

TOPOLOGICAL METHODS IN NONLINEAR ANALYSIS

Vol. 67, No. 2

June 2026

A NEW CONSTRUCTION OF COUNTEREXAMPLES TO THE BOUNDED ORBIT CONJECTURE

JIEHUA MAI — ENHUI SHI — KESONG YAN — FANPING ZENG

Topol. Methods Nonlinear Anal. **67** (2026), 347–356
DOI: 10.12775/TMNA.2025.055

Published by the
Juliusz Schauder Center
TORUŃ, 2026

ISSN 1230-3429
ISBN 977-1230-342-60-4

This e-offprint is for personal use only and shall not be self-archived in electronic repositories. If you wish to self-archive your article, you may use the accepted manuscript pre-print version for positioning on your own website, provided that the journal reference to the published version (with DOI and published page numbers) is given. You may further deposit the accepted manuscript pre-print version in any repository, provided it is only made publicly available 12 months after official publication or later and provided acknowledgement is given to the original source of publication and a link is inserted to the published article on the TMNA website.

A NEW CONSTRUCTION OF COUNTEREXAMPLES TO THE BOUNDED ORBIT CONJECTURE

JIEHUA MAI — ENHUI SHI — KESONG YAN — FANPING ZENG

ABSTRACT. The bounded orbit conjecture says that every homeomorphism on the plane with each of its orbits being bounded must have a fixed point. Brouwer's translation theorem asserts that the conjecture is true for orientation preserving homeomorphisms, but Boyles' counterexample shows that it is false for the orientation reversing case. In this paper, we give a more comprehensible construction of counterexamples to the conjecture. Roughly speaking, we construct an orientation reversing homeomorphism f on the square $J^2 = [-1, 1]^2$ with $\omega(x, f) = \{(-1, 1), (1, 1)\}$ and $\alpha(x, f) = \{(-1, -1), (1, -1)\}$ for each $x \in (-1, 1)^2$. Then, by a semi-conjugacy defined by pushing an appropriate part of ∂J^2 into $(-1, 1)^2$, f induces a homeomorphism on the plane, which is a counterexample.

1. Introduction

In 1912, Brouwer [2] obtained his translation theorem, which has been a fundamental tool in studying surface homeomorphisms nowadays (see e.g. [7]).

2020 *Mathematics Subject Classification.* 37E30.

Key words and phrases. Fixed point; periodic point; plane homeomorphism; bounded orbit; ω -limit set; α -limit set.

Jiehua Mai and Fanping Zeng are supported by NNSF of China (Grant No. 12261006) and Project of Guangxi First Class Disciplines of Statistics (No. GJKY-2022-01).

Enhui Shi is supported by NNSF of China (Grant No. 12271388).

Kesong Yan is supported by NNSF of China (Grant No. 12561032) and the Talent Research Startup Foundation of Hainan Normal University (HSZK-KYQD-202519).

THEOREM 1.1 (Brouwer translation theorem). *If $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is an orientation preserving homeomorphism and has no fixed point, then f is a translation, that is each orbit of f is unbounded.*

Some simple modern proofs or extensions of the theorem can be found in [3]–[6], [8]. The following example shows that Brouwer translation theorem does not hold for orientation reversing homeomorphisms (see [1]).

EXAMPLE 1.2. Let $h: \mathbb{R}^2 \rightarrow \mathbb{R}^2, (x, y) \mapsto (-x, y)$ be the reflection across y -axis. Let $g: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $g(x, y) = (x, y - |x| + 1)$ for $|x| < 1$ and $g(x, y) = (x, y)$ for $|x| \geq 1$. Then $g \circ h$ is an orientation reversing homeomorphism and has no fixed point, but each point (x, y) with $|x| \geq 1$ is a 2-periodic point; in particular, the orbit of (x, y) is bounded.

Note that the homeomorphism $g \circ h$ in Example 1.2 has many points with an unbounded orbit, say the points in the strip determined by $|x| < 1$. This naturally leads to the following conjecture.

CONJECTURE 1.3 (Bounded orbit conjecture). *If $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a homeomorphism and each orbit of f is bounded, then f has a fixed point.*

In 1981, Boyles [1] solved this long outstanding conjecture by constructing a counterexample. Checking this counterexample carefully, we found that it is essentially a very clever improvement of Example 1.2. The core idea of the construction is to force each point in the strip $|x| < 1$ to move upwards as well as to the left and right, so that its limit sets lie in the vertical lines $|x| = 1$. In order to achieve this construction process, the author divided the strip $|x| < 1$ into infinitely many irregular blocks, and carefully adjusted the distances of movements of each block to ensure the convergence of relevant series. These intricate constructions brought the readers some difficulties in understanding why such counterexamples do exist.

The aim of the paper is to give a more comprehensible construction of counterexamples. The style of our construction is more topological rather than geometrical, so that we can avoid the difficulties caused by irregularity of blocks and possible divergence of series appearing in [1]. In addition, the ideas of our construction are also available to other problems around plane homeomorphisms (see e.g. [9]).

In Section 3, we construct an orientation reversing homeomorphism f on the square $J^2 = [-1, 1]^2$ with $\omega(x, f) = \{(-1, 1), (1, 1)\}$ and $\alpha(x, f) = \{(-1, -1), (1, -1)\}$ for each $x \in (-1, 1)^2$ (see Lemma 3.1 for the details). Then by a topological semi-conjugacy defined by pushing an appropriate part of ∂J^2 into $(-1, 1)^2$, f induces a homeomorphism on the square, and then induces a homeomorphism on the plane by a topological conjugacy, which meets our requirements. Explicitly, we get the following theorem.

THEOREM 1.4. *There exists an orientation reversing fixed point free homeomorphism $h: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that:*

- (a) *The set of all periodic points of h consists of 2-periodic points and is equal to the subset $(-\infty, -1] \cup [1, \infty)$ of x -axis.*
- (b) *For each non-periodic point x , the orbit of x is bounded and the limit sets $\omega(x, h) = \alpha(x, h) = \{(-1, 0), (1, 0)\}$.*

Noting that both the homeomorphism given in [1] and that given in Theorem 1.4 have infinitely many periodic points, the following problem is left.

PROBLEM 1.5. Does there exist an orientation reversing fixed point free homeomorphism on the plane which has no unbounded orbit and has only finitely many periodic orbits?

2. Definitions and notations

Let $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{Z}_-$, and $\mathbb{N}$ be the sets of integers, nonnegative integers, non-positive integers, and positive integers, respectively. For any $n \in \mathbb{N}$ write $\mathbb{N}_n = \{1, \dots, n\}$. For $r, s \in \mathbb{R}$, we use (r, s) to denote a point in $\mathbb{R}^2$. If $r < s$, we also use (r, s) to denote an open interval in $\mathbb{R}$. These will not lead to confusion. For example, if we write $(r, s) \in X$, then (r, s) will be a point; if we write $t \in (r, s)$, then (r, s) will be a set, and hence is an open interval.

Let X be a metric space and $f: X \rightarrow X$ be a homeomorphism. Let f^0 be the identity map of X , and let $f^n = f \circ f^{n-1}$ be the composition map of f and f^{n-1} . For $x \in X$, the sets $O(x, f) \equiv \{f^n(x) : n \in \mathbb{Z}\}$, $O_+(x, f) \equiv \{f^n(x) : n \in \mathbb{Z}_+\}$, and $O_-(x, f) \equiv \{f^{-n}(x) : n \in \mathbb{Z}_+\}$ are called the *orbit*, *positive orbit*, and *negative orbit* of x , respectively. A point $x \in X$ is called a *fixed point* of f if $f(x) = x$. We call $x \in X$ a *periodic point* of f if $f^n(x) = x$ for some $n \in \mathbb{N}$; the smallest such n is called the *period* of x under f . A periodic point of period n is also called an *n -periodic point*. Denote by $P(f)$ the set of periodic points of f . For $x, y \in X$, if there is a sequence of positive integers $n_1 < n_2 < \dots$ such that $f^{n_i}(x) \rightarrow y$ then we call y an *ω -limit point* of x . We denote by $\omega(x, f)$ the set of all ω -limit points of x and call it the *ω -limit set* of x . If $y \in \omega(x, f^{-1})$, then we call y an *α -limit point* of x . We denote by $\alpha(x, f)$ the set of all α -limit points of x and call it the *α -limit set* of x .

Let X and Y be topological spaces, and $\beta: X \rightarrow X$ and $\gamma: Y \rightarrow Y$ be continuous maps. If there exists a continuous surjection (resp. a homeomorphism) $\eta: X \rightarrow Y$ such that $\eta \circ \beta = \gamma \circ \eta$, then β and γ are said to be *topologically semi-conjugate* (resp. *topologically conjugate*), and η is called a *topological semi-conjugacy* (resp. a *topological conjugacy*) from β to γ .

The following lemma is well known (see e.g. [9, Lemma 3.2]).

LEMMA 2.1. *Let $\beta: X \rightarrow X$ and $\gamma: Y \rightarrow Y$ with a topological semi-conjugacy $\eta: X \rightarrow Y$ be as above. If both X and Y are compact metric spaces, then:*

- (a) *For any $x \in X$, the ω -limit set $\omega(\eta(x), \gamma) = \eta(\omega(x, \beta))$.*
- (b) *If both β and γ are homeomorphisms, then, for any $x \in X$, the α -limit set $\alpha(\eta(x), \gamma) = \eta(\alpha(x, \beta))$.*

We always write $J = [-1, 1]$. For any compact connected manifold M , denote by ∂M the boundary, and by $\overset{\circ}{M}$ the interior of M . In particular, we have $\partial J = \{-1, 1\}$, $\overset{\circ}{J} = (-1, 1)$, $\partial J^2 = \{-1, 1\} \times J \cup J \times \{-1, 1\}$, and $\overset{\circ}{J}^2 = (-1, 1)^2$.

Define the homeomorphism $f_{01}: J \rightarrow J$ by

$$f_{01}(s) = \begin{cases} \frac{1}{2}(s+1) & \text{if } 0 \leq s \leq 1, \\ s + \frac{1}{2} & \text{if } -\frac{1}{2} \leq s \leq 0, \\ 2s+1 & \text{if } -1 \leq s \leq -\frac{1}{2}. \end{cases}$$

Then we define the *standard vertical shift* $f_{02}: J^2 \rightarrow J^2$ by

$$f_{02}(r, s) = (r, f_{01}(s)) \quad \text{for any } (r, s) \in J^2.$$

For any $s \in J$, write $J_s = J \times \{s\}$. A homeomorphism $\zeta: J^2 \rightarrow J^2$ is said to be *normally rising* if

$$\zeta(J_s) = f_{02}(J_s), \quad \text{for any } s \in J.$$

Define the *level reflection* $\Psi: J^2 \rightarrow J^2$ by

$$\Psi(r, s) = (-r, s) \quad \text{for any } (r, s) \in J^2.$$

Then Ψ^2 is the identity on J^2 , and for any $(r, s) \in J^2$, we have

$$\Psi f_{02}(r, s) = f_{02} \Psi(r, s) = (-r, f_{01}(s)),$$

that is $\Psi \circ f_{02} = f_{02} \circ \Psi$.

3. Normally rising orientation reversing homeomorphisms on J^2

Write $v_1 = (-1, 1)$, $v_2 = (1, 1)$, $v_3 = (-1, -1)$, $v_4 = (1, -1)$. Then v_1, v_2, v_3 and v_4 are the four vertices of the square J^2 .

LEMMA 3.1. *There exists a normally rising orientation reversing homeomorphism $f: J^2 \rightarrow J^2$ such that $\omega(x, f) = \{v_1, v_2\}$ and $\alpha(x, f) = \{v_3, v_4\}$ for any $x \in \overset{\circ}{J}^2$, and $f|_{\partial J^2} = \Psi f_{02}|_{\partial J^2}$.*

PROOF. Take $a_1 = 1/4 > a_2 > a_3 > \dots > 0$ such that $a_n \rightarrow 0$ as $n \rightarrow \infty$. Let

$$D_0 = J \times [0, 1/2], \quad B_{0n} = J \times [a_n, 1/2], \quad F_{0n} = J \times [0, a_n].$$

Then $D_0 = B_{0n} \cup F_{0n}$ for any $n \in \mathbb{N}$, and $B_{01} \subset B_{02} \subset B_{03} \subset \dots$ and $F_{01} \supset F_{02} \supset F_{03} \supset \dots$. For any $i \in \mathbb{Z}$, let

$$D_i = f_{02}^i(D_0), \quad B_{in} = f_{02}^i(B_{0n}), \quad F_{in} = f_{02}^i(F_{0n}).$$

Then $D_{i-1} \cap D_i = f_{02}^i(J_0)$ and $D_i = B_{in} \cup F_{in}$ for any $n \in \mathbb{N}$ and any $i \in \mathbb{Z}$. Write

$$R_0 = J \times [0, 1], \quad R_1 = J \times [1/2, 1].$$

Then $R_0 = D_0 \cup R_1$, $R_1 = \bigcup \{D_i : i \in \mathbb{N}\} \cup J_1$, and $\Psi f_{02}(R_0) = f_{02}(R_0) = R_1$.

Take $b_1 = 1/2 < b_2 < b_3 < \dots < 1$ such that $b_n \rightarrow 1$ as $n \rightarrow \infty$. For any $n \in \mathbb{N}$, take a homeomorphism $\varphi_n : J \rightarrow J$ such that

$$0 < \varphi_n(r) - r \leq \frac{2b_n}{n} \quad \text{for any } r \in \overset{\circ}{J},$$

$$\varphi_n(r) - r = \frac{2b_n}{n} \quad \text{for any } r \in \left[-b_n, b_n - \frac{2b_n}{n} \right].$$

Then $\varphi_n^n(-b_n) = b_n$ and $\varphi_n^n([-b_n, 1]) = [b_n, 1]$.

For any $n \in \mathbb{N}$, write $k(n) = n(n+1)$. Then $k(n)$ is an even number, $k(1) = 2$, $k(2) = 6$, $k(3) = 12$, $\dots$, and $k(n+1) = k(n) + 2n + 2$. Let

$$\mathbb{N}(n) = \{k(n), k(n) + 1, \dots, k(n+1) - 1\}.$$

Then $\mathbb{N}(n)$ is a subset of $\mathbb{N}$ containing $2n + 2$ elements, and $\{\mathbb{N}(n) : n \in \mathbb{N}\}$ is a partition of $\mathbb{N} - \{1\}$.

For any given $i \in \mathbb{N} - \{1\}$, let $n(i) = \max\{m \in \mathbb{N} : k(m) \leq i\}$. Then $n(i)$ is the number in $\mathbb{N}$ such that $i \in \mathbb{N}(n(i))$, and we have $n(i) = 1$ for $i = 2, \dots, 5$, $n(i) = 2$ for $i = 6, \dots, 11$, $n(i) = 3$ for $i = 12, \dots, 19$, and so on.

Clearly, we can uniquely define a homeomorphism $\Phi : R_1 \rightarrow R_1$ by the following conditions:

- ($\Phi.1$) $\Phi|(D_1 \cup \partial R_1) = \text{id}$, and $\Phi(J_s) = J_s$ for any $s \in [1/2, 1]$.
- ($\Phi.2$) For any $i \in \mathbb{N} - \{1\}$ and any $(r, s) \in B_{in(i)}$, if i is odd then $\Phi(r, s) = (r, s)$;
if i is even then $\Phi(r, s) = (\varphi_{n(i)}(r), s)$.
- ($\Phi.3$) For any $i \in \mathbb{N} - \{1\}$ and any $r \in J$, $\Phi|((\{r\} \times J) \cap F_{in(i)})$ is linear.

Note that in the above definition of $\Phi : R_1 \rightarrow R_1$, we divide R_1 in the manner shown by the following equality:

$$R_1 = D_1 \cup D_2 \cup \dots \cup D_i \cup \dots \cup J_1$$

$$= D_1 \cup F_{2n(2)} \cup B_{2n(2)} \cup \dots \cup F_{in(i)} \cup B_{in(i)} \cup \dots \cup J_1,$$

and the definition of $\Phi|B_{in(i)}$ is directly given by means of $\varphi_{n(i)}$ if i is even, and the definition of $\Phi|F_{in(i)}$ is given by means of $\Phi|B_{i-1, n(i-1)}$ and $\Phi|B_{in(i)}$.

Let $\eta = \Phi \Psi f_{02}|R_0 : R_0 \rightarrow R_1$. Then η is a homeomorphism, $\eta|(D_0 \cup \partial R_0) = \Psi f_{02}|(D_0 \cup \partial R_0)$, and $\eta(J_s) = f_{02}(J_s)$ for any $s \in [0, 1]$.

CLAIM 1. For any given $x_0 = (r_0, s_0) \in \overset{\circ}{J} \times (0, 1/2]$, $\lim_{i \rightarrow \infty} \eta^{2^i}(x_0) = v_2$.

PROOF OF CLAIM 1. For any $i \in \mathbb{N}$, let $x_i = (r_i, s_i) = \eta^i(x_0)$. Then $r_i \in \overset{\circ}{J}$ and $s_i = f_{01}^i(s_0) \rightarrow 1$ as $i \rightarrow \infty$. Take a $\mu \in \mathbb{N}$ such that $x_0 \in B_{0\mu}$. Then for any $i \geq k(\mu)$, we have $n(i) \geq \mu$ and $x_i \in B_{i\mu} \subset B_{in(i)}$. Hence, for any odd $i \geq k(\mu)$, we have

$$\eta(x_i) = \Phi\Psi f_{02}(r_i, s_i) = \Phi(-r_i, s_{i+1}) = (\varphi_{n(i+1)}(-r_i), s_{i+1}) = (r_{i+1}, s_{i+1})$$

with $r_{i+1} = \varphi_{n(i+1)}(-r_i) > -r_i$, and for any even $i \geq k(\mu)$, we have

$$\eta(x_i) = \Phi\Psi f_{02}(r_i, s_i) = \Phi(-r_i, s_{i+1}) = (-r_i, s_{i+1})$$

with $r_{i+1} = -r_i$, and

$$\eta^2(x_i) = \Phi\Psi f_{02}(-r_i, s_{i+1}) = \Phi(r_i, s_{i+2}) = (r_{i+2}, s_{i+2})$$

with

$$(3.1) \quad r_{i+2} = \varphi_{n(i+2)}(r_i) > r_i.$$

Therefore, for any even $i \geq k(\mu)$, we have

$$(3.2) \quad -1 < r_i < r_{i+2} < r_{i+4} < r_{i+6} < \dots < 1.$$

In order to prove $\lim_{i \rightarrow \infty} r_{2i} = 1$, consider any given $\varepsilon > 0$. Take an $m = m(\varepsilon) \in \mathbb{N} - \mathbb{N}(\mu)$ such that $b_m > 1 - \varepsilon$ and $-b_m < r_{k(\mu)}$. Then $r_{k(m)} > r_{k(\mu)} > -b_m$. Consider the point $y_{k(m)} = (-b_m, s_{k(m)}) \in B_{k(m), m}$. Let $\beta_{k(m)} = -b_m$. For any $i \in \mathbb{N}$, let

$$y_{k(m)+i} = (\beta_{k(m)+i}, s_{k(m)+i}) = \eta^i(y_{k(m)}).$$

Similar to (3.1), for any even $i \geq k(m)$, we have

$$(3.3) \quad \beta_{i+2} = \varphi_{n(i+2)}(\beta_i) > \beta_i.$$

Note that $n(i) = m$ and $\varphi_{n(i)} = \varphi_m$ for $k(m) \leq i < k(m+1) = k(m) + 2m + 2$, and

$$(3.4) \quad \varphi_m(r) = r + \frac{2b_m}{m} \quad \text{for any } r \in \left[-b_m, b_m - \frac{2b_m}{m}\right].$$

In (3.3) and (3.4), taking $i = k(m), k(m) + 2, \dots, k(m) + 2n - 2$ and $r = \beta_{k(m)}, \beta_{k(m)+2}, \dots, \beta_{k(m)+2n-2}$, we get

$$\begin{aligned} \beta_{k(m)+2} &= \varphi_{n(k(m)+2)}(\beta_{k(m)}) = \beta_{k(m)} + \frac{2b_m}{m}, \\ \beta_{k(m)+4} &= \varphi_{n(k(m)+4)}(\beta_{k(m)+2}) = \beta_{k(m)+2} + \frac{2b_m}{m} = \beta_{k(m)} + \frac{4b_m}{m}, \\ &\dots\dots\dots \\ \beta_{k(m)+2m} &= \varphi_{n(k(m)+2m)}(\beta_{k(m)+2m-2}) = \beta_{k(m)+2m-2} + \frac{2b_m}{m} \\ &= \beta_{k(m)} + \frac{2mb_m}{m} = -b_m + 2b_m = b_m > 1 - \varepsilon. \end{aligned}$$

Define a projection $p: J^2 \rightarrow J$ by $p(x) = r$ for any $x = (r, s) \in J^2$, that is, $p(x)$ is the abscissa of x . Clearly, for any $s \in [0, 1]$ and any $x, y \in J_s$, if $p(x) > p(y)$ then $p\eta^2(x) > p\eta^2(y)$. Thus we have

$$r_{k(m)+2m} > \beta_{k(m)+2m} > 1 - \varepsilon$$

since $r_{k(m)} > -b_m = \beta_{k(m)}$, and hence, by (3.2), we have

$$\varepsilon > 1 - r_{k(m)+2m} > 1 - r_{k(m)+2m+2} > 1 - r_{k(m)+2m+4} > \dots > 0.$$

This means that $\lim_{i \rightarrow \infty} r_{2i} = 1$, which with $\lim_{i \rightarrow \infty} s_i = 1$ implies $\lim_{i \rightarrow \infty} \eta^{2i}(x_0) = \lim_{i \rightarrow \infty} (r_{2i}, s_{2i}) = (1, 1) = v_2$. $\square$

Since $\eta(v_2) = v_1$, from Claim 1 we get

CLAIM 2. For any given $x_0 = (r_0, s_0) \in \overset{\circ}{J} \times (0, 1/2]$, $\lim_{i \rightarrow \infty} \eta^{2i+1}(x_0) = v_1$, and $\omega(x_0, \eta) = \{v_1, v_2\}$.

Define the vertical reflection $\Psi_v: J^2 \rightarrow J^2$ by

$$\Psi_v(r, s) = (r, -s) \quad \text{for any } (r, s) \in J^2.$$

Then $\Psi_v^2 = \text{id}$, $\Psi_v f_{02} = f_{02}^{-1} \Psi_v$, and $\Psi_v(D_i) = D_{-i-1}$ for any $i \in \mathbb{Z}$. Write

$$R_{-1} = J \times [-1, 0], \quad R_{-2} = J \times [-1, -1/2].$$

Then $J^2 = R_0 \cup R_{-1}$, $R_{-1} = \Psi_v(R_0) = D_{-1} \cup R_{-2}$, and $R_{-2} = \Psi_v(R_1) = \bigcup \{D_{-i-1} : i \in \mathbb{N}\} \cup J_{-1}$. Let

$$\zeta = \Psi_v \eta \Psi_v|_{R_{-1}}: R_{-1} \rightarrow R_{-2}.$$

Then ζ is a homeomorphism. From Claims 1 and 2 we get

CLAIM 3. For any $x \in \overset{\circ}{J} \times [-1/2, 0)$, one has $\lim_{i \rightarrow \infty} \zeta^{2i}(x) = v_4$, $\lim_{i \rightarrow \infty} \zeta^{2i+1}(x) = v_3$, and $\omega(x, \zeta) = \{v_3, v_4\}$.

We now define $f: J^2 \rightarrow J^2$ by

$$f|_{R_0} = \eta, \quad f|_{D_{-1}} = \Psi f_{02}|_{D_{-1}}, \quad \text{and} \quad f|_{R_{-2}} = \zeta^{-1}.$$

Then f is a normally rising orientation reversing homeomorphism, and $f|\partial J^2 = \Psi f_{02}|\partial J^2$. Further, for any $x \in \overset{\circ}{J}^2$, there exists i and $j \in \mathbb{Z}$ such that $f^i(x) \in \overset{\circ}{J} \times (0, 1/2]$ and $f^j(x) \in \overset{\circ}{J} \times [-1/2, 0)$, and hence we always have

$$\omega(x, f) = \omega(f^i(x), f) = \omega(f^i(x), \eta) = \{v_1, v_2\},$$

and

$$\alpha(x, f) = \alpha(f^j(x), f) = \omega(f^j(x), f^{-1}) = \omega(f^j(x), \zeta) = \{v_3, v_4\}.$$

Lemma 3.1 is proved. $\square$

4. Homeomorphisms of the plane which have neither fixed point nor unbounded orbit

In this section, we will use Lemma 3.1 to construct an orientation reversing homeomorphism on the plane which has neither fixed point nor unbounded orbit.

THEOREM 4.1. *There exists an orientation reversing fixed point free homeomorphism $h: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that:*

- (a) *The set of all periodic points of h consists of 2-periodic points and is equal to the subset $(-\infty, -1] \cup [1, \infty)$ of x -axis.*
- (b) *For each non-periodic point x , the orbit of x is bounded and the limit sets $\omega(x, h) = \alpha(x, h) = \{(-1, 0), (1, 0)\}$.*

PROOF. Let the four vertices $v_1 = (-1, 1)$, $v_2 = (1, 1)$, $v_3 = (-1, -1)$ and $v_4 = (1, -1)$ of J^2 be as in Lemma 3.1. Write $v_5 = (-1, 0)$, $v_6 = (1, 0)$, $v_7 = (0, 1)$, $v_8 = (0, -1)$, $v_9 = (-1/2, 0)$, $v_{10} = (1/2, 0)$. Then v_5, v_6, v_7, v_8 are the four midpoints of the four sides of J^2 , respectively. Let the level reflection Ψ and the vertical reflection Ψ_v be the same as in the proof of Lemma 3.1. Clearly, we can take a continuous map $\xi: J^2 \rightarrow J^2$ satisfying the following conditions:

- (1) $\xi(0, s) = (0, s)$ for any $s \in J$, and $\xi(r, 0) = (r/2, 0)$ for any $r \in J$.
- (2) $\xi([v_6, v_2]) = \{v_0\}$, $\xi([v_7, v_2]) = [v_7, v_2] \cup [v_2, v_6] \cup [v_6, v_0]$, and $\xi|_{[v_7, v_2]}$ is an injection.
- (3) $\xi\Psi = \Psi\xi$ and $\xi\Psi_v = \Psi_v\xi$.
- (4) $\xi|_{\overset{\circ}{J}^2}$ is an injection, and $\xi(\overset{\circ}{J}^2) = \overset{\circ}{J}^2 - [v_5, v_9] - [v_0, v_6]$.

Let the homeomorphism $f: J^2 \rightarrow J^2$ be as in Lemma 3.1. Define a map $g: J^2 \rightarrow J^2$ by $g = \xi f \xi^{-1}$. Note that, if $x \in [v_5, v_9] \cup [v_0, v_6]$, then $\xi^{-1}(x)$ contains more than one point, but $\xi f \xi^{-1}(x)$ still contains only one point. Thus g is well defined. It is easy to check that g is a bijection, and is continuous. Thus $g: J^2 \rightarrow J^2$ is a homeomorphism. Moreover, from the definition $g = \xi f \xi^{-1}$ we see that g has the following properties:

- (P.A) g is orientation reversing.
- (P.B) f and g are topologically semi-conjugate, and ξ is a topological semi-conjugacy from f to g .
- (P.C) The set of periodic points of g is $P(g) = \partial J^2 \cup [v_5, v_9] \cup [v_0, v_6] = \xi(\partial J^2)$, with $g|_{P(g)} = \Psi|_{P(g)}$. Hence, in $P(g)$, only v_7 and v_8 are fixed points, and other points of $P(g)$ are 2-periodic points.
- (P.D) For any $x \in J^2 - P(g) = \xi(\overset{\circ}{J}^2)$, from Lemmas 3.1 and 2.1 we get

$$\begin{aligned}\omega(x, g) &= \xi(\omega(\xi^{-1}(x), f)) = \xi(\{v_1, v_2\}) = \{v_9, v_0\}, \\ \alpha(x, g) &= \xi(\alpha(\xi^{-1}(x), f)) = \xi(\{v_3, v_4\}) = \{v_9, v_0\}.\end{aligned}$$

(P.E) By (P.D) it is easy to see that, for any $x \in J^2 - P(g)$, there exists an $\varepsilon = \varepsilon(x) \in (0, 1/4]$ such that $O(x, g) \subset [-1 + \varepsilon, 1 - \varepsilon]^2$.

We now define a homeomorphism $\psi: \overset{\circ}{J}^2 \rightarrow \mathbb{R}^2$ by $\psi(r, s) = (\text{tg}(\pi r/2), \text{tg}(\pi s/2))$ for any $(r, s) \in \overset{\circ}{J}^2$, and then define a homeomorphism $h: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $h = \psi g \psi^{-1}$. From (P.A)–(P.E) we see that h has the following properties:

(P.F) h is orientation reversing.

(P.G) h and $g|(-1, 1)^2$ are topologically conjugate, with ψ being a topological conjugacy from g to h .

(P.H) The set of periodic points of h is

$$P(h) = \psi(P(g) \cap \overset{\circ}{J}^2) = (-\infty, -1] \times \{0\} \cup [1, \infty) \times \{0\},$$

with $h(r, 0) = (-r, 0)$ for any $(r, 0) \in P(h)$. Hence h has no fixed point, and all points in $P(h)$ are 2-periodic points. Particularly, write $w_1 = (-1, 0)$, $w_2 = (1, 0)$, and let $O_2 = \{w_1, w_2\}$. Then O_2 is a periodic orbit of h .

(P.I) For any $x \in \mathbb{R}^2 - P(h) = \psi(J^2 - P(g))$, by (P.D) it holds that

$$\omega(x, h) = \psi(\omega(\psi^{-1}(x), g)) = \psi(\{v_9, v_0\}) = \{w_1, w_2\} = O_2,$$

$$\alpha(x, h) = \psi(\alpha(\psi^{-1}(x), g)) = \psi(\{v_9, v_0\}) = \{w_1, w_2\} = O_2.$$

(P.J) For any $x \in \mathbb{R}^2$, the orbit $O(x, h)$ is bounded.

From (P.F)–(P.J) we see that h has the properties mentioned in Theorem 4.1, and the proof is complete. $\square$

Acknowledgements. We would like to thank the referees for their valuable comments and suggestions that improved the manuscript.

REFERENCES

- [1] S.M. BOYLES, *A counterexample to the bounded orbit conjecture*, Trans. Amer. Math. Soc. **266** (1981), 415–422.
- [2] L.E.J. BROUWER, *Beweiss des ebenen Translationssatzes*, Math. Ann. **72** (1912), 37–54.
- [3] M. BROWN, *A new proof of Brouwer's lemma on translation arcs*, Houston J. Math. **10** (1984), 35–41.
- [4] A. FATHI, *An orbit closing proof of Brouwer's lemma on translation arcs*, L'enseignement Mathématique **33** (1987), 315–322.
- [5] J. FRANKS, *A new proof of the Brouwer plane translation theorem*, Ergodic Theory Dynam. Systems **12** (1992), 217–226.
- [6] L. GUILLOU, *Théorème de translation plane de Brouwer et généralisations du théorème de Poincaré–Birkhoff*, Topology **33** (1994), 331–351.
- [7] P. LE CALVEZ, *From Brouwer theory to the study of homeomorphisms of surfaces*, European Mathematical Society (EMS), Zürich, 2006, pp. 77–98.
- [8] P. LE CALVEZ, *Une version feuilletée équivariante du théorème de translation de Brouwer*, Inst. Hautes Études Sci. Publ. Math. **102** (2005), 1–98.

- [9] J. MAI, E. SHI, K. YAN AND F. ZENG, *Can points of bounded orbits surround points of unbounded orbits?*, arXiv: 2404.13642.

Manuscript received August 5, 2024

accepted November 23, 2025

JIEHUA MAI
School of Mathematics and Quantitative Economics
Guangxi University of Finance and Economics
Nanning, Guangxi, 530003, P.R. CHINA
and
Institute of Mathematics
Shantou University
Shantou, Guangdong, 515063, P.R. CHINA
E-mail address: jiehuamai@163.com; jhmai@stu.edu.cn

ENHUI SHI
School of Mathematics and Sciences
Soochow University
Suzhou, Jiangsu 215006, P.R. CHINA
E-mail address: ehshi@suda.edu.cn

KESONG YAN (corresponding author)
 <https://orcid.org/0000-0003-0352-1293>
School of Mathematics and Statistics
Hainan Normal University
Haikou, Hainan, 571158, P.R. CHINA
E-mail address: ksyang@mail.ustc.edu.cn

FANPING ZENG
School of Mathematics and Quantitative Economics
Guangxi University of Finance and Economics
Nanning, Guangxi, 530003, P.R. CHINA
E-mail address: fpzeng@gxu.edu.cn