

Equicontinuity of minimal sets for amenable group actions on dendrites

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ABSTRACT. We show that if G is an amenable group acting on a dendrite X , then the restriction of G to any minimal set K is equicontinuous, and K is either finite or homeomorphic to the Cantor set.

1. Introduction

It is well known that every continuous action of a topological group G on a compact metric space X must have a minimal set K . A natural question is to ask what can be said about the topology of K , and the dynamics of the subsystem (K, G) . The answer to this question certainly depends on the topology of X and involves the algebraic structure of G . We assume throughout that groups are topological groups, and that the actions are continuous.

In the case of an orientation-preserving group action on the circle \mathbb{S}^1 , the topology of minimal sets and the dynamics on them are well understood. In fact, for any action of a topological group G on \mathbb{S}^1 , the minimal set K can only be a finite set, a Cantor set, or the whole circle (see, for example, [13]). The interaction between the topology of K and the algebraic structure of G arises as follows.

- If K is a Cantor set, then (K, G) is semi-conjugate to a minimal action of G on \mathbb{S}^1 .
- If $K = \mathbb{S}^1$, then (K, G) is either equicontinuous, or (K, G) is ε -strongly proximal for some $\varepsilon > 0$, and G contains a free non-commutative subgroup (so, in particular, G cannot be amenable; see [8]).

The classes of minimal group actions on the circle up to topological conjugacy have been classified by Ghys using bounded Euler class (see [4, 5]).

Recently, there has been considerable progress in the study of group actions on dendrites. Minimal group actions on dendrites appear naturally in the theory of 3-dimensional hyperbolic geometry (see, for example, [2, 10]). Shi proved that every minimal group action on a dendrite is strongly proximal, and the acting group cannot be amenable (see [15, 16]). Based on the results obtained by Marzougui and Naghmouchi in [9], Shi and Ye showed that an amenable group action on a dendrite always has a minimal set consisting

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of 1 or 2 points (see [17]), which is also implied by the work of Malyutin and Duchesne–Monod (see [7, 3]). For group actions on dendrites with no finite orbits, Glasner and Megrelishvili showed the extreme proximality of minimal subsystems and the strong proximality of the whole system; for amenable group actions on dendrites, they showed that every infinite minimal subsystem is almost automorphic (see [6]). For \mathbb{Z} actions on dendrites, Naghmouchi proved that every minimal set is either finite or an adding machine (see [12]).

We prove the following theorem in this paper, which extends the corresponding result for \mathbb{Z} actions in [12], and answers a question proposed by Glasner and Megrelishvili in [6].

THEOREM 1.1. *Let G be an amenable group acting on a dendrite X , and suppose that K is a minimal set for the action. Then (K, G) is equicontinuous, and K is either finite or homeomorphic to the Cantor set.*

Recently, Shi and Ye have shown that every amenable group action on uniquely arcwise connected continua (without the assumption of local connectedness) must have a minimal set consisting of 1 or 2 points (see [18]). We end this introduction with the following general question:

What results holding for group actions on dendrites can be extended to actions on uniquely arcwise connected continua?

In the following, we assume all the groups appearing in this paper are countable.

2. Preliminaries

2.1. Group actions. Let X be a compact metric space, $\text{Homeo}(X)$ its homeomorphism group, and let G be a group. A group homomorphism $\phi : G \rightarrow \text{Homeo}(X)$ is called an *action* of G on X ; we also write (X, G) to denote an action of G on X . For brevity, we usually write gx or $g(x)$ instead of $\phi(g)(x)$.

The *orbit* of $x \in X$ under the action of G is the set

$$Gx = \{gx \mid g \in G\}.$$

For a subset $A \subseteq X$, set $GA = \bigcup_{x \in A} Gx$; a set A is said to be *G -invariant* if $GA = A$; finally, a point $x \in X$ is called a *fixed point* of the action if $Gx = \{x\}$. If A is a G -invariant closed subset of X and $\overline{Gx} = A$ for every $x \in A$ (that is, the orbit of each point is dense), then A is called a *minimal set for the action*. In this setting every action has a minimal set by Zorn's lemma.

A Borel probability measure μ on X is called *G -invariant* if $\mu(g(A)) = \mu(A)$ for every Borel set $A \subset X$ and every $g \in G$. The following lemma follows directly from the G -invariance of the support $\text{supp}(\mu)$ (which is automatic).

LEMMA 2.1. *If (X, G) is minimal and μ is a G -invariant Borel probability measure on X , then $\text{supp}(\mu) = X$.*

LEMMA 2.2. *Suppose that a group G acts on a compact metric space X , and that K is a minimal set in X carrying a G -invariant Borel probability measure μ . If U and V are open sets in X such that $V \supset U$ and $g(V \cap K) \subset U \cap K$ for some $g \in G$, then $K \cap (V \setminus \overline{U}) = \emptyset$.*

PROOF. Assume to the contrary that there is some $u \in K \cap (V \setminus \overline{U})$. Then there is an open neighborhood $W \ni u$ with $W \subset V \setminus \overline{U}$. By Lemma 2.1, we have $\mu(W \cap K) > 0$. This then implies that $\mu(V \cap K) = \mu(g(V \cap K)) \leq \mu(U \cap K) < \mu(V \cap K)$, a contradiction. \square

2.2. Amenable groups. Amenability was first introduced by von Neumann. Recall that a countable group G is said to be *amenable* if there is a sequence of finite sets F_i ($i = 1, 2, 3, \dots$) such that

$$\lim_{i \rightarrow \infty} \frac{|gF_i \triangle F_i|}{|F_i|} = 0$$

for every $g \in G$, where $|F_i|$ is the number of elements in F_i . The sequence (F_i) is called a *Følner sequence* and each F_i a *Følner set*. It is well known that solvable groups and finite groups are amenable and that any group containing a free non-commutative subgroup is not amenable. One may consult the monograph of Paterson [14] for the proofs of the following lemmas.

LEMMA 2.3. *Every subgroup of an amenable group is amenable.*

LEMMA 2.4. *A group G is amenable if and only if every action of G on a compact metric space X has a G -invariant Borel probability measure on X .*

2.3. Dendrites. A *continuum* is a non-empty connected compact metric space. A continuum is said to be *non-degenerate* if it is not a single point. An *arc* is a continuum which is homeomorphic to the closed interval $[0, 1]$. A continuum X is *uniquely arcwise connected* if for any two points $x \neq y \in X$ there is a unique arc $[x, y]$ in X connecting x and y .

A *dendrite* X is a locally connected, uniquely arcwise connected, continuum. If Y is a subcontinuum of a dendrite X , then Y is called a *subdendrite* of X . For a dendrite X and a point $c \in X$, if $X \setminus \{c\}$ is not connected, then c is called a *cut point* of X ; if $X \setminus \{c\}$ has at least 3 components, then c is called a *branch point* of X .

Lemmas 2.5 to 2.8 are taken from [11].

LEMMA 2.5. *Let X be a dendrite with metric d . Then, for every $\varepsilon > 0$, there is a $\delta > 0$ such that $\text{diam}([x, y]) < \varepsilon$ whenever $d(x, y) < \delta$.*

LEMMA 2.6. *Let X be a dendrite. If A_i ($i = 1, 2, 3, \dots$) is a sequence of mutually disjoint sub-dendrites of X , then $\text{diam}(A_i) \rightarrow 0$ as $i \rightarrow \infty$.*

LEMMA 2.7. *Let X be a dendrite. Then X has at most countably many branch points. If X is nondegenerate, then the cut point set of X is uncountable.*

LEMMA 2.8. *Let X be a dendrite and $c \in X$. Then each component U of $X \setminus \{c\}$ is open in X , and $\bar{U} = U \cup \{c\}$.*

Now we give a proof of the following technical lemma.

LEMMA 2.9. *Let X be a dendrite and let $f : X \rightarrow X$ be a homeomorphism. Suppose o is a fixed point of f , and let c_1, c_2 be cut points of X different from o . Suppose that U is a component of $X \setminus \{c_1\}$ not containing o , that V is a component of $X \setminus \{c_2\}$ not containing o , and that $f(c_1) \in V$. Then $f(U) \subset V$.*

PROOF. Assume to the contrary that there is some $u \in U$ with $f(u) \notin V$. Since c_2 is a cut point, $f(c_1) \in V$, $o \notin V$, and $f(o) = o$, we have $c_2 \in [f(o), f(c_1)]$ and $c_2 \in [f(u), f(c_1)]$. This implies that $f^{-1}(c_2) \in [o, c_1] \cap [u, c_1] = \{c_1\}$ since $o \notin U$. Thus $f(c_1) = c_2$, which contradicts the assumption that $f(c_1) \in V$. \square

If $[a, b]$ is an arc in a dendrite X , denote by $[a, b)$, $(a, b]$, and (a, b) the sets $[a, b] \setminus \{b\}$, $[a, b] \setminus \{a\}$, and $[a, b] \setminus \{a, b\}$, respectively.

2.4. Equicontinuity. Let X be a compact metric space with metric d , and let G be a group acting on X . Two points $x, y \in X$ are said to be *regionally proximal* if there are sequences $(x_i), (y_i)$ in X and (g_i) in G such that $x_i \rightarrow x$ and $y_i \rightarrow y$ as $i \rightarrow \infty$, and $\lim g_i x_i = \lim g_i y_i = w$ for some $w \in X$. If x, y are regionally proximal and $x \neq y$, then $\{x, y\}$ is said to be a *non-trivial regionally proximal pair*. The action (X, G) is *equicontinuous* if, for every $\varepsilon > 0$, there is a $\delta > 0$ such that $d(gx, gy) < \varepsilon$ for all $g \in G$ whenever $d(x, y) < \delta$.

The following lemma can be found in [1].

LEMMA 2.10. *Suppose (X, G) is a group action. Then (X, G) is equicontinuous if and only if it contains no non-trivial regionally proximal pair.*

3. Proof of the main theorem

In this section we are going to show our main result. Before doing this we state two simple lemmas.

LEMMA 3.1. *Suppose a group G acts on the closed interval $[0, 1]$. If $K \subset [0, 1]$ is minimal, then K contains at most 2 points.*

PROOF. Let $x = \inf K$ and $y = \sup K$. Then G preserves the set $\{x, y\}$, so $K = \{x, y\}$ by the minimality of K . \square

LEMMA 3.2 (See [17]). *Let G be an amenable group acting on a dendrite X . Then there is a G -invariant set consisting of 1 or 2 points.*

Now we are ready to prove the main result.

PROOF OF THEOREM 1.1. We first show that (K, G) is equicontinuous.

Assume to the contrary that (K, G) is not equicontinuous. Then by Lemma 2.10, there are $u \neq v \in K$ such that u, v are regionally proximal; that is, there are sequences $(u_i), (v_i)$ in X and (g_i) in G with

$$(3.1) \quad u_i \rightarrow u, v_i \rightarrow v, \lim g_i x_i = \lim g_i y_i = w$$

as $i \rightarrow \infty$ for some $w \in K$.

By Lemma 3.2, there are $o_1, o_2 \in X$ such that $\{o_1, o_2\}$ is a G -invariant set. Then $[o_1, o_2]$ is G -invariant by the unique arcwise connectedness of X . From the assumption, K is infinite so $K \cap [o_1, o_2] = \emptyset$ by Lemma 3.1. Without loss of generality, we may suppose that $o_1 = o_2$ and denote this common point by o ; otherwise, we need only collapse $[o_1, o_2]$ to one point. Then o is a fixed point for the action.

Case 1. $[u, o] \cap [v, o] = \{o\}$ (see Fig.1(1)). By Lemma 2.7, we can choose cut points $c_1 \in (u, o)$ and $c_2 \in (v, o)$. Let D_u be the component of $X \setminus \{c_1\}$, which contains u ; let D_v be the component of $X \setminus \{c_2\}$, which contains v . From minimality and Lemma 2.8, there is some $g' \in G$ with $g'w \in D_u$. From (3.1) and Lemma 2.5, we have

$$(3.2) \quad u_i \in D_u, v_i \in D_v \text{ and } g'_i g_i [u_i, v_i] \subset D_u$$

for large enough i . Write $g = g' g_i$. Then $o \in [u_i, v_i]$ and $g(o) \in D_u$. This is a contradiction, since o is fixed by G .

Case 2. $[u, o] \cap [v, o] = [z, o]$ for some $z \neq o$.

Subcase 2.1. $z = v$ (see Fig.1(2)). Then $u \neq z$ and $z \in K$. Take a cut point $c_1 \in (u, z)$ and let D_u be the component of $X \setminus \{c_1\}$ which contains u . Then $v \notin D_u$, and there is some $g \in G$ with $gz \in D_u$ by the minimality of K . Take a cut point $c_2 \in (z, o)$ which is sufficiently close to z to ensure that $g(c_2) \in D_u$. Let D_z be the component of $X \setminus \{c_2\}$ which contains z . By

Lemma 2.4, there is a G -invariant Borel probability measure on K . Applying Lemma 2.9, we get $g(D_z) \subset D_u$, which contradicts Lemma 2.2, since $z \in D_z \setminus \overline{D_u}$.

Subcase 2.2. $z = u$. In this case we can deduce a contradiction along the lines of the argument in Subcase 2.1.

Subcase 2.3. $z \neq u$ and $z \neq v$ (see Fig.1(3)). Take a cut point $c_1 \in (u, z)$. Let D_u be the component of $X \setminus \{c_1\}$, which contains u . Similar to the argument in Case 1, there is some $g \in G$ with $g(z) \in D_u$. Take a cut point $c_2 \in (z, o)$ which is sufficiently close to z to ensure that $g(c_2) \in D_u$. Let D_z be the component of $X \setminus \{c_2\}$, which contains z . Then $g(D_z) \subset D_u$ by Lemma 2.9. This contradicts Lemma 2.2 since $v \in D_z \setminus \overline{D_u}$.

Now we prove that if K is not finite, then K is homeomorphic to the Cantor set. If not, then there is some non-degenerate connected component Y of K . Clearly, for any $g, g' \in G$, either $g(Y) = g'(Y)$ or $g(Y) \cap g'(Y) = \emptyset$. This, together with Lemma 2.6 and the equicontinuity of (K, G) , implies that the subgroup $H = \{g \in G : g(Y) = Y\}$ has finite index in G . It follows that (Y, H) is minimal. This contradicts Lemma 3.2 and Lemma 2.3, since Y is a non-degenerate dendrite. \square

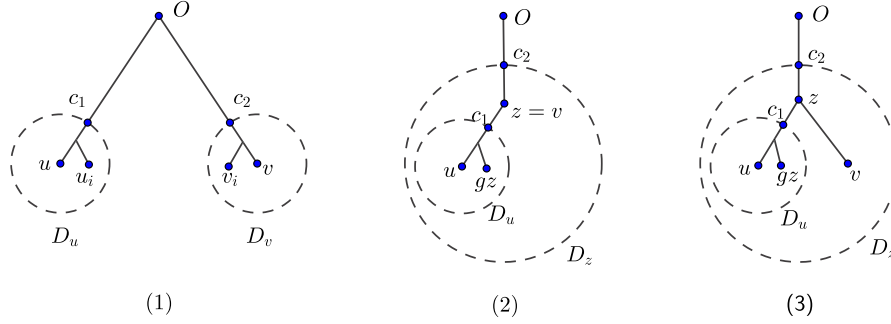


Fig. 1

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