatorname{int}(I_N)$. By the definition of infinite tower and Fact 2.3, we see that $B, f_{N+1}(B), f_{N+1}^2(B), ...$ are pairwise disjoint and are all contained in I_{N+1}. Since μ is G-invariant, we have

$$\mu(B) = \mu(f_{N+1}(B)) = \mu(f_{N+1}^2(B)) = \cdots$$

and then

$$\mu(I_{N+1}) \geq \sum_{i=0}^{\infty} \mu(f_{N+1}^i(B)) = \infty,$$

which contradicts the assumption that μ is a Radon measure.

Claim *((2) $\Longrightarrow$ (3)). For any subgroup G of* $\mathrm{Homeo}_+(\mathbb{R})$ *without crossed elements, if there exists a nonempty minimal closed subset, then there does not exist an infinite tower covering the line.*

Assume that Λ is a nonempty closed minimal subset of $\mathbb{R}$. Fix a point $x \in \Lambda$. If there exists an infinite tower $\{(I_i, f_i)\}_{i=1}^{\infty}$ such that $\bigcup_{n=1}^{\infty} I_n = \mathbb{R}$, then there exists $N \in \mathbb{N}^+$ such that $x \in \mathrm{int}(I_N)$. Write $I_N = [a, b]$. We may assume that $f_N(x) > x$, otherwise replace f_N by f_N^{-1}. Then $\lim_{n\to+\infty} f_N^n(x) = b$. Then $b \in \mathrm{Fix}(f_N) \cap \Lambda$. Since Λ is minimal, there must be some $g \in G$ sending b to (a, b). Then f_N and g are crossed, which contradicts the hypothesis.

Claim *((3) $\Longrightarrow$ (1) + (2)). For any subgroup G of* $\mathrm{Homeo}_+(\mathbb{R})$ *without crossed elements, if G has no infinite tower covering the line, then there exists a G-invariant Radon measure and a nonempty minimal closed subset.*

Case 1 $\mathrm{Fix}(G) \neq \emptyset$. Then take any fixed point $x \in \mathrm{Fix}(G)$, the Dirac measure δ_x is a G invariant Radon measure and $\{x\}$ is a minimal closed subset.

Case 2 $\mathrm{Fix}(G) = \emptyset$. Let $F = \{f \in G : \mathrm{Fix}(f) \neq \emptyset\}$ and let $\Gamma = \langle F\rangle$. By Lemma 2.6 and Lemma 2.7, $\mathrm{Fix}(\Gamma) \neq \emptyset$. Hence $\Gamma = F$ and Γ is a proper normal subgroup of G, by Fact 2.2. Thus $\mathrm{Fix}(\Gamma)$ is G-invariant.

Subcase 2a $\Gamma = \{\mathrm{id}\}$. In this case, the G-action is free. By Hölder's Lemma 2.8, this action is conjugate to translations on the line. Note that the Lebesgue measure is translation invariant and there always exists a minimal closed subset M for any group consisting of translations. Pulling back the Lebesgue measure and the minimal subset M by the conjugation, we obtained a G-invariant Radon measure and a minimal closed subset.

Subcase 2b Γ is nontrivial and $\mathrm{Fix}(\Gamma)$ is uncountable. This case has been proved in Lemma 2.9.

Subcase 2c Γ is nontrivial and $\mathrm{Fix}(\Gamma)$ is countable. Choose $g \in G \setminus \Gamma$ and $x_0 \in \mathrm{Fix}(\Gamma)$. We may assume that $g(x) > x$ for any $x \in \mathbb{R}$. Set $x_n = g^n(x_0), n \in \mathbb{Z}$. Since $\mathrm{Fix}(g) = \emptyset$, $\{x_n : n \in \mathbb{Z}\}$ has no accumulating points. Set

$$X = [x_{-1}, x_2], \quad Y = \mathrm{Fix}(\Gamma) \cap [x_{-1}, x_2].$$

Then Y and $Y \cap [x_0, x_1]$ are countable compact nonempty subsets of X.

Define Y_0 to be the set of isolated points in Y, which is nonempty since Y is countable and compact. Moreover, $Y_0 \cap [x_0, x_1]$ is nonempty. Set $Y_1 = Y \setminus Y_0$ which is a proper closed subset of Y. Define $Y_2 = Y_1 \setminus \{$isolated points in Y_1 under the subspace topology$\}$. For an ordinal β, suppose that we have defined the nonempty closed subsets Y_α for all $\alpha < \beta$. If $\beta = \alpha + 1$, define $Y_\beta = Y_\alpha \setminus \{$isolated points in Y_α under subspace topology$\}$. If β is a limit ordinal, then define $Y_\beta = \bigcap_{\alpha<\beta} Y_\alpha$, which is nonempty by compactness. Since Y is countable, there must exist a countable ordinal γ such that

$$Y_\gamma \cap [x_0, x_1] \neq \emptyset, \text{ and } Y_{\gamma+1} \cap [x_0, x_1] = \emptyset.$$

Thus every point of $Y_\gamma \cap [x_0, x_1]$ is isolated in Y_γ under the subspace topology.

Take $y \in Y_\gamma \cap [x_0, x_1]$. We claim that Gy is a closed subset of $\mathbb{R}$ without accumulating points. Otherwise, there exist $f_n \in G$ such that $f_n(y) \to z \in \mathrm{Fix}(\Gamma)$ as $n \to \infty$. Let $k \in \mathbb{Z}$ be such that $z \in [x_k, x_{k+1}]$. Then $g^{-k} f_n(y) \in (x_{-1}, x_2)$, for sufficiently large n and $g^{-k} f_n(y) \to g^{-k}(z) \in [x_0, x_1]$ as $n \to \infty$. Then Y_γ has an accumulating point in $[x_0, x_1]$ which is a contradiction (note that for any $\alpha \leq \gamma$, $x \in Y_\alpha \cap [x_0, x_1]$ and $f \in G$, if $f(x) \in [x_0, x_1]$ then $f(x) \in Y_\alpha \cap [x_0, x_1]$, since f is a homeomorphism). Thus Gy is a discrete sequence $(y_n)_{n\in\mathbb{Z}}$ which is unbounded in both directions. Let $\mu = \sum_{n\in\mathbb{Z}} \delta_{y_n}$. Then μ is a G-invariant Radon measure and Gy is a minimal closed subset.

4. Proof of Theorem 1.5

For a nilpotent subgroup of $\mathrm{Homeo}_+(\mathbb{R})$, if it does not have an invariant Radon measure, then we can construct an infinite tower which is available for us to deal with the smooth case. Precisely, we have the following lemma. (We use $\mathbb{N}^+$ to denote the set of positive integers.)

Lemma 4.1. *Let G be a nilpotent subgroup of $\mathrm{Homeo}_+(\mathbb{R})$. If there does not exist G-invariant Radon measure of $\mathbb{R}$, then there exist subgroups A, B of G, a closed interval I_0 and an infinite tower $(I_i, h_i)_{i=1}^{\infty}$ such that*

(1) *for any $i \in \mathbb{N}^+$, I_i is a closed interval and I_i is contained in the interior of I_{i+1};*
(2) *$\forall j \in \mathbb{N}^+$, $\mathrm{Fix}(h_j) \cap I_j = \mathrm{End}(I_j)$;*
(3) *$I_0 \subseteq \mathrm{int}(I_1)$ and $\mathrm{Fix}(A) \cap I_0 = \mathrm{End}(I_0)$;*
(4) *$A \lhd B, [B, B] \leq A$, and $h_j \in B$, $\forall j \in \mathbb{N}^+$.*

Proof. Let H be a subgroup of G generated by the elements that have fixed points. Then, by Lemma 2.7, every element of H has fixed points and H is a normal subgroup of G, by Fact 2.2.

Claim 1. $H \neq \{e\}$.

If $H = \{e\}$, then the action of G is free. Thus, by Hölder's Theorem 2.8, we may assume that G consists of the translations of the line. Then the Lebesgue measure is an invariant Radon measure, which contracts the hypothesis of the lemma.

Claim 2. $\mathrm{Fix}(H) = \emptyset$.

If $\mathrm{Fix}(H) \neq \emptyset$, then we conclude that G have an invariant Radon measure and a minimal subset by Lemma 2.9 for case that $\mathrm{Fix}(H)$ is uncountable and by Subcase 2c in the proof of Theorem 1.4 for the case that $\mathrm{Fix}(H)$ is countable. Thus we get a contraction again.

Since H is nilpotent, there exist a finite normal series $\{e\} = H_0 \lhd H_1 \lhd \cdots \lhd H_n = H$ of H, for some positive integer n, such that $[H, H_{i+1}] \leq H_i$, for any $i = 1, \cdots, n-1$. Since $\mathrm{Fix}(H) = \emptyset$ by Claim 2, we can take $m \in \{1, \cdots, n\}$ to be the least integer such that $\mathrm{Fix}(H_m) = \emptyset$.

Case 1. $m = 1$.

In this case, take a nontrivial element $h_0 \in H_1$. Then take a connected component (a_0, b_0) of $\mathbb{R} \setminus \mathrm{Fix}(h_0)$. We claim that $a_0, b_0 \in \mathbb{R}$. In fact, at least one of a_0, b_0 is finite, since h_0 is nontrivial. We may assume that $a_0 \in \mathbb{R}$. By the assumption that $\mathrm{Fix}(H_1) = \emptyset$, there exists some $h \in H_1$ such that $h(a_0) > a_0$. Note that H_1 is commutative. Thus we have $h(a_0) \in \mathrm{Fix}(h_0)$, and then $b_0 \leq h(a_0)$. Therefore, $a_0, b_0 \in \mathbb{R}$ and $\mathrm{Fix}(h_0) \cap [a_0, b_0] = \{a_0, b_0\}$.

Take $h_1 \in H_1$ such that $h_1(b_0) > b_0$. Then $h_1((a_0, b_0)) \cap (a_0, b_0) = \emptyset$, by the commutativity of H_1. Thus $\mathrm{Fix}(h_1) \cap [a_0, b_0] = \emptyset$. Let (a_1, b_1) be the connected component of $\mathbb{R} \setminus \mathrm{Fix}(h_1)$ containing $[a_0, b_0]$. By the similar arguments as above, we have $a_1, b_1 \in \mathbb{R}$ and $\mathrm{Fix}(h_1) \cap [a_1, b_1] = \{a_1, b_1\}$. Proceeding in this way, we obtain an infinite tower $([a_i, b_i], h_i)_{i=1}^{\infty}$ in the end.

Take $A = \langle h_0 \rangle$, $B = H_1$, $I_i = [a_i, b_i]$, and h_i to be as above. Then A, B, I_0 and $(I_i, h_i)_{i=1}^{\infty}$ such defined satisfy the requirements.

Case 2. $m > 1$.

In this case, we take $A = H_{m-1}$ and $B = H_m$. Take a connected component (a_0, b_0) of $\mathbb{R} \setminus \mathrm{Fix}(A)$. By similar arguments as above, we have $a_0, b_0 \in \mathbb{R}$ and $\mathrm{Fix}(A) \cap [a_0, b_0] = \{a_0, b_0\}$ (note that $\mathrm{Fix}(A)$ is B-invariant, since A is normal in B).

Since $\mathrm{Fix}(B) = \emptyset$, there exists $h_1 \in B$ such that $h_1(b_0) > b_0$, which implies that $h_1(a_0, b_0) \cap (a_0, b_0) = \emptyset$. Thus $\mathrm{Fix}(h_1) \cap [a_0, b_0] = \emptyset$. Let (a_1, b_1) be the connected component of $\mathbb{R} \setminus \mathrm{Fix}(h_1)$ containing $[a_0, b_0]$. Then $a_1, b_1 \in \mathbb{R}$ and $\mathrm{Fix}(h_1) \cap [a_1, b_1] = \{a_1, b_1\}$. Moreover, $a_1, b_1 \in \mathrm{Fix}(A)$, since $h_1^i(a_0), h_1^i(b_0) \in \mathrm{Fix}(A)$ for all i, and $\lim_{i \to -\infty} h_1^i(a_0) = a_1, \lim_{i \to +\infty} h_1^i(b_0) = b_1$.

Now $b_1 \in \mathrm{Fix}(A) \cap \mathrm{Fix}(h_1)$. Then we can take $h_2 \in B$ such $h_2(b_1) > b_1$ by $\mathrm{Fix}(B) = \emptyset$. Similarly, we can take an interval $[a_2, b_2]$ such that $[a_1, b_1] \subseteq (a_2, b_2)$ and $\mathrm{Fix}(h_2) \cap [a_2, b_2] = \{a_2, b_2\}$. Since $[B, B] \subset A$, the group $\langle A, h_1 \rangle$ is normal in B. Then we have further $\{a_2, b_2\} \subset \mathrm{Fix}(A) \cap \mathrm{Fix}(h_1) \cap \mathrm{Fix}(h_2)$. Inductively, we can obtain an infinite tower $([a_i, b_i], h_i)_{i=1}^{\infty}$ which satisfies

(i) $\forall i \in \mathbb{N}^+, [a_i, b_i] \subseteq (a_{i+1}, b_{i+1})$,
(ii) $\forall i \in \mathbb{N}^+$, $h_i \in B$, and $\mathrm{Fix}(h_i) \cap [a_i, b_i] = \{a_i, b_i\}$,
(iii) $\forall i \in \mathbb{N}^+$, $\{a_i, b_i\} \subset \mathrm{Fix}(A) \cap \mathrm{Fix}(\langle h_1, \cdots, h_i \rangle)$.

Thus we complete the proof by taking $A = H_{m-1}$, $B = H_m$ $I_i = [a_i, b_i]$, and h_i to be as above. $\square$

To prove Theorem 1.5, we need the following version of generalised Kopell Lemma.

Lemma 4.2 *([6] Proposition 2.8). Given an integer $k \geq 3$, let $\{L_{l_1, \cdots, l_k} : (l_1, \cdots, l_k) \in \mathbb{Z}^k\}$ be a family of closed intervals with disjoint interiors and disposed on $[0, 1]$ respecting the lexicographic order, that is, $L_{l_1, \cdots, l_k}$ is to the left of $L_{l'_1, \cdots, l'_k}$ if and only if $(l_1, \cdots, l_k)$ is lexicographically smaller that $(l'_1, \cdots, l'_k)$. Let $h_1, \cdots, h_k$ be C^1 diffeomorphisms of $[0, 1]$ such that for each $j \in \{1, \cdots, k\}$ and each $(l_1, \cdots, l_k) \in \mathbb{Z}^k$ one has*

$$h_j(L_{l_1, \cdots, l_{j-1}, l_j, \cdots, l_k}) = L_{l_1, \cdots, l_{j-1}, l'_j, \cdots, l'_k},$$

for some $(l'_j, l'_{j+1}, \cdots, l'_k) \in \mathbb{Z}^{k-j+1}$ satisfying $l'_j \neq l_j$. If $\alpha > 0$ satisfies

$$\alpha(1 + \alpha)^{k-2} \geq 1,$$

then $h_1, \cdots, h_{k-1}$ cannot be simultaneously contained in $\mathrm{Diff}_+^{1+\alpha}([0, 1])$.

Proof of Theorem 1.5. Let G be a nilpotent subgroup of $\mathrm{Diff}_+^{1+\alpha}(\mathbb{R})$, for some $\alpha > 0$. By Theorem 1.4, it suffices to show that there exists an invariant Radon measure.

To the contrary, if there does not exist any invariant Radon measure, then, by Lemma 4.1, there exist subgroups A, B of G, a closed interval I_0 and an infinite tower $(I_i, h_i)_{i=1}^{\infty}$ satisfying the properties $(1) - (4)$ in Lemma 4.1. Moreover, we may assume that $h_i(x) > x$, for any $i \in \mathbb{N}^+$ and any $x \in I_0$; otherwise, we can replace it by its inverse. Take a positive integer $k \geq 3$ such that $\alpha(1 + \alpha)^{k-2} \geq 1$ and set

$$\mathcal{L} = \{g(I_0) : g \in \langle A \cup \{h_1, \cdots, h_k\} \rangle\}.$$

Claim. For each $L \in \mathcal{L}$, L is contained in the interior of I_k and there exists a unique $(l_1, \cdots, l_k) \in \mathbb{Z}^k$ such that $L = h_1^{l_1} \cdots h_i^{l_i} \cdots h_k^{l_k}(I_0)$.

In fact, we set $\Gamma = \langle A \cup \{h_1, \cdots, h_k\}\rangle$. Since $\Gamma \leq B$ and $[B, B] \leq A$, we have that $A \lhd \Gamma$ and Γ/A is an Abelian group with finite rank. Note that $\mathrm{Homeo}_+(\mathbb{R})$ is torsion-free. Thus, for any $g \in \Gamma$, there exists a unique $(l_1, \cdots, l_k) \in \mathbb{Z}^k$ such that $gA = h_1^{l_1} \cdots h_k^{l_k} A$. Hence the second part of the claim holds by the fact that I_0 is A-invariant. The first part is easily followed from the observation that the end points of I_k are contained in $\mathrm{Fix}(\Gamma)$. Thus the claim holds.

For every $(l_1, \cdots, l_k) \in \mathbb{Z}^k$, it is clear that

$$\begin{aligned} h_1^{l_1} \cdots h_k^{l_k}(I_0) &\subseteq h_2^{l_2} \cdots h_k^{l_k}(I_1) \\ &\subseteq \cdots \subseteq h_i^{l_i} \cdots h_k^{l_k}(I_{i-1}) \subseteq \cdots \\ &\subseteq h_k^{l_k}(I_{k-1}) \subseteq I_k. \end{aligned}$$

Set $L_{l_1,\cdots,l_k} = h_1^{l_1} \cdots h_k^{l_k}(I_0)$. Thus $\{L_{l_1,\cdots,l_k} : (l_1, \cdots, l_k) \in \mathbb{Z}^k\}$ is a family of closed intervals contained in I_k with disjoint interiors. By the assumption that $h_i(x) > x$, for every $i \in \mathbb{N}^+$ and every $x \in I_0$, $\{L_{l_1,\cdots,l_k} : (l_1, \cdots, l_k) \in \mathbb{Z}^k\}$ are disposed on I_{k+1} respecting the lexicographic order. By the Claim, we have $\mathcal{L} = \{L_{l_1,\cdots,l_k} : (l_1, \cdots, l_k) \in \mathbb{Z}^k\}$. Now for each $j \in \{1, \cdots, k\}$ and each $(l_1, \cdots, l_k) \in \mathbb{Z}^k$ one has

$$h_j(L_{l_1,\cdots,l_{j-1},l_j,l_{j+1},\cdots,l_k}) = L_{l_1,\cdots,l_{j-1},l_j+1,l_{j+1},\cdots,l_k}.$$

By the choice of k and Lemma 4.2, we know that $h_1, \cdots, h_{k-1}$ cannot be contained in $\mathrm{Diff}_+^{1+\alpha}(I_k)$ simultaneously, which contradicts the hypothesis that G is a subgroup of $\mathrm{Diff}_+^{1+\alpha}(\mathbb{R})$. This completes the proof. □

5. A counterexample of C^1 subgroup

In this section, we construct an example which shows that Theorem 1.5 does not hold for C^1 commutative subgroups of $\mathrm{Homeo}_+(\mathbb{R})$. The following construction is due to Yoccoz ([3, Lemma 2.1]).

Lemma 5.1. *For any closed intervals $I = [a, b]$, $J = [c, d]$ there exists a C^1 orientation preserving diffeomorphism $\phi_{I,J} : I \longrightarrow J$ with the following properties:*

(1) $\phi'_{I,J}(a) = \phi'_{I,J}(b) = 1$;
(2) *Given $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in [a, b]$,*

$$\left|\phi'_{I,J}(x) - 1\right| < \varepsilon, \quad \textit{whenever} \quad \left|\frac{d-c}{b-a} - 1\right| < \delta;$$

(3) *For any closed interval K and for any $x \in I$,*

$$\phi_{I,K}(x) = \phi_{J,K}(\phi_{I,J}(x)).$$

Theorem 5.2. *There exists a non-finitely generated abelian group G consisting of C^1 orientation preserving diffeomorphisms of $\mathbb{R}$ such that there exists an infinite tower $\{(I_j, f_j)\}_{j=1}^{\infty}$ with $f_j \in G, j = 1, 2, \cdots$, such that $\bigcup_{j=1}^{\infty} I_j = \mathbb{R}$.*
(Then, by Theorem 1.4, there exists neither G invariant Radon measure nor nonempty closed minimal set.)

Proof. Firstly, we define $f_1 : [-1,1] \longrightarrow [-1,1]$ by

$$f_1(x) = \begin{cases} \exp\left(\frac{1}{x-1} - \frac{1}{x+1}\right) + x, & x \in (-1,1) \\ -1, & x = -1 \\ 1, & x = 1. \end{cases}$$

Then f_1 satisfies

- f_1 is a C^1 orientation preserving diffeomorphism of $[-1,1]$;
- $f_1(\pm 1) = \pm 1$ and $f_1(x) > x$ for any $x \in (-1,1)$;
- $f_1'(-1) = f_1'(1) = 1$.

Next, choose two infinite sequences $-2 < \cdots < a_2 < a_1 < a_0 = -1$ and $1 = b_0 < b_1 < b_2 < \cdots < 2$ such that

$$\lim_{n\to\infty} a_n = -2, \quad \lim_{n\to\infty} b_n = 2,$$

and

$$\lim_{n\to\infty} \frac{a_{n-1} - a_n}{a_n - a_{n+1}} = 1, \ \lim_{n\to\infty} \frac{b_{n+1} - b_n}{b_n - b_{n-1}} = 1.$$

For example, we can take

$$a_n = -2 + \frac{1}{n+1}, \ b_n = 2 - \frac{1}{n+1}, \ n = 1, 2, \cdots.$$

Define

$$f_2(x) = \begin{cases} \phi_{[a_{n+1},a_n],[a_n,a_{n-1}]}(x), & x \in [a_{n+1}, a_n], n = 1, 2, \cdots \\ \phi_{[a_1,a_0],[-1,1]}(x), & x \in [a_1, a_0] \\ \phi_{[-1,1],[b_0,b_1],}(x), & x \in [-1,1] \\ \phi_{[b_n,b_{n+1}],[b_{n+1},b_{n+2}]}(x), & x \in [b_n, b_{n+1}], n = 0, 1, 2, \cdots \\ \pm 2, & x = \pm 2. \end{cases}$$

Then, by Lemma 5.1 and the choices of $\{a_n\}$ and $\{b_n\}$, f_2 satisfies

- f_2 is a C^1 orientation preserving diffeomorphism of $[-2,2]$;
- $f_2(\pm 2) = \pm 2$ and $f_2(x) > x$ for any $x \in (-2,2)$;
- $f_2'(-2) = f_2'(2) = 1$.

Then we extend f_1 to a diffeomorphism $\tilde{f}_1$ of $[-2,2]$:

$$\tilde{f}_1(x) = \begin{cases} f_2^{-(n+1)} f_1 f_2^{n+1}(x), & x \in [a_{n+1}, a_n], n = 1, 2, \cdots \\ f_1(x), & x \in [-1,1] \\ f_2^{n+1} f_1 f_2^{-(n+1)}(x), & x \in [b_n, b_{n+1}] \\ \pm 2, & x = \pm 2. \end{cases}$$

We denote $\tilde{f}_1$ by f_1 for $x \in [-2,2]$. Then

$$f_1 f_2(x) = f_2 f_1(x), \quad \forall x \in [-2, 2].$$

Continuing the above process, we can construct a sequence of commuting C^1 orientation preserving diffeomorphisms $f_1, f_2, \cdots$ of $\mathbb{R}$. More precisely, assume that we have constructed pairwise commuting C^1 orientation preserving diffeomorphisms $f_1, \cdots, f_k$ of $[-k, k]$ for $k \in \mathbb{N}^+$ with the following properties:

(1) $f_i(\pm i) = \pm i$ and $\forall x \in (-i, i)$, $f_i(x) > x$, for $i = 1, 2, \cdots, k$;
(2) $f_i'(-i) = f_i'(i) = 1$, for $i = 1, 2, \cdots, k$;
(3) $f_i f_j(x) = f_j f_i(x)$ for all $x \in [-k, k]$ and $1 \le i, j \le k$.

Then choose two infinite sequences $-(k+1) < \cdots < c_2 < c_1 < c_0 = -k$ and $k = d_0 < d_1 < d_2 < \cdots < k+1$ such that

$$\lim_{n\to\infty} c_n = -(k+1), \quad \lim_{n\to\infty} d_n = k+1,$$

and

$$\lim_{n\to\infty} \frac{c_{n-1} - c_n}{c_n - c_{n+1}} = 1, \quad \lim_{n\to\infty} \frac{d_{n+1} - d_n}{d_n - d_{n-1}} = 1.$$

For example, we can take

$$c_n = -(k+1) + \frac{1}{n+1}, \; d_n = k+1 - \frac{1}{n+1}, \; n = 1, 2, \cdots.$$

Define

$$f_{k+1}(x) = \begin{cases} \phi_{[c_{n+1},c_n],[c_n,c_{n-1}]}(x), & x \in [c_{n+1}, c_n], n = 1, 2, \cdots \\ \phi_{[c_1,c_0],[-k,k]}(x), & x \in [c_1, c_0] \\ \phi_{[-k,k],[d_0,d_1],}(x), & x \in [-k, k] \\ \phi_{[d_n,d_{n+1}],[d_{n+1},d_{n+2}]}(x), & x \in [d_n, d_{n+1}], n = 0, 1, 2, \cdots \\ \pm(k+1), & x = \pm(k+1). \end{cases}$$

Then by Lemma 5.1 and the choices of $\{c_n\}$ and $\{d_n\}$, f_{k+1} satisfies

- f_{k+1} is a C^1 orientation preserving diffeomorphism of $[-k-1, k+1]$;
- $f_{k+1}(\pm(k+1)) = \pm(k+1)$ and $f_{k+1}(x) > x$ for any $x \in (-(k+1), k+1)$;
- $f_{k+1}'(-k-1) = f_{k+1}'(k+1) = 1$.

We extend $f_1, \cdots, f_k$ to diffeomorphisms $\tilde{f}_1, \cdots, \tilde{f}_k$ of $[-(k+1), k+1]$: for $i = 1, \cdots, k$,

$$\tilde{f}_i(x) = \begin{cases} f_{k+1}^{-(n+1)} f_i f_{k+1}^{n+1}(x), & x \in [c_{n+1}, c_n], n = 1, 2, \cdots \\ f_k(x), & x \in [-k, k], \\ f_{k+1}^{n+1} f_i f_{k+1}^{-(n+1)}(x), & x \in [d_n, d_{n+1}], \\ \pm(k+1), & x = \pm(k+1). \end{cases}$$

Denote $\tilde{f}_i$ by f_i for $x \in [-(k+1), k+1]$. Then $f_1, \cdots, f_{k+1}$ are commuting orientation preserving C^1 diffeomorphisms of $[-(k+1), k+1]$.

From the constructing process, we see that $[-1,1] \subsetneq [-2,2] \subsetneq \cdots$ and $f_1, f_2, \cdots$ form an infinite tower, and the group G generated by $f_1, f_2, \cdots$ is a non-finitely generated abelian group consisting of C^1 orientation preserving diffeomorphisms of $\mathbb{R}$. This completes the proof. □

It was pointed out by the referee that this example was essentially in [2]. However, it is worthwhile to construct it explicitly here, especially for the convenience of the readers.

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References

[1] L.A. Beklaryan, On analogues of the Tits alternative for groups of homeomorphisms of the circle and the line, Mat. Zametki 71 (3) (2002) 334–347 (in Russian); translation in Math. Notes 71 (3–4) (2002) 305–315.
[2] C. Bonatti, E. Farinelli, Centralizers of C^1-contractions of the half line, Groups Geom. Dyn. 9 (3) (2015) 831–889.
[3] B. Farb, J. Franks, Groups of homeomorphisms of one-manifolds III: nilpotent subgroups, Ergod. Theory Dyn. Syst. 23 (2003) 1467–1484.
[4] N. Guelman, C. Rivas, Quasi-invariant measures for some amenable groups acting on the line, Algebraic Geom. Topol. 18 (2) (2018) 1067–1076.
[5] A. Navas, Groups of Circle Diffeomorphisms, Chicago Lectures in Math., 2011.
[6] A. Navas, Growth of groups and diffeomorphisms of the interval, Geom. Funct. Anal. 18 (2008) 988–1028.
[7] J. Plante, On solvable groups acting on the real line, Trans. Am. Math. Soc. 278 (1983) 401–414.
[8] V.V. Solodov, Homeomorphisms of the circle and foliations, Izv. Akad. Nauk SSSR, Ser. Mat. (Math. USSR-Izv.) 48 (3) (1984) 599–613.