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The Nonexistence of Expansive $\mathbb{Z}^d$ Actions on Graphs

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Abstract It is well known that if X is an arc or a circle, then there is no expansive homeomorphism on X. In this paper we prove that there is no expansive $\mathbb{Z}^d$ action on X, which answers the two questions raised by us before. In 1979, Mañé proved that there is no expansive homeomorphism on infinite dimensional spaces. Contrary to this result, we construct an expansive $\mathbb{Z}^2$ action on an infinite dimensional space. We also construct an expansive $\mathbb{Z}^2$ action on a zero dimensional space but no element in $\mathbb{Z}^2$ is expansive.

Keywords Expansive homeomorphism, Arc, Circle, Group action
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1 Introduction

Let X be a compact metric space with metric d. A homeomorphism f of X is expansive if there exists $c > 0$ such that for any $x, y \in X$ with $x \neq y$, there is an integer $n \in \mathbb{Z}$ with $d(f^n(x), f^n(y)) > c$, where c is called an expansive constant for f. Whether a continuum (compact connected metric space) admits expansive homeomorphisms is an interesting problem in topological dynamics and continuum theory. Many results have been obtained (see [1–10]). Especially there is no expansive homeomorphism on an arc or a circle (see [1], [2] or [11]).

In recent years there has been a considerable progress in the study of higher dimensional actions (i.e., action of $\mathbb{Z}^d$ or $\mathbb{R}^d$ with $d > 1$) (see [12]). Stimulated by this progress, we started to consider the existence of expansive $\mathbb{Z}^d$ actions on continua. In [13] we proved that there is no expansive $\mathbb{Z}^2$ action on the closed interval $I = [0, 1]$ and constructed an expansive G action on I, where G is generated by two noncommutative homeomorphisms. Furthermore, we proved that there is no expansive $\mathbb{Z}^2$ action on the unit circle in [14]. In this paper, we prove that there is no expansive $\mathbb{Z}^d$ action on an arc or a circle for any $d \geq 1$. This result also answers the two questions in [13].

In [7] Mañé proved that there is no expansive homeomorphism on infinite dimensional spaces. Contrary to this result we construct an expansive $\mathbb{Z}^2$ action on an infinite dimensional space. This example indicates that some new phenomena arise when we consider higher dimensional expansive actions. At the same time, this example also shows that the conclusion "a space which admits no expansive $\mathbb{Z}$ actions must admit no expansive $\mathbb{Z}^d$ actions" is not true in general. We also construct an expansive $\mathbb{Z}^2$ action on a zero dimensional space but no element in $\mathbb{Z}^2$ is expansive.

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2 Preliminaries

Let G be a discrete group (a topological group with discrete topology), X be a compact metric space with metric d and let $\mathrm{Hom}(X)$ be the group of selfhomeomorphisms of X. Recall that a group homomorphism $\alpha : G \to \mathrm{Hom}(X)$ is called an action of (the discrete group) G on X. This concept is a generalization for a discrete dynamical system, for if (X, f) is a discrete system, where f is a homeomorphism on X, then the map $n \mapsto f^n$ defines a group homomorphism from the integer additive group $\mathbb{Z}$ to $\mathrm{Hom}(X)$.

The action $\alpha : G \to \mathrm{Hom}(X)$ is called expansive if there is a constant $c > 0$ (called an expansive constant for α) such that, for any $x \neq y \in X$, there is a $g \in G$ with $d(\alpha(g)(x), \alpha(g)(y)) > c$. From this definition we know a space admits no expansive homeomorphisms if and only if it admits no expansive $\mathbb{Z}$ actions.

In this paper we are mainly interested in the case $G = \mathbb{Z}^d$ where $\mathbb{Z}^d$ is the free abelian group of rank d.

In the following, we use the symbol I to denote the unit closed interval $[0, 1]$, and $\mathbb{S}^1$ stands for the unit circle in the complex plane:

Definition 2.1 *A homeomorphism $f : I \to I$ is call positive if for any interior point $x \in I$ we have $f(x) > x$, and f is called negative if for any interior point x we have $f(x) < x$.*

Let $\overline{R} = \{-\infty\} \cup R \cup \{+\infty\}$ be the two points compactification of the real line R. Then $\overline{R}$ is homeomorphic to I. Define $L : \overline{R} \to \overline{R}$ by $L(x) = x + 1$, for all $x \in R$, $L(+\infty) = +\infty$, and $L(-\infty) = -\infty$. Obviously L is a homeomorphism.

Lemma 2.2 *Let $f : I \to I$ be a positive homeomorphism. Then f and L are topologically conjugate, i.e., there exists a homeomorphism $\phi : \overline{R} \to I$ such that $\phi^{-1} \circ f \circ \phi = L$.*

Proof We define $\phi : \overline{R} \to I$ as follows: For any $x \in [0, 1)$ let $\phi(x) = 1/2 + x(f(1/2) - 1/2)$ and for any $n \in \mathbb{Z}$ and any $x \in [0, 1)$, let $\phi(x + n) = f^n(\phi(x))$. Let $\phi(-\infty) = 0$ and $\phi(+\infty) = 1$. It is easy to check that ϕ is a homeomorphism.

For any $x \in R$, there exists a unique $n \in \mathbb{Z}$ such that $x = \{x\} + n$, where $\{x\}$ means the decimal part of a real number x. By the definition of ϕ, we have

$$\begin{aligned}\phi^{-1} \circ f \circ \phi(x) &= \phi^{-1} \circ f \circ \phi(\{x\} + n) = \phi^{-1} \circ f^{n+1} \circ \phi(\{x\}) \\ &= \phi^{-1} \circ \phi(\{x\} + n + 1) = \{x\} + n + 1 = x + 1 = L(x).\end{aligned}$$

Obviously $\phi^{-1} \circ f \circ \phi(+\infty) = +\infty$ and $\phi^{-1} \circ f \circ \phi(-\infty) = -\infty$. So f and L are topologically conjugate.

Recall that if H is a subgroup of G, then the cardinality of the coset G/H is called the coset index of H in G and is denoted by $[G : H]$.

Proposition 2.3 *Let G be a discrete group, X be a compact metric space with metric d and let $\alpha : G \to \mathrm{Hom}(X)$ be an action of G on X. Let H be a subgroup of G. If the coset index $[G : H]$ is finite, then α is expansive if and only if the restriction $\alpha|_H : H \to \mathrm{Hom}(X)$ is expansive.*

Proof Sufficiency is clear. Suppose $\alpha|_H$ is not expansive. Let $G = g_1H \cup g_2H \cup \cdots \cup g_nH$ be the coset decomposition of G, where $n = [G : H]$. By uniform continuity, for any $c > 0$ there exists $\delta > 0$ such that if $d(x, y) \leq \delta$ then for all $1 \leq i \leq n$, $d(\alpha(g_i)(x), \alpha(g_i)(y)) \leq c$. Since $\alpha|_H$ is not expansive, there exist $x', y' \in X$ with $x' \neq y'$, such that for any $h \in H$, $d(\alpha(h)(x'), \alpha(h)(y')) \leq \delta$. It follows that

$$d(\alpha(g_ih)(x'), \alpha(g_ih)(y')) = d(\alpha(g_i)\alpha(h)(x'), \alpha(g_i)\alpha(h)(y')) \leq c,$$

for all $1 \leq i \leq n$, and for all $h \in H$, i.e., $d(\alpha(g)(x'), \alpha(g)(y')) \leq c$, for all $g \in G$. So α is not expansive.

Lemma 2.4 *Let $\mathscr{F} = \{f_i | 1 \leq i \leq n\}$ be n pairwise commutative homeomorphisms on I which satisfy $f_i(0) = 0$ and $f_i(1) = 1$, for all $1 \leq i \leq n$. If every f_i is not the identity, then there exist a closed interval $[\alpha, \beta] \subset I$ and some $f_{i'} \in \mathscr{F}$ such that $f_i([\alpha, \beta]) = [\alpha, \beta]$, for all $1 \leq i \leq n$, and the restriction of $f_{i'}$ to $[\alpha, \beta]$ is either positive or negative.*

Proof We use the symbol $\mathrm{Fix}(f_i)$ to denote the fixed points set of f_i. Since each f_i is not the identity, $I \setminus \mathrm{Fix}(f_i) \neq \emptyset$, for all $1 \le i \le n$. As the two end points 0 and 1 belong to $\mathrm{Fix}(f_i)$, we may write $\Gamma_i = \{(\alpha_{i,j}, \beta_{i,j}) | \bigcup_{j\in\Lambda_i}(\alpha_{i,j}, \beta_{i,j}) = I \setminus \mathrm{Fix}(f_i)\}$, $1 \le i \le n$, where Λ_i is some index set and for any fixed i these open intervals $(\alpha_{i,j}, \beta_{i,j})$ are pairwise disjoint. By commutativity, we have $f_i(\mathrm{Fix}(f_j)) = \mathrm{Fix}(f_j)$, for all $1 \le i \le n$, for all $1 \le j \le n$, which implies that either

$$(\alpha_{i,j}, \beta_{i,j}) \cap (\alpha_{i',j'}, \beta_{i',j'}) = \emptyset, \text{ or } (\alpha_{i,j}, \beta_{i,j}) \subset (\alpha_{i',j'}, \beta_{i',j'}), \text{ or } (\alpha_{i,j}, \beta_{i,j}) \supset (\alpha_{i',j'}, \beta_{i',j'}),$$

for all $1 \le i \le n$, for all $1 \le i' \le n$, for all $j \in \Lambda_i$, and for all $j' \in \Lambda_{i'}$. We claim that there exists some open interval $(\alpha, \beta) = (\alpha_{i',j'}, \beta_{i',j'})$ that satisfies

$$(\alpha, \beta) \cap (\alpha_{i,j}, \beta_{i,j}) = \emptyset \text{ or } (\alpha, \beta) \supset (\alpha_{i,j}, \beta_{i,j}), \tag{1}$$

for all $1 \le i \le n$, for all $j \in \Lambda_i$. We take the following steps to find (α, β):

Step 1. Select an arbitrary $(\alpha_{1,j_1}, \beta_{1,j_1}) \in \Gamma_1$ and write $(\alpha, \beta) = (\alpha_{1,j_1}, \beta_{1,j_1})$.

Step 2. If there is some $(\alpha_{2,j_2}, \beta_{2,j_2}) \in \Gamma_2$ such that $(\alpha, \beta) \subset (\alpha_{2,j_2}, \beta_{2,j_2})$, then write $(\alpha, \beta) = (\alpha_{2,j_2}, \beta_{2,j_2})$. Otherwise we do not change the value of (α, β).

Continue this process $\cdots\cdots$.

Step n. If there is some $(\alpha_{n,j_n}, \beta_{n,j_n}) \in \Gamma_n$ such that $(\alpha, \beta) \subset (\alpha_{n,j_n}, \beta_{n,j_n})$ then write $(\alpha, \beta) = (\alpha_{n,j_n}, \beta_{n,j_n})$. Otherwise we do not change the value of (α, β).

Then we get $(\alpha, \beta) = (\alpha_{i',j'}, \beta_{i',j'})$ for some $1 \le i' \le n$ and some $j' \in \Lambda_{i'}$. It is easy to see that (α, β) satisfies (1), and thus $[\alpha, \beta]$ and $f_{i'}$ satisfy the requirement.

In the following, we always write intervals on $\mathbb{S}^1$ anticlockwise, so $[\alpha, \beta] \subset \mathbb{S}^1$ denotes the anticlockwise closed interval beginning at α and ending at β (α may be equal to β).

Lemma 2.5 *Let $\{f_i | 1 \le i \le n\}$ be n pairwise commutative homeomorphisms on $\mathbb{S}^1$ such that each f_i preserves the orientation of $\mathbb{S}^1$. If for every $1 \le i \le n$, f_i is not the identity and $\mathrm{Fix}(f_i) \neq \emptyset$, then there is a closed interval $[\alpha, \beta] \subset \mathbb{S}^1$ such that $f_i([\alpha, \beta]) = [\alpha, \beta]$, for all $1 \le i \le n$.*

Proof The proof is similar to that of Lemma 2.4 so we omit it here. We need to note only that the requirement of preserving orientation is to exclude the following case: α, β are the only two fixed points of some f_i and $f_i([\alpha, \beta]) = [\beta, \alpha]$.

Let $f : X \to X$ be a homeomorphism on X. Recall that a point $x \in X$ is called nonwandering if, for every open neighborhood U of x, there is an integer $n \neq 0$ such that $f^n(U) \cap U \neq \emptyset$.

Lemma 2.6 *Let f and g be two commutative homeomorphisms on X and $\Omega(g)$ be the nonwandering set of g. Then $f(\Omega(g)) = \Omega(g)$.*

Proof For any $x \in X \setminus \Omega(g)$, there is an open neighborhood U of x such that $g^n(U) \cap g^m(U) = \emptyset$, for all $n \neq m \in \mathbb{Z}$. Since f and g are commutative, we have $g^n(f(U)) \cap g^m(f(U)) = f(g^n(U) \cap g^m(U)) = \emptyset$, for all $n \neq m \in \mathbb{Z}$. Since $f(U)$ is an open neighborhood of $f(x)$, $f(x) \in X \setminus \Omega(g)$. Hence $f(X \setminus \Omega(g)) \subset X \setminus \Omega(g)$. Similarly by the commutativity of f^{-1} and g we get $f^{-1}(X \setminus \Omega(g)) \subset X \setminus \Omega(g)$. So $f(\Omega(g)) = \Omega(g)$.

The following well-known result will be used in the proof of the main theorem:

Theorem 2.7 (See [11]) *Let $T : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be a homeomorphism with no periodic points. Then T is semi-conjugate to a minimal rotation P of $\mathbb{S}^1$, i.e., there is a continuous surjection $\Psi : \mathbb{S}^1 \to \mathbb{S}^1$ such that $\Psi \circ T = P \circ \Psi$. The map Ψ has the property that for each $z \in \mathbb{S}^1$, $\Psi^{-1}(z)$ is either a point or a closed sub-interval of $\mathbb{S}^1$.*

3 No expansive $\mathbb{Z}^d$ Actions on Graphs

With the preparation in the previous section now we are ready to prove the main theorem in this paper.

Theorem 3.1 *There is no expansive $\mathbb{Z}^d$ action on the unit interval I and the unit circle $\mathbb{S}^1$.*

Proof We shall prove this by induction on d. The case $d = 1$ is just the well-known result that there is no expansive homeomorphism on I and on $\mathbb{S}^1$. Suppose that the conclusion is true for $d < n$. We shall establish it for $d = n$ in two steps.

Step 1. There is no expansive $\mathbb{Z}^n$ action on I.

Otherwise, suppose $\lambda : \mathbb{Z}^n \to \mathrm{Hom}(I)$ is an expansive $\mathbb{Z}^n$ action with expansive constant $c > 0$. Let $f_i = \lambda(\vec{e}_i)$, for all $1 \le i \le n$, where $\vec{e}_i$ is the i-th vector of the canonical base of $\mathbb{Z}^n$. By Proposition 2.3 we may assume that $f_i(0) = 0, f_i(1) = 1$, for all $1 \le i \le n$. Otherwise we need to consider only the $\mathbb{Z}^n$ action generated by $\{f_i^2 | 1 \le i \le n\}$. By Lemma 2.4 we may also assume that f_1 is positive on I. By Lemma 2.2 there is a homeomorphism $\phi : \overline{R} \to I$ such that $\phi^{-1} f_1 \phi = L$, where L is defined as in Lemma 2.2. Let $\tilde{f}_i = \phi^{-1} f_i \phi$, for all $1 \le i \le n$. Then $\{\tilde{f}_i | 1 \le i \le n\}$ are n pairwise commutative homeomorphisms on $\overline{R}$ and $\tilde{f}_1 = L$. Obviously there is a constant $\delta > 0$ such that for any $x, y \in R$, if $|x - y| \le \delta$ then

$$d(\phi(x), \phi(y)) \le c. \tag{2}$$

Let $\pi : R \longrightarrow \mathbb{S}^1$ be the exponent map, i.e., $\pi(x) = e^{2\pi i x}$, for all $x \in R$. Let $\hat{d}$ be the arclength metric on $\mathbb{S}^1$. Under the Euclidean metric on R, π is locally isometric. For arbitrary $x, y \in R$, if there is some integer n such that $x - y = n$, then for any $2 \le i \le n$, we have

$$\pi \tilde{f}_i(x) = \pi \tilde{f}_i(y + n) = \pi \tilde{f}_i \tilde{f}_1^n(y) = \pi \tilde{f}_1^n \tilde{f}_i(y) = \pi(\tilde{f}_i(y) + n) = \pi \tilde{f}_i(y).$$

So $\tilde{f}_i$ induces naturally a continuous map $g_i : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ with $g_i(\pi(x)) = \pi(\tilde{f}_i(x))$, for all $2 \le i \le n$. Since each $\tilde{f}_i$ is a homeomorphism and $\tilde{f}_i(x+1) = \tilde{f}_i(x) + 1$ for any $x \in R$, g_i is a homeomorphism on $\mathbb{S}^1$ and preserves the orientation of $\mathbb{S}^1$. Thus $\{g_i | 2 \le i \le n\}$ generate a $\mathbb{Z}^{n-1}$ action on $\mathbb{S}^1$. Since for any $x \in R$, $\tilde{f}_i(x+1) = \tilde{f}_i(x) + 1$ and $\tilde{f}_i^{-1}(x+1) = \tilde{f}_i^{-1}(x) + 1$, the restrictions $\tilde{f}_i|_R$ and $\tilde{f}_i^{-1}|_R$ are uniformly continuous under the Euclidean metric on R, for all $2 \le i \le n$. So there is a $\delta' > 0$ such that, if $|x - y| < \delta'$, then, for any $2 \le i \le n$,

$$|\tilde{f}_i(x) - \tilde{f}_i(y)| < \frac{1}{2}, \quad |\tilde{f}_i^{-1}(x) - \tilde{f}_i^{-1}(y)| < \frac{1}{2}. \tag{3}$$

Let $\delta'' = \min\{\delta, \delta', \frac{1}{2}\}$. By the inductive assumption, there exist $x, y \in \mathbb{S}^1$ and $x \ne y$ such that

$$\hat{d}(g_2^{m_2} g_3^{m_3} \cdots g_n^{m_n}(x), g_2^{m_2} g_3^{m_3} \cdots g_n^{m_n}(y)) \le \delta'', \tag{4}$$

for any $\vec{m} = (m_2, m_3, \ldots, m_n) \in \mathbb{Z}^{n-1}$. Select $\tilde{x}, \tilde{y} \in R$ so that $\pi(\tilde{x}) = x, \pi(\tilde{y}) = y$ and $|\tilde{x} - \tilde{y}| = \hat{d}(x, y) \le \delta''$. Since π is a locally isometry, by (3) and (4) we get

$$|\tilde{f}_2^{m_2} \tilde{f}_3^{m_3} \cdots \tilde{f}_n^{m_n}(\tilde{x}) - \tilde{f}_2^{m_2} \tilde{f}_3^{m_3} \cdots \tilde{f}_n^{m_n}(\tilde{y})| = \hat{d}(g_2^{m_2} g_3^{m_3} \cdots g_n^{m_n}(x), g_2^{m_2} g_3^{m_3} \cdots g_n^{m_n}(y)) \le \delta'' \le \delta.$$

By the definition of $\tilde{f}_1$, we have

$$|\tilde{f}_1^{m_1} \tilde{f}_2^{m_2} \cdots \tilde{f}_n^{m_n}(\tilde{x}) - \tilde{f}_1^{m_1} \tilde{f}_2^{m_2} \cdots \tilde{f}_n^{m_n}(\tilde{y})| = |\tilde{f}_2^{m_2} \cdots \tilde{f}_n^{m_n}(\tilde{x}) - \tilde{f}_2^{m_2} \cdots \tilde{f}_n^{m_n}(\tilde{y})| \le \delta.$$

Furthermore by (2), we have

$$d(f_1^{m_1} \cdots f_n^{m_n} \phi(\tilde{x}), f_1^{m_1} \cdots f_n^{m_n} \phi(\tilde{y})) = d(\phi \tilde{f}_1^{m_1} \cdots \tilde{f}_n^{m_n}(\tilde{x}), \phi \tilde{f}_1^{m_1} \cdots \tilde{f}_n^{m_n}(\tilde{y})) \le c,$$

for all $(m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n$, a contradiction. Therefore there is no expansive $\mathbb{Z}^n$ action on I.

Step 2. There is no expansive $\mathbb{Z}^n$ action on $\mathbb{S}^1$.

Otherwise suppose $\theta : \mathbb{Z}^n \longrightarrow \mathrm{Hom}(\mathbb{S}^1)$ is an expansive action with expansive constant $c > 0$. Let $f_i = \theta(\vec{e}_i)$, for all $1 \le i \le n$, where $\vec{e}_i$ is the i-th vector of the canonical base of $\mathbb{Z}^n$. If each f_i has a periodic point, then by Proposition 2.3 we may assume that each f_i has fixed points in $\mathbb{S}^1$ and preserves the orientation, otherwise we need to consider only the $\mathbb{Z}^n$ action generated by some suitable iterate of f_i. By Lemma 2.5 there is some closed subinterval $[a, b] \subset \mathbb{S}^1$ satisfying $f_i([a, b]) = [a, b]$, $1 \le i \le n$. Thus the restrictions $\{f_i|_{[a,b]} \mid 1 \le i \le n\}$ generate a $\mathbb{Z}^n$ expansive action on $[a, b]$, which contradicts the conclusion of Step 1. So there is some f_i with no periodic points. Without loss of generality, we may assume f_1 has no periodic points. By Theorem 2.7, f_1 is semi-conjugate to a minimal rotation P on $\mathbb{S}^1$, i.e., there is a continuous surjection $\Psi : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ such that $\Psi f_1 = P \Psi$. If for every $x \in \mathbb{S}^1$ $\Psi^{-1}(x)$ is a single point, then Ψ is a homeomorphism. Thus f_1 is conjugate to an isometric action on $\mathbb{S}^1$. In this case, it is not difficult to show that the $\mathbb{Z}^{n-1}$ action generated by $\{f_i | 2 \le i \le n\}$ is still expansive (expansive constant may be changed), which contradicts the inductive assumption. Therefore there is some $x \in \mathbb{S}^1$ such that $\Psi^{-1}(x)$ is a non-degenerate subinterval of $\mathbb{S}^1$. Let $\Gamma = \{(\alpha_i, \beta_i) \subset \mathbb{S}^1 | \exists x \in I \text{ s.t. } [\alpha_i, \beta_i] = \Psi^{-1}(x)\}$ and let Ω be the nonwandering set of f_1. It is easy to see that $\mathbb{S}^1 \setminus \Omega = \bigcup_{(\alpha_i, \beta_i) \in \Gamma} (\alpha_i, \beta_i)$. By Lemma 2.6, $f_i(\mathbb{S}^1 \setminus \Omega) = \mathbb{S}^1 \setminus \Omega$, $1 \le i \le n$.

Since the family of open intervals (α_i, β_i) are pairwise disjoint, each f_i permutes these open intervals in Γ, i.e., f_i maps each $(\alpha_j, \beta_j) \in \Gamma$ onto some $(\alpha_{j'}, \beta_{j'}) \in \Gamma$. Fix some $(\alpha_0, \beta_0) \in \Gamma$. Let $H = \{\vec{m} \in \mathbb{Z}^d | \theta(\vec{m})((\alpha_0, \beta_0)) = (\alpha_0, \beta_0)\}$. It is clear that H is a subgroup of $\mathbb{Z}^d$. Consider the coset decomposition $\mathbb{Z}^d = H \cup (\vec{m}_1 + H) \cup \cdots \cup (\vec{m}_k + H) \cup \cdots$. It is easy to see that if $\vec{m}_i \neq \vec{m}_j$ then $\theta(\vec{m}_i)((\alpha_0, \beta_0)) \neq \theta(\vec{m}_j)((\alpha_0, \beta_0))$. So the cardinality of $\mathbb{Z}^d/H$ is infinite. Since the length of $\mathbb{S}^1$ is finite, there is some $k > 0$ such that, for any $i > k$ and any $\vec{h} \in H$, we have

$$\operatorname{diam}(\theta(\vec{m}_i + \vec{h})((\alpha_0, \beta_0))) = \operatorname{diam}(\theta(\vec{m}_i)((\alpha_0, \beta_0))) \leq c. \tag{5}$$

By uniform continuity, there is a $\delta > 0$ such that $d(x, y) \leq \delta$ implies

$$d(\theta(\vec{m}_i)(x), \theta(\vec{m}_i)(y)) \leq c, \text{ for all } 1 \leq i \leq k. \tag{6}$$

Since H is a subgroup of $\mathbb{Z}^n$, $H \cong \mathbb{Z}^l$ for some $0 \leq l \leq n$ (see [15, Ch. 1-Thm. 7.3]). By Step 1 we know that the restriction of H to $[\alpha_0, \beta_0]$ is not expansive. So there are $x, y \in [\alpha_0, \beta_0]$ with

$$d(\theta(\vec{h})(x), \theta(\vec{h})(y)) \leq \delta, \text{ for all } h \in H. \tag{7}$$

By (5), (6) and (7), we get that for any $\vec{m} \in \mathbb{Z}^n$, $d(\theta(\vec{m})(x), \theta(\vec{m})(y)) \leq c$, which contradicts our assumption. So there exists no expansive $\mathbb{Z}^n$ action on $\mathbb{S}^1$. The proof is complete.

Recall that a graph is a continuum which can be written as a union of finitely many arcs in such a way that each two arcs are either disjoint or intersect only in one or both of their end points.

Corollary 3.2 *There is no expansive $\mathbb{Z}^d$ action on graphs.*

Proof This corollary can be easily obtained from Proposition 2.3 and Theorem 3.1.

4 Two Examples

One may expect that a space which admits no expansive $\mathbb{Z}$ actions must admit no expansive $\mathbb{Z}^d$ actions for any $d \geq 1$. But this is not true in general as the following example shows.

Mañé proved that there is no expansive $\mathbb{Z}$ action on infinite dimensional spaces [7]. In the following we will construct an expansive $\mathbb{Z}^2$ action on an infinite dimensional space.

Let $K^2 = \mathbb{S}^1 \times \mathbb{S}^1$ be a two-dimensional torus and $A : K^2 \to K^2$ be an homeomorphism on K^2 which is induced by a matrix $[A] = (a_{ij})_{2\times 2}$, where a_{ij} are integers, $\det([A]) = 1$ and $[A]$ has no eigenvalues of modulus 1. Then A is an expansive homeomorphism on K^2([11, p. 143]).

Now let $X = (K^2)^Z = \{(x_i)_{i=-\infty}^{\infty} | x_i \in K^2\}$ be the product of countable many copies of K^2. Let d be a compactible metric on K^2. Define a compactible metric $\tilde{d}$ on X by $\tilde{d}(x, y) = \sum_{i=-\infty}^{\infty} \frac{1}{2^{|i|}} d(x_i, y_i)$, for any $x = (x_i) \in X$ and $y = (y_i) \in X$. Now we define two homeomorphisms on X by $f(x)_i = A(x_i)$ and $\sigma(x)_i = x_{i+1}$ for any $x \in X$. Obviously f and σ are commutative, so they generate a $\mathbb{Z}^2$ action on X.

Proposition 4.1 *The $\mathbb{Z}^2$ actions generated by f and σ on X are expansive.*

Proof If $x \neq y \in X$, then there is some integer m with $x_m \neq y_m$. Since $A : K^2 \to K^2$ is expansive, there is a constant $c > 0$ such that for any $a \neq b \in K^2$, there is an integer n with $d(A^n(a), A^n(b)) > c$. It follows that there is some integer k with $d(f^k(x)_m, f^k(y)_m) = d(A^k(x_m), A^k(y_m)) > c$. So $\tilde{d}(\sigma^m f^k(x), \sigma^m f^k(y)) = \sum_{i=-\infty}^{\infty} \frac{1}{2^{|i|}} d(A^k x_{i+m}, A^k y_{i+m}) > d(A^k x_m, A^k y_m) > c$. Thus the $\mathbb{Z}^2$ action generated by f and σ is expansive with expansive constant c.

Although we give a space which admits expansive $\mathbb{Z}^2$ actions but admits no expansive homeomorphisms in the above proposition, the dimension of the space is infinite. We don't know whether there is a finite dimensional space which admits an expansive $\mathbb{Z}^2$ action but admits no expansive $\mathbb{Z}$ actions. However, in the following, we can give an expansive $\mathbb{Z}^2$ action on a zero-dimensional space but none of the elements in $\mathbb{Z}^2$ is expansive.

Let $X = \{0, 1\}^{\mathbb{Z}^2} = \{(x_{(i,j)})_{(i,j)\in\mathbb{Z}^2} | x_{(i,j)} \in \{0, 1\}\}$. Let σ_1 and σ_2 be two shifts on X defined by $\sigma_1(x)_{(i,j)} = x_{(i+1,j)}$ and $\sigma_2(x)_{(i,j)} = x_{(i,j+1)}$, for any $x = (x_{(i,j)}) \in X$. Obviously σ_1 and σ_2 are two commutative homeomorphisms on X, so they generate a $\mathbb{Z}^2$ action on X. Define a compactible metric d on X by $d(x, x) = 0$ and $d(x, y) = 2^{-\min\{|i|+|j| \mid x_{(i,j)} \neq y_{(i,j)}\}}$ for $x \neq y$.

Proposition 4.2 *The $\mathbb{Z}^2$ action generated by σ_1 and σ_2 on X is expansive but no elements in $\mathbb{Z}^2$ are expansive.*

Proof If $x \neq y \in X$, then there is some $(m,n) \in \mathbb{Z}^2$ with $x_{(m,n)} \neq y_{(m,n)}$. So

$$d(\sigma_1^m \sigma_2^n(x), \sigma_1^m \sigma_2^n(y)) = 2^{-\min\{|i|+|j| \mid x_{(i+m,j+n)} \neq y_{(i+m,j+n)}\}} = 1.$$

Therefore the $\mathbb{Z}^2$ action generated by σ_1 and σ_2 is expansive with expansive constant 1.

If $(m,n) \neq (0,0)$, we will prove that the homeomorphism $h = \sigma_1^m \sigma_2^n$ on X is not expansive. For any $r > 0$, let $A_r = \{(m',n') \in \mathbb{Z}^2 \mid |mn' - nm'| \geq r\}$. It is easy to see that for any $(m',n') \in A_r$, we have $|m'| + |n'| \geq (m'^2 + n'^2)^{\frac{1}{2}} \geq \frac{r}{(m^2+n^2)^{\frac{1}{2}}}$. It follows that for any $\epsilon > 0$, there is a sufficiently large r with

$$2^{-(|m'|+|n'|)} \leq 2^{-r(m^2+n^2)^{\frac{-1}{2}}} < \epsilon, \quad \text{for all } (m',n') \in A_r. \tag{8}$$

Let $x \in X$ with $x_{(i,j)} = 1$ if $(i,j) \in A_r$ and $x_{(i,j)} = 0$ if $(i,j) \notin A_r$, and let $y \in X$ with $y_{(i,j)} = 0$, for all $(i,j) \in \mathbb{Z}^2$. Since $(m',n') \in A_r$ implies $(m' - km, n' - kn) \in A_r$ for any $k \in \mathbb{Z}$, we have

$$\{(i,j) \mid x_{(i+km,j+kn)} \neq y_{(i+km,j+kn)}\} \subset A_r. \tag{9}$$

By (8) and (9) we get $d(h^k(x), h^k(y)) = 2^{-\min\{|i|+|j| \mid x_{(i+km,j+kn)} \neq y_{(i+km,j+kn)}\}} < \epsilon$. Since ϵ is arbitrary, h is not expansive.

5 Some Questions

The following questions are still open:

Question 5.1. Can a commutative group, which is not finitely generated, act on an arc or a circle expansively?

Question 5.2. Can a nilpotent group act on an arc or a circle expansively?

Question 5.3. There are many continua that admit no expansive $\mathbb{Z}$ actions besides the arc and circle, which of these do not admit expansive $\mathbb{Z}^d$ actions with $d \geq 1$?

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