# ERDŐS' PROBLEM AND $(n, \frac{1}{3})$-SEPARATED SETS

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ABSTRACT. Inspired by the Erdős' problem in Ramsey theory, we propose a dynamical version of the problem and answer it positively for circle maps.



## 1. INTRODUCTION

The following problem is due to Erdős, which is a major open problem in Ramsey theory (see e.g. [4, p.4]).

**Problem 1.1** (Erdős' problem)**.** *Is there a constant $C > 0$ so that if $N \geq C^r$, then every edge-coloring of $K_N$ using $r$ colors contains a monochromatic triangle?*

Here, $K_N$ denotes the complete graph on $N$ vertices. The following proposition indicates that the above problem has a negative answer if $C = 2$ (see e.g. [4, p.3] for a very simple combinatoric proof).

**Proposition 1.2.** *For each positive integer $r$, there exists an edge-coloring of $K_{2^r}$ using $r$ colors with no monochromatic triangle.*

To illustrate the main ideas in the paper, we provide a dynamical proof of Proposition 1.2.

*Proof.* Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the circle endowed with the metric $d$ induced by the Euclidean metric on $\mathbb{R}$; that is $d(x,y) = \min\{|x-y|, 1-|x-y|\}$ for any $x, y \in [0,1)$. For each positive integer $r$, let $A_r = \{\frac{i}{2^r} : i = 0, \ldots, 2^r - 1\}$. Then the cardinality $|A_r| = 2^r$. Let $f : \mathbb{T} \to \mathbb{T}$ be defined by $f(x) = 2x$. Clearly, for any $i \neq j$, there is $n_{ij} \in \{0, \ldots, r-1\}$ with $d(f^{n_{ij}}(\frac{i}{2^r}), f^{n_{ij}}(\frac{j}{2^r})) > \frac{1}{3}$; then color the edge $\{\frac{i}{2^r}, \frac{j}{2^r}\}$ by $n_{ij}$. Thus we get an edge-coloring of the complete graph $K_{2^r}$ with vertex set $A_r$ by $r$ colors. If $K_{2^r}$ contains a monochromatic triangle $\{x, y, z\}$, then there is some $n \in \{0, \ldots, r-1\}$ with $d(f^n(a), f^n(b)) > \frac{1}{3}$ for any $\{a, b\} \subset \{x, y, z\}$. This is a contradiction. $\square$

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Now we recall some definitions in dynamical system (see [3, p.168]). Let $X$ be a compact metric space with metric $d$ and let $f : X \to X$ be continuous. For positive integer $n$, the *$n$-step Bowen metric* $d_n$ is defined by $d_n(x,y) = \max\{d(f^i(x), f^i(y)) : i = 0, \ldots, n-1\}$. For $\varepsilon > 0$ and $A \subset X$, $A$ is said to be an *$(n,\varepsilon)$-separated set* of $X$ with respect to $f$, if $d_n(x,y) > \varepsilon$ for any $x \neq y \in A$. We denote by $s_f(n,\varepsilon)$ the maximal cardinality of any $(n,\varepsilon)$-separated set of $X$. We use $C_d(X,\varepsilon)$ to denote the maximal cardinality of subsets of $X$ any two points of which have distance $> \varepsilon$, and call it the *capacity* of $X$ with respect to $d$ and $\varepsilon$.

For positive integers $k$ and $r$, the *Ramsey number* $R(k,r)$ is the minimal $R$ such that if each edge of the complete graph $K_R$ is colored using one of $r$ colors, then there is a monochromatic $K_k$.

The following simple principle establishes a connection between Ramsey theory and dynamical system. Similar ideas also appeared in [2].

**Principle 1.3.** Let $X$ be a compact metric space with metric $d$ and $f : X \to X$ be continuous. For some $\varepsilon > 0$, suppose $C_d(X,\varepsilon) = k$ and $s_f(n,\varepsilon) = m$. Then $R(k+1,n) > m$.

The proof of Principle 1.3 follows the same lines as in Proposition 1.2.

*Proof.* Let $\{x_1, \ldots, x_m\}$ be a maximal $(n,\varepsilon)$-separated set. Regard $\{x_1, \ldots, x_m\}$ as the vertices of the complete graph $K_m$. Then for any $x_i \neq x_j$, there is some $n_{ij} \in \{0, \ldots, n-1\}$ with $d(f^{n_{ij}}(x_i), f^{n_{ij}}(x_j)) > \varepsilon$; color the edge $\{x_i, x_j\}$ by $n_{ij}$. If there is a monochromatic $K_{k+1}$, then there is some $q \in \{0, \ldots, n-1\}$ with any two elements of $\{f^q(x) : x \in V(K_{k+1})\}$ having distance $> \varepsilon$. This contradicts $C_d(X,\varepsilon) = k$. So $R(k+1,n) > m$. $\square$

Now we propose the following problem.

**Problem 1.4.** *Is there a constant $C > 0$ so that for any compact metric space $(X,d)$ with $C_d(X, \frac{1}{3}) = 2$, any positive integer $n$, and any continuous $f : X \to X$, we always have $s_f(n, \frac{1}{3}) \leq C^n$?*

**Remark 1.5.** (1) If the answer to Problem 1.4 is negative, then there are a sequence $0 < C_1 < C_2 < \ldots$ with $C_i \to \infty$, and metric spaces $(X_i, d_i)$ with $C_{d_i}(X_i, \frac{1}{3}) = 2$, and positive integers $n_i$, and $f_i : X_i \to X_i$, such that $s_{f_i}(n_i, \frac{1}{3}) > C_i^{n_i}$. By Principle 1.3, we get that $R(3, n_i) > C_i^{n_i}$. Thus the answer to Problem 1.1 is also negative. So, we may view Problem 1.4 as a dynamical version of Erdős' problem.

(2) There are many compact metric spaces $(X,d)$ satisfying the condition $C_d(X,\frac{1}{3})=2$, such as the circle $\mathbb{T}$ in the proof of Proposition 1.2 and $[0,1]$ with the Euclidean metric. In fact, for any compact metrizable space $X$, if $X$ is not a single point, then there always exits a compatible metric $d$ on $X$ with $C_d(X,\frac{1}{3})=2$. This can be seen by topologically embedding $X$ into the Hilbert Cub $H=\prod_{i=1}^{\infty} I_i$ with each $I_i=[0,1]$ (see [1, p.241]). Take such an imbedding $\eta$ and we may suppose there are $x\neq y\in X$ with $\eta(x)_1=0$ and $\eta(y)_1=1$, and take a compatible metric $\rho$ on $H$ defined by $\rho((x_i),(y_i))=\sum_{i=1}^{\infty}\frac{|x_i-y_i|}{2\cdot 100^{i-1}}$. Then the metric $d$ on $\eta(X)$ by restricting $\rho$ to $\eta(X)$ satisfies the requirement.

The aim of the paper is to give a positive answer toward Problem 1.4 for the case $X=\mathbb{T}$ (with the metric $d$ given in the proof of Proposition 1.2).

We obtained the following theorem.

**Theorem 1.6.** *There is a constant $C>0$ so that for any positive integer n, and any continuous $f:\mathbb{T}\to\mathbb{T}$, we always have $s_f(n,\frac{1}{3})\leq C^n$.*

**Remark 1.7.** From the proof in the following section, we see that $C$ can be taken to be 3. The proof of Theorem 1.6 heavily relies on some special properties of the circle and the maps on it. However, the constant $C$ in Problem 1.4 is almost completely independent of spaces and maps, so if the answer to Problem 1.4 is yes, there must be some more clever ideas to prove it. In fact, the edge-coloring of any complete graph coming from a dynamical system is very restrictive. Intuitively, such colorings cannot vary suddenly (near edges have near colorings), due to the continuity of the maps. So, Problem 1.4 should have a great chance to have a positive answer and have an easy combinatoric proof.

## 2. Proof of Theorem 1.6

In the following, we always assume the metric $d$ on $\mathbb{T}=\mathbb{R}/\mathbb{Z}$ is as in the proof of Proposition 1.2 and $\mu$ is the Lebesgue measure on $\mathbb{T}$ induced by $d$.

For positive integer $p$, the $\times p$ map $f_p:\mathbb{T}\to\mathbb{T}$ is defined by $f_p(x)=px$ for any $x\in\mathbb{T}$. If $d_n$ is the Bowen metric on $\mathbb{T}$ with respect to $f_p$, then we use $B_n(x,\varepsilon)$ to denote the open ball centered at $x$ of radius $\varepsilon$ with respect to the metric $d_n$, that is

$$B_n(x,\varepsilon)=\{y\in\mathbb{T}:d(f_p^i(x),f_p^i(y))<\varepsilon,\ \forall\, i=0,\dots,n-1\}.$$

From the definition, the following lemma is clear.

**Lemma 2.1.** *For any $x \in \mathbb{T}$, $B_n(x,\varepsilon) = x + B_n(0,\varepsilon)$.*

**Lemma 2.2.** *If $J = (a,b)$ is a connected component of $B_n(0,\frac{1}{6})$ with $f_p^n(a) = f_p^n(b) = 0$, then $\mu(B_{n+1}(0,\frac{1}{6}) \cap J) = \frac{1}{3}\mu(J)$.*

*Proof.* We identify $\mathbb{T}$ with $[-\frac{1}{2},\frac{1}{2})$. WLOG, we may suppose $a < b$. Let $k = \frac{b-a}{p^n}$ and $A = \{a + \frac{i}{6p^n} : i = 0,\dots,6kp^{2n}\}$. By the definition, we see that

$$B_{n+1}(0,\frac{1}{6}) \cap J - A = \{x \in J : d(f_p^n(x),0) < \frac{1}{6}\} - A$$
$$= \bigsqcup_{i=6j \text{ or } 6j+5;\ j=0,\dots,kp^{2n}-1} (a + \frac{i}{6p^n}, a + \frac{i+1}{6p^n}),$$

which implies the conclusion. □

**Proposition 2.3.** *For any $p = 6^\ell$ with $\ell \geq 1$ and any positive integer $n$, $s_{f_p}(n,1/3) \leq 3^n$.*

*Proof.* It is clear that $B_1(0,\frac{1}{6}) = (-\frac{1}{6},\frac{1}{6})$. By Lemma 2.2 and an induction argument, we have

$$\mu(B_n(0,1/6)) = \frac{1}{3^{n-1}}\mu(B_1(0,1/6)) = \frac{1}{3^n}.$$

Then by Lemma 2.1, for any $y \in \mathbb{T}$, $\mu(B_n(y,1/6)) = \frac{1}{3^n}$. This shows that

$$3^n \geq t_n := \max\{l : \exists\, y_1,\dots,y_l \in \mathbb{T}, s.t.\ B_n(y_1,1/6),\dots,B_n(y_l,1/6) \text{ are disjoint}\}.$$

Clearly, we have $s_{f_P}(n,1/3) \leq t_n$, which means $s_{f_P}(n,1/3) \leq 3^n$. □

**Lemma 2.4.** *Let $n$ be a positive integer and $\{x^{(0)},\dots,x^{(n-1)}\} \subset \mathbb{T}$. Then for any $\delta > 0$, there is some $p_0$ such that for any $p \geq p_0$, there is $y = y(p) \in \mathbb{T}$ with*

$$d(f_p^i(y), x^{(i)}) < \delta,\ \forall\, i = 0,\dots,n-1.$$

*Proof.* Take $p_0$ sufficiently large such that for any open interval $J$ of length $2\delta$, $f_p(J)$ covers the whole circle for any $p \geq p_0$. Let $f = f_p$ and let $I_j = (x^{(j)} - \delta, x^{(j)} + \delta)$, $j = 0,\dots,n-1$. Then

$$I_0 \cap f^{-1}I_1 \cap \cdots \cap f^{-(n-1)}I_{n-1} \neq \emptyset,$$

any point $y$ of which satisfies the requirement. □

The following proposition is implied immediately by Lemma 2.4.

**Proposition 2.5.** *Let* $n,k \in \mathbb{N}$ *and* $\{x_j^{(0)}, x_j^{(1)}, \ldots, x_j^{(n-1)}\} \subset \mathbb{T}$ *for* $j = 1, \ldots, k$. *For any* $\delta > 0$, *there is some* $p_0$ *such that for any* $p \geq p_0$, *there are* $y_1, \ldots, y_k \in \mathbb{T}$ *with*

$$d(f_p^i(y_j), x_j^{(i)}) < \delta, \; \forall\, i = 0, \ldots, n-1, j = 1, \ldots, k.$$

**Proposition 2.6.** *Let* $g$ *be a continuous map on* $\mathbb{T}$ *and* $\varepsilon > 0$ . *Then for sufficiently large* $p$, *we have*

$$s_g(n,\varepsilon) \leq s_{f_p}(n,\varepsilon).$$

*Proof.* Write $k = s_g(n,\varepsilon)$ and let $\{x_1, \ldots, x_k\} \subset \mathbb{T}$ be an $(n,\varepsilon)$-separated set of $g$. Then by the continuity, there is $\delta > 0$ such that $\{x_1, \ldots, x_k\}$ is still $(n, \varepsilon + \delta)$-separated for $g$. By Proposition 2.5, there is some $p_0$ such that if $p > p_0$, there are $y_1, \ldots, y_k \in \mathbb{T}$ with

$$d(g^i(x_j), f_p^i(y_j)) < \delta/2, \; \forall\, i = 0, \ldots, n-1, \; j = 1, \ldots, k,$$

which implies that for any $i = 0, \ldots, n-1$ and $u \neq v$, if $d(g^i(x_u), g^i(x_v)) > \varepsilon + \delta$, then we have

$$d(f_p^i(y_u), f_p^i(y_v)) \geq d(g^i(x_u), g^i(x_v)) - d(g^i(x_u), f^i(y_u)) - d(g^i(x_v), f^i(y_v)) > \varepsilon.$$

Thus $y_1, \ldots, y_k$ is an $(n,\varepsilon)$-separated set for $f_p$. Therefore $s_g(n,\varepsilon) = k \leq s_{f_p}(n,\varepsilon)$. □

Now we end the proof of Theorem 1.6 by applying Proposition 2.3 and Proposition 2.6 to $p = 6^\ell$ with sufficiently large $\ell$ and $\varepsilon = \frac{1}{3}$.

## References


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