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SOME ERGODIC AND RIGIDITY PROPERTIES OF DISCRETE HEISENBERG GROUP ACTIONS

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ABSTRACT

The goal of this paper is to study ergodic and rigidity properties of smooth actions of the discrete Heisenberg group $\mathcal{H}$. We establish the decomposition of the tangent space of any C^∞ compact Riemannian manifold M for Lyapunov exponents, and show that all Lyapunov exponents for the central elements are zero. We obtain that if an $\mathcal{H}$ action contains an Anosov element, then under certain conditions on the eigenvalues of this element, the action of each central element is of finite order. In particular, there is no faithful codimension one Anosov Heisenberg group action on any compact manifold, and there is no faithful codimension two Anosov Heisenberg group action on tori. In addition, we show smooth local rigidity for higher rank ergodic $\mathcal{H}$ actions by toral automorphisms, using a generalization of the KAM (Kolmogorov–Arnold–Moser) iterative scheme.

0. Introduction

In the past few decades, there has been considerable progress in studying the ergodic theory and smooth rigidity of higher rank abelian group actions. Smooth (local) global rigidity for higher rank abelian algebraic actions has been extensively studied; some of the highlights are [DK1], [DK] and [KKH, HW]. We refer the reader to [Sc] to a systematic introduction of the dynamics of algebraic $\mathbb{Z}^d$ actions. A natural question is how to extend these theories to noncommutative group actions. The discrete Heisenberg group is a 2-step nilpotent group, which is most close to being abelian. So studying the dynamical properties of discrete Heisenberg group actions is the first step toward extending what we have known about $\mathbb{Z}^d$ actions.

Throughout the paper, we use the symbol $\mathcal{H}$ to denote the discrete Heisenberg group (see Section 1 for the explicit definition). We first establish a tangent space decomposition into subspaces related to Lyapunov exponents on any compact manifolds M in Theorem A, which can be viewed as an extension of the corresponding theorem established for $\mathbb{Z}^2$ actions in [Hu]. As corollaries of this theorem, we obtain that the action of central elements of $\mathcal{H}$ must have 0 Lyapunov exponents with respect to any $\mathcal{H}$ invariant measure, and have 0 topological entropy when the action is C^∞. This indicates that the action of central elements in $\mathcal{H}$ cannot be chaotic for any $\mathcal{H}$ action on compact manifolds.

The second part of our work concerns faithfulness of $\mathcal{H}$ actions. We show in Theorem B that if an $\mathcal{H}$ action is C^r, $r > 1$, and contains an Anosov element which has simple eigenvalues on the stable direction with $\lambda_- > \lambda_+^{\min\{r,2\}}$ (see (1.4)), then the action of any central element of $\mathcal{H}$ is of finite order. In particular, it is true for any codimension 1 action. For Anosov $\mathcal{H}$ actions on tori, we show further in Theorem D that the action of any central element is either conjugate to a translation of finite order or conjugate to an affine transformation of order 2 when the action is of codimension one or two. This implies especially that there is no faithful Anosov $\mathcal{H}$ action on $\mathbb{T}^n$ with $n \leq 5$, though there are faithful Anosov $\mathcal{H}$ actions on $\mathbb{T}^n$ with $n = 6$ or $n \geq 8$ (see Example 1.4 and Remark 1.5 in Section 1).

Lastly we obtain some rigidity results for $\mathcal{H}$ actions on tori. We prove that all such actions are topologically conjugate to affine ones in Theorem C, using the results in [AP], [Fr] and [Ma]. Further, we extend an approach for proving local differentiable rigidity of Heisenberg group action by toral automorphisms, based on a KAM-type iteration scheme that was first introduced in [DK] and was later developed in [DK1].

We note that the expansiveness and homoclinic points for Heisenberg algebraic actions have recently been investigated by M. Göll, K. Schmidt, and E. Verbitskiy in [GSV]. One may consult [GS, Li, OW] for the study of abstract ergodic theory about nilpotent group or amenable group actions. It was known in the 1970s that if $M = \mathbb{R}, \mathbb{S}^1$ or $I = [0,1]$, then any nilpotent subgroup of $\mathrm{Diff}^2(M)$ must be abelian ([PT]), which implies that there is no faithful C^2 action of $\mathcal{H}$ on $\mathbb{S}^1$. In this century it was discovered out that every finitely generated, torsion-free nilpotent group has a faithful C^1 action on M ([FF]). For the case $\dim M = 2$, there are many faithful analytic Heisenberg group actions on $\mathbb{S}^2$, closed disks, closed annulus and torus ([Pa]). However, Franks and Handel [FH] showed that a nilpotent group of C^1 diffeomorphisms which are isotopic to the identity and preserve a measure whose support is all of $\mathbb{T}^2$ must be abelian.

The paper is organized as follows. We state the results of the paper in Section 1. Section 2 contains the proof of Theorem A concerning Lyapunov exponents, while Section 3 contains the proof of Theorem B concerning faithfulness. Theorems C and D are proved in Section 4. The last section deals with smooth rigidity: the proof of Theorem E.

1. Background and statement of results

Let M be a C^∞ compact manifold. We denote by $\mathrm{Diff}^r(M)$ the group of C^r diffeomorphisms on M for $r > 0$. Let $\mathcal{H}$ be the discrete Heisenberg group with generators A, B and $\mathcal{C}$ satisfying the relations

$$AC = \mathcal{C}A, \quad B\mathcal{C} = \mathcal{C}B, \quad AB = BA\mathcal{C}. \tag{1.1}$$

Then for every $K \in \mathcal{H}$, there is a unique triple $(n_1, n_2, n_3) \in \mathbb{Z}^3$ such that $K = A^{n_1}B^{n_2}\mathcal{C}^{n_3}$. Clearly, $\mathcal{H}$ is a 2-step nilpotent group with center $\langle \mathcal{C} \rangle$. In this paper, we examine ergodic and rigidity properties of $\mathcal{H}$ actions on compact manifolds.

Let $\alpha : \mathcal{H} \to \mathrm{Diff}^r(M)$ be a C^r action of $\mathcal{H}$ on a C^∞ compact Riemannian manifold M, i.e., $\alpha : G \to \mathrm{Diff}^r(M)$ is a group homomorphism. Write $f = \alpha(A)$, $g = \alpha(B)$, and $h = \alpha(\mathcal{C})$; then

$$fh = hf, \quad gh = hg, \quad \text{and} \quad fg = gfh. \tag{1.2}$$

Throughout the paper, we always use f, g and h to denote $\alpha(A)$, $\alpha(B)$ and $\alpha(\mathcal{C})$ respectively, for a fixed $\mathcal{H}$ action α.

The Heisenberg group $\mathcal{H}$ naturally has an action on $\mathbb{T}^3$ since $\mathcal{H}$ embeds into $\mathrm{SL}(3, \mathbb{Z})$. We can obtain more general examples such as the following.

Example 1.1: Let

$$A = \begin{bmatrix} X & I_n & O \\ O & X & O \\ O & O & X \end{bmatrix}, \quad B = \begin{bmatrix} Y & O & O \\ O & Y & I_n \\ O & O & Y \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} I_n & O & X^{-1}Y^{-1} \\ O & I_n & O \\ O & O & I_n \end{bmatrix},$$

where $X, Y \in \mathrm{SL}(n, \mathbb{Z})$ with $XY = YX$, and I_n and O are the $n \times n$ identity and zero matrix, respectively.

It is easy to check that condition (1.1) is satisfied. If $M = \mathbb{T}^{3n}$, then A, B and $\mathcal{C}$ induce automorphisms f, g and h on M, respectively, that generate a Heisenberg group action α.

1.1. Dynamical properties of the central element h. A probability measure μ on M is said to be α**-invariant** if $\alpha(k)_*\mu = \mu$ for every $k \in \mathcal{H}$. We denote by $\mathcal{M}(M, \alpha)$ the set of all α-invariant Borel probability measures. Since $\mathcal{H}$ is amenable, $\mathcal{M}(M, \alpha) = \mathcal{M}(M, f) \cap \mathcal{M}(M, g) \cap \mathcal{M}(M, h) \neq \emptyset$, where $\mathcal{M}(M, f)$ is the set of f-invariant probability measures (see, e.g., [Zi, Corollary 4.1.7]).

Let T_xM be the tangent space of M at $x \in M$ and $\phi \in \mathrm{Diff}^2(M)$; then ϕ induces a map $D\phi_x : T_xM \to T_{\phi x}M$. It is well known that there exists a measurable set Γ_ϕ with $\mu\Gamma_\phi = 1$ for all $\mu \in \mathcal{M}(M,\phi)$, such that for all $x \in \Gamma_\phi$, $0 \neq u \in T_xM$, the limit

$$\chi(x,u,\phi) = \lim_{n\to\infty} \frac{1}{n} \log \parallel D\phi_x^n u \parallel$$

exists and is called the **Lyapunov exponent** of u at x.

Let $\lambda_1(x,\phi) > \cdots > \lambda_{r(x,\phi)}(x,\phi)$ denote all Lyapunov exponents of ϕ at x with multiplicities $m_1(x,\phi), \ldots, m_{r(x,\phi)}(x,\phi)$, respectively, and

$$T_xM = \bigoplus_{i=1}^{r(x,\phi)} E_i(x,\phi)$$

be the corresponding decomposition of tangent space at $x \in M$.

The first result shows that the Lyapunov space decomposition of $\mathcal{H}$-action is similar to the case of $\mathbb{Z}^2$ action:

Theorem A: *There exists an $\mathcal{H}$ invariant measurable set Γ such that $\mu\Gamma = 1$ for all $\mu \in \mathcal{M}(M,\alpha)$, and there is an $\mathcal{H}$ invariant decomposition of the tangent bundle over Γ,*

$$T_xM = \bigoplus_{i=1}^{r(x,f)} \bigoplus_{j=1}^{r(x,g)} E_{ij}(x), \quad x \in \Gamma,$$

such that if $E_{ij}(x) \neq \{0\}$, then for all $0 \neq u \in E_{ij}(x)$ and all $s,t,r \in \mathbb{Z}$,

$$\lim_{n\to\infty} \frac{1}{n} \log \parallel D(f^s g^t h^r)_x^n u \parallel = s\lambda_i(x,f) + t\lambda_j(x,g). \tag{1.3}$$

By this theorem, we have some immediate corollaries, which indicate that the action of central elements in $\mathcal{H}$ cannot be chaotic for any $\mathcal{H}$ action on compact manifolds. The concept of chaos was first introduced by T. Li and J. Yorke in [LJ]. Up to today, there are many definitions of chaos based on different viewpoints. In general, having positive Lyapunov exponents or having positive topological entropy is regarded as an important feature of chaos for diffeomorphisms.

Corollary A.1: *All Lyapunov exponents of h are zero with respect to any measure $\mu \in \mathcal{M}(M,\alpha)$. In particular, for any $n > 0$, if h has finitely many periodic points p of period n, then all eigenvalues of Dh_p^n have modulus 1.*

Corollary A.2: *If the action α is C^∞, then the topological entropy $h_{\mathrm{top}}(h)=0$.*

Remark 1.2: As we mentioned in Introduction, Corollary A.1 and Corollary A.2 indicate that the central elements in $\mathcal{H}$ cannot be chaotic for any C^∞ $\mathcal{H}$ action on manifolds.

Observe that $f^n g^n = g^n f^n h^{n^2}$. We have naturally the following question.

Question 1.3: For any $\mu \in \mathcal{M}(M,\alpha)$, is $\|Dh_x^n v\|$ bounded by $e^{\sqrt{n}\varepsilon}$ for some $\varepsilon > 0$, or even by a polynomial in n for μ-a.e. x?

1.2. Existence of a faithful Anosov $\mathcal{H}$ action. A group action $\alpha : G \to \mathrm{Diff}^r(M)$ is called **faithful** if the map α is injective. We give some conditions under which an $\mathcal{H}$ action cannot be faithful.

A diffeomorphism ϕ on a compact manifold M is called **Anosov** if the whole tangent bundle has a uniformly hyperbolic decomposition into $T_xM = E_x^s \oplus E_x^u$ invariant under the differential $D\phi : TM \to TM$; that is, there are constants $C > 0$ and $0 < \lambda < 1$ such that, for all $n \geq 0$,

$$||D\phi^n|_{E_x^s}|| \leq C\lambda^n \quad \text{and} \quad ||D\phi^{-n}|_{E_x^u}|| \leq C\lambda^n,$$

$\forall x \in M$. For a group action $\alpha : G \to \mathrm{Diff}^r(M)$, if $\alpha(G)$ contains an Anosov element, then the action α is called **Anosov**. The following is an example of an Anosov $\mathcal{H}$ action:

Example 1.4: In Example 1.1, if X or Y is a hyperbolic matrix, then f or g is a hyperbolic diffeomorphisms on $M = \mathbb{T}^{3n}$. Hence α is an Anosov action. In particular, we can take $M = \mathbb{T}^6$, and

$$X = Y = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and then we get an Anosov $\mathcal{H}$ action on $M = \mathbb{T}^6$.

We say that an Anosov diffeomorphism ϕ has **simple eigenvalues on the stable direction** if for every periodic point p of period n, all eigenvalues of $D\phi_p^n|_{E_p^s}$ are real and has algebraic multiplicity 1. An Anosov diffeomorphism is said to be of **codimension** 1 if either $\dim E^u = 1$ or $\dim E^s = 1$. Clearly, either ϕ or ϕ^{-1} has simple eigenvalues on the stable direction if ϕ is of codimension 1.

If $D\phi_p^n|_{E_p^s}$ has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{\dim E^s(\phi)}$ on the stable direction, we denote

$$\lambda_- = \min_{1 \leq i \leq \dim E^s(\phi)} |\lambda_i|, \quad \lambda_+ = \max_{1 \leq i \leq \dim E^s(\phi)} |\lambda_i|. \tag{1.4}$$

Theorem B: *Let $\alpha : \mathcal{H} \to \mathrm{Diff}^r(M)$, $r > 1$, be an action of $\mathcal{H}$ on compact manifold M such that $\alpha(\mathcal{H})$ contains an Anosov diffeomorphism that has simple eigenvalues with $\lambda_- > \lambda_+^{\min\{2,r\}}$ on the stable direction. Then $h^k = \mathrm{id}$ for some $k \geq 1$.*

Following from Theorem B, we have immediately

Corollary B.1: *There is no faithful C^r, $r > 1$, Heisenberg group action that contains an Anosov element with simple eigenvalues with $\lambda_- > \lambda_+^{\min\{2,r\}}$ on the stable direction. In particular, there is no faithful C^r codimension 1 Anosov Heisenberg group action on any compact manifold.*

Also the proof of Theorem B gives

Corollary B.2: *For a C^r, $r > 1$, $\mathcal{H}$ action on M, if any element is a codimension 1 Anosov diffeomorphism that has exactly one fixed point, then $h^2 = \mathrm{id}$.*

The next statement is a consequence of Theorem D. We state it here since it concerns faithfulness.

Corollary D.1: *There is no faithful C^2 Anosov Heisenberg group action on $\mathbb{T}^n$ with $n \leq 5$.*

As supplements to Corollary D.1, we give the following remark.

Remark 1.5: Example 1.4 indicates that $\mathbb{T}^6$ does admit faithful Anosov Heisenberg group $\mathcal{H}$ actions. So the number “5” that appeared in the above corollary is the best. In fact, it is easy to see that there exist faithful Anosov Heisenberg group actions on $\mathbb{T}^n$ for any integer $n \geq 8$ by using the action in Example 1.4 crossing an Anosov diffeomorphism of a manifold of dimension two or higher. However, the authors do not know whether such group actions exist on $\mathbb{T}^7$.

1.3. Rigidity of $\mathcal{H}$ action. Let α, α' be two actions of group G on M. Then α and α' are said to be **topologically conjugate** if there is a homeomorphism $T : M \to M$ such that

$$T \circ \alpha(g) = \alpha'(g) \circ T$$

for any $g \in G$.

The first two results are about global topological rigidity of an $\mathcal{H}$ action on $\mathbb{T}^N$. Before the statement of Theorem C, we introduce some related notions. Let $\mathbb{T}^n \equiv \mathbb{R}^n/\mathbb{Z}^n$ be the n-dimensional torus. For any $A \in \mathrm{GL}(n, \mathbb{Z})$ and $a \in \mathbb{R}^n$, define $T_{A,a} : \mathbb{T}^n \to \mathbb{T}^n$ by

$$T_{A,a}([x]) = [Ax + a]$$

for any $x \in \mathbb{R}^n$. Such $T_{A,a}$ is called an **affine transformation** on $\mathbb{T}^n$. Especially, $T_{A,0}$ is called a **linear automorphism** of $\mathbb{T}^n$ induced by A and $T_{\mathrm{id},a}$ is called a **translation** on $\mathbb{T}^n$ by a. All the affine transformations on $\mathbb{T}^n$ form a group which is called the **affine transformation group** of $\mathbb{T}^n$ and is denoted by $\mathrm{Aff}(\mathbb{T}^n)$. If A has no eigenvalue with modular 1, then $T_{A,a}$ is an Anosov diffeomorphism.

Theorem C: *Every Anosov Heisenberg group action on $\mathbb{T}^n$ is topologically conjugate to an affine one.*

In some cases, the form of h can be completely determined as the following theorem shows.

Theorem D: *If f is a codimension 1 Anosov diffeomorphism of a Heisenberg group action on $\mathbb{T}^n$, then h is topologically conjugate to a translation of finite order. If f is a codimension 2 Anosov diffeomorphism for a Heisenberg group action on $\mathbb{T}^n$, then h is either topologically conjugate to a translation of finite order or topologically conjugate to an affine transformation $T_{-\mathrm{id},c}$ of order 2 for some $c \in \mathbb{T}^n$.*

For the codimension 1 case, the following example indicates that h in the above theorem can be non-trivial.

Example 1.6: Let

$$A = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}, \quad b = \begin{pmatrix} 2/5 \\ 3/5 \end{pmatrix}, \quad c = \begin{pmatrix} 2/5 \\ 4/5 \end{pmatrix}.$$

Define affine transformations f, g, h on $\mathbb{T}^2$ by

$$f([x]) = [Ax], \quad g([x]) = [Ax + b], \quad \text{and} \quad h([x]) = [x + c]$$

for all $x \in \mathbb{R}^2$. Then $fh = hf$, $gh = hg$, $fg = gfh$, and $h^5([x]) = [x]$. Thus we get a Heisenberg group action on $\mathbb{T}^2$ with h being a translation of order 5.

The following example shows that there do exist examples such that h is conjugate to $T_{-Id,c}$ as shown in Theorem D in the codimension 2 case.

Example 1.7: Take

$$X = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Let

$$A = \begin{bmatrix} X & 0 \\ 0 & -X \end{bmatrix}, \quad B = \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} -I & 0 \\ 0 & -I \end{bmatrix},$$

where I is the 2×2 identity matrix. For any $c \in \mathbb{R}^4$, let $a = -\frac{1}{2}(A - I)c$ and $b = -\frac{1}{2}(B - I)c$. Then it is easy to check that the affine transformations $T_{A,a}, T_{B,b}$ and $T_{-\mathrm{id},c}$ on $\mathbb{T}^4$ satisfy the relations

$$T_{A,a}T_{-\mathrm{id},c} = T_{-\mathrm{id},c}T_{A,a}, \; T_{B,b}T_{-\mathrm{id},c} = T_{-\mathrm{id},c}T_{B,b} \text{ and } T_{A,a}T_{B,b} = T_{B,b}T_{A,a}T_{-\mathrm{id},c}.$$

Thus we get an affine Anosov $\mathcal{H}$ action on $\mathbb{T}^4$ with h being of the form $T_{-\mathrm{id},c}$.

An action α of a finitely generated discrete group G on a manifold M is $C^{k,r,\ell}$ **locally rigid** if any C^k perturbation $\widetilde{\alpha}$ which is sufficiently C^r close to α on a finite generating set is C^ℓ conjugate to α; i.e., there exists a diffeomorphism T of M C^ℓ close to identity which conjugates $\widetilde{\alpha}$ to α: $T \circ \alpha(g) = \alpha'(g) \circ T$ for any $g \in G$.

An $\mathcal{H}$ action α by automorphisms on $\mathbb{T}^N$ is called an **ergodic higher rank action** if it contains two elements h_1, h_2 such that $\alpha(h_1^m h_2^n) \in SL(N, \mathbb{Z})$ is ergodic for all $(m, n) \neq 0$ in $\mathbb{Z}^2$.

THEOREM E: *Let α be an ergodic higher rank $\mathcal{H}$ action by automorphisms of the N-dimensional torus. Then there exists a constant $l = l(\alpha, N) \in \mathbb{N}$ such that α is $C^{\infty,l,\infty}$ locally rigid.*

Remark 1.8: In this paper, we only consider smooth actions of the three-dimensional discrete Heisenberg group $\mathcal{H}$. Certainly, some results obtained here can be easily extended to actions of higher-dimensional ones; for example, having 0 Lyapunov exponents still holds for central elements in higher-dimensional Heisenberg groups, since every central element of which is contained in a three dimensional Heisenberg subgroup; it seems that a decomposition theorem similar to Theorem A can also be established for such actions following the same ideas in the paper. However, we feel that it is not direct to extend the other results to actions of general Heisenberg groups. We hope the ideas and results in the paper may shine some light on the study of smooth actions of higher-dimensional Heisenberg, nilpotent, or more generally solvable groups.

2. Lyapunov exponents: Proof of Theorem A and its corollaries

Since g and h are commuting maps, by Theorem A in [Hu] there exists a measurable set Γ_0 with $g^s h^t \Gamma_0 = \Gamma_0$, $\forall s, t \in \mathbb{Z}$, and $\mu\Gamma = 1$, $\forall \mu \in \mathcal{M}(M, g, h)$, such that for all $x \in \Gamma_0$, there is a (unique) decomposition of the tangent space into

$$T_x M = \bigoplus_{j=1}^{r(x,g)} \bigoplus_{k=1}^{r(x,h)} E_{jk}(x) \tag{2.1}$$

such that for all $s, t \in \mathbb{Z}$ and for all $0 \neq u \in E_{jk}(x)$ when $E_{jk}(x) \neq 0$,

$$\lim_{n\to\infty} \frac{1}{n} \log \| D(g^s h^t)_x^n u \| = s\lambda_j(x, g) + t\lambda_k(x, h). \tag{2.2}$$

Moreover,

$$D(g^s h^t)_x (E_{jk}(x)) = E_{jk}(g^s h^t x)$$

and

$$\lambda_j(g^s h^t x, g) = \lambda_j(x, g), \quad \lambda_k(g^s h^t x, h) = \lambda_k(x, h).$$

Let

$$\Gamma_1 = \bigcap_{i\in\mathbb{Z}} f^i \Gamma_0.$$

By (1.2) it is clear that $f\Gamma_1 = \Gamma_1$, $g\Gamma_1 = \Gamma_1$, and $h\Gamma_1 = \Gamma_1$. So Γ_1 is an $\mathcal{H}$-invariant measurable set and $\mu\Gamma_1 = 1$, $\forall \mu \in \mathcal{M}(M, \alpha)$.

Lemma 2.1: *For all $x \in \Gamma_1$ and all $0 \neq u \in E_{jk}(x)$, we have*

$$\chi(fx, Df_x u, g^{\pm}) = \pm\lambda_j(x, g) \mp \lambda_k(x, h), \quad \chi(fx, Df_x u, h^{\pm}) = \pm\lambda_k(x, h).$$

Proof. By (1.2) we have

$$\| Dg^{\pm n} Df_x u \| = \| Df Dg^{\pm n} Dh_x^{\mp n} u \|, \quad \| Dh^{\pm n} Df_x u \| = \|Df Dh_x^{\pm n} u\|.$$

So there is a constant $c > 0$ such that for all $n \geq 0$,

$$c^{-1}\|Dg^{\pm n} Dh_x^{\mp n} u\| \leq \|Dg^{\pm n} Df_x u\| \leq c\|Dg^{\pm n} Dh_x^{\mp n} u\|,$$
$$c^{-1}\|Dh_x^{\pm n} u\| \leq \|Dh^{\pm n} Df_x u\| \leq c\|Dh_x^{\pm n} u\|.$$

Then by (2.2) we get that

$$\begin{aligned} \chi(fx, Df_x u, g^{\pm 1}) &= \lim_{n\to\infty} \frac{1}{n} \log \|Dg^{\pm n} Df_x u\| = \pm\lambda_j(x, g) \mp \lambda_k(x, h), \\ \chi(fx, Df_x u, h^{\pm 1}) &= \lim_{n\to\infty} \frac{1}{n} \log \|Dh^{\pm n} Df_x u\| = \pm\lambda_k(x, h), \end{aligned} \tag{2.3}$$

which are what we need. ∎

LEMMA 2.2: *For every $x \in \Gamma_1$, every (j,k) with $E_{jk}(x) \neq 0$ and every $0 \neq u \in E_{jk}(x)$, we have $\chi(x,u,h) = 0$.*

Proof. Assume to the contrary that there exist some $x_0 \in \Gamma_1$ and some (j_0, k_0) with $E_{j_0k_0}(x_0) \neq 0$ and some $0 \neq u_0 \in E_{j_0k_0}(x_0)$ such that $\chi(x_0, u_0, h) \neq 0$. From Lemma 2.1 we have

$$\begin{aligned}\chi(fx_0, Df_{x_0}u_0, g) =&\lambda_{j_0}(x_0, g) - \lambda_{k_0}(x_0, h),\\ \chi(fx_0, Df_{x_0}u_0, h^{\pm 1}) =& \pm\, \lambda_{k_0}(x_0, h).\end{aligned}$$

It follows that in the decomposition

$$T_{fx_0}M = \bigoplus_{j=1}^{r(fx_0,g)} \bigoplus_{k=1}^{r(fx_0,h)} E_{jk}(fx_0),$$

there is some $E_{j_1k_1}(fx_0) \neq 0$ such that for all $0 \neq u \in E_{j_1k_1}(fx_0)$,

$$\begin{aligned}\chi(fx_0, u, g) =&\lambda_{j_0}(x_0, g) - \lambda_{k_0}(x_0, h),\\ \chi(fx_0, u, h) =&\lambda_{k_0}(x_0, h).\end{aligned}$$

Then by the induction process, we get that in the decomposition

$$T_{f^nx_0}M = \bigoplus_{j=1}^{r(f^nx_0,g)} \bigoplus_{k=1}^{r(f^nx_0,h)} E_{jk}(f^nx_0),$$

there is some $E_{j_nk_n}(f^nx_0) \neq 0$ such that for all $0 \neq u \in E_{j_nk_n}(f^nx_0)$,

$$\begin{aligned}\chi(f^nx_0, u, g) =&\lambda_{j_0}(x_0, g) - n\lambda_{k_0}(x_0, h),\\ \chi(f^nx_0, u, h) =&\lambda_{k_0}(x_0, h).\end{aligned}$$

Since $\lambda_{k_0}(x_0, h) = \chi(x_0, u_0, h) \neq 0$, $|\chi(f^nx_0, u, g)| \to \infty$ as $n \to \infty$, contradicting the fact that $|\chi(p, v, g)| \leq \sup_{x\in M} \|Dg_x\| < \infty$, $\forall p \in \Gamma_1$, $\forall v \in T_pM$. ∎

PROPOSITION 2.3: *For all $x \in \Gamma_1$, there is a decomposition of the tangent space into $T_xM = \bigoplus_{j=1}^{r(x,g)} E_j(x, g)$ such that for all $0 \neq u \in E_j(x, g)$,*

(i) *the spectrum $\{\lambda_j(x, g),\ m_j(x, g),\ j = 1, \ldots, r(x, g)\}$ is $\mathcal{H}$ invariant;*
(ii) *$D(f^sg^th^r)_xE_j(x, g) = E_j(f^sg^th^rx, g)$, $\forall j = 1, \ldots, r(x, g)$, $\forall\, s, t, r \in \mathbb{Z}$;*
(iii) *$\lim_{n\to\infty} \frac{1}{n} \log \|D(g^th^r)^n_x u\| = t\lambda_j(x, g)$, $\forall\, t, r \in \mathbb{Z}$.*

Proof. By Lemma 2.2, we get that the number $r(x,h)=1$ and the decomposition (2.1) becomes, for all $x\in\Gamma_1$,

$$T_xM=\bigoplus_{j=1}^{r(x,g)}E_j(x,g),$$

where

$$E_j(x,g)=\bigoplus_{k=1}^{r(x,h)}E_{jk}(x)=E_{j1}(x).$$

It follows from Lemma 2.2 again that equations (2.3) become

$$\begin{aligned}\chi(fx,Df_xu,g^{\pm1})&=\pm\,\lambda_j(x,g),\\ \chi(fx,Df_xu,h^{\pm1})&=0.\end{aligned}$$

So (i) and (ii) hold, and (iii) follows from (2.2). ∎

Exchanging f and g, we get the following similar proposition:

Proposition 2.4: *There is an $\mathcal{H}$ invariant measurable set Γ_2 with $\mu\Gamma_2=1$, $\forall\mu\in\mathcal{M}(M,\alpha)$ such that for all $x\in\Gamma_2$ there is a decomposition of the tangent space into $T_x=\bigoplus_{i=1}^{r(x,f)}E_i(x,f)$ satisfying that for every $0\neq u\in E_i(x,f)$,*

(i) *the spectrum $\{\lambda_i(x,f),\ m_i(x,f),\ i=1,\dots,r(x,f)\}$ is $\mathcal{H}$ invariant;*
(ii) *$D(f^sg^th^r)_xE_i(x,f)=E_i(f^sg^th^rx,f)$, $\forall i=1,\dots,r(x,f)$, $\forall\ s,t,r\in\mathbb{Z}$;*
(iii) *$\lim_{n\to\infty}\frac{1}{n}\log\|D(f^sh^r)^n_xu\|=s\lambda_i(x,f)$, $\forall\ s,r\in\mathbb{Z}$.*

Proposition 2.5: *There is an $\mathcal{H}$ invariant measurable set $\Gamma_3\subset\Gamma_1\cap\Gamma_2$ with $\mu\Gamma_3=1$ $\forall\mu\in\mathcal{M}(M,\alpha)$ such that for all $x\in\Gamma_3$ there is a decomposition of the tangent space into*

$$T_xM=\bigoplus_{i=1}^{r(x,f)}\bigoplus_{j=1}^{r(x,g)}E_{ij}(x)$$

satisfying that if $E_{ij}(x)\neq0$, then for all $0\neq u\in E_{ij}(x)$ and all $s,t,r\in\mathbb{Z}$,

(i) $$\lim_{n\to\infty}\frac{1}{n}\log\|D(f^sh^r)^n_xu\|=s\lambda_i(x,f),$$
$$\lim_{n\to\infty}\frac{1}{n}\log\|D(g^th^r)^n_xu\|=t\lambda_j(x,g);$$
(ii) *$D(f^sg^th^r)_xE_{ij}(x)=E_{ij}(f^sg^th^rx)$;*
(iii) *$\lambda_i(f^sg^th^rx,f)=\lambda_i(x,f)$, $\lambda_j(f^sg^th^rx,g)=\lambda_j(x,g)$.*

Proof. For all $x \in \Gamma_1 \cap \Gamma_2$, let $T_xM = \bigoplus_{j=1}^{r(x,g)} E_j(x,g)$ be the decomposition given in Proposition 2.3. By Proposition 2.3(ii),

$$Df_x(E_j(x,g)) = E_j(fx,g).$$

Restricted to $\{E_j(x,g)\}$, $\{Df_x^n\}$ is a cocycle on M with respect to f. Thus, similar to the proof of Proposition 2.3 in [Hu], we obtain an $\mathcal{H}$ invariant measurable set $\Gamma_3 \subset \Gamma_1 \cap \Gamma_2$ with $\mu\Gamma_3 = 1$ $\forall \mu \in \mathcal{M}(M,\alpha)$, and a decomposition of the tangent space into

$$T_xM = \bigoplus_{i=1}^{r(x,f)} \bigoplus_{j=1}^{r(x,g)} E_{ij}(x), \quad x \in \Gamma_3.$$

Clearly $E_{ij}(x) = E_i(x,f) \cap E_j(x,g)$. Thus (i) (ii) and (iii) are direct corollaries of Proposition 2.3 and Proposition 2.4. ∎

Proof of Theorem A. For any $s,t,r \in \mathbb{Z}$ and any $\varepsilon > 0$, set

$$A_\varepsilon^+ = \{x : \exists 0 \neq u \in E_{ij}(x) \text{ s.t. } \chi(x,u,f^sg^th^r) - s\lambda_i(x,f) - t\lambda_j(x,g) > (|\lambda|+1)\varepsilon\},$$
$$A_\varepsilon^- = \{x : \exists 0 \neq u \in E_{ij}(x) \text{ s.t. } \chi(x,u,f^sg^th^r) - s\lambda_i(x,f) - t\lambda_j(x,g) < (|\lambda|+1)\varepsilon\},$$

where $\lambda = 6s + 6t + |2r - st|$. We need to prove that for all $\mu \in \mathcal{M}(M,\alpha)$ and for all $\varepsilon > 0$, $\mu(A_\varepsilon^\pm) = 0$.

Now we prove $\mu(A_\varepsilon^+) = 0$; the other one can be obtained similarly.

Assume to the contrary that $\mu(A_\varepsilon^+) > 0$ for some $\mu \in \mathcal{M}(M,\alpha)$ and some $\varepsilon > 0$. Then there exists a sufficiently large constant $C > 0$ such that the set

$$\begin{aligned} A_{\varepsilon,C} := \{x :\exists\, 0 \neq u \in E_{ij}(x) \\ \text{s.t. } \|D(f^sg^th^r)_x^n u\| > C^{-1}\|u\| \exp n(s\lambda_i(x,f) \\ + t\lambda_j(x,g) + |\lambda|\varepsilon)\,\forall n \geq 0\} \end{aligned} \tag{2.4}$$

satisfies $\mu(A_{\varepsilon,C}) > 0$. Let

$$\delta = \mu(A_{\varepsilon,C}).$$

By (1.2) we have

$$\|D(f^sg^th^r)_x^n u\| = \|Dh^{st\frac{n(n-1)}{2}} Df^{sn} Dg^{tn} Dh_x^{rn} u\|.$$

Then

$$\|D(f^sg^th^r)_x^{2n} u\| = \|Dh^{2stn^2} Df^{2sn} Dg^{2tn} Dh_x^{(2r-st)n} u\| \quad \forall n \geq 0.$$

For $l>0$, let

$$\begin{aligned} A_f^l = \{x : & l^{-1}\|u\| \exp n(\lambda_i(x,f)-\varepsilon) \\ & \le \|Df_x^n u\| \le l\|u\|\exp n(\lambda_i(x,f)+\varepsilon)\ \forall u\in E_{ij}(x)\ \forall n\ge 0\}; \\ A_g^l = \{x : & l^{-1}\|u\| \exp n(\lambda_j(x,g)-\varepsilon) \\ & \le \|Dg_x^n u\| \le l\|u\|\exp n(\lambda_j(x,g)+\varepsilon)\ \forall u\in E_{ij}(x)\ \forall n\ge 0\}; \\ A_h^l = \{x : & l^{-1}\|u\|\exp(-|n|\varepsilon) \le \|Dh_x^n u\| \le l\|u\|\exp(|n|\varepsilon)\ \forall u\in E_{ij}(x)\ \forall n\in\mathbb{Z}\}. \end{aligned}$$

Choose l sufficiently large so that

$$\mu(A_i^l) > 1-\frac{1}{26}\delta, \quad i=f,g,h.$$

Let

$$B_n = A_g^l \cap g^{-tn}(A_f^l)\cap A_h^l \cap h^{-stn^2} f^{-sn}(A_g^l)\cap h^{-stn^2}(A_f^l).$$

Then $\mu(B_n) > 1-\frac{5}{26}\delta$, and for all $x\in B_n$ and all $0\ne u\in E_{ij}(x)$, we have

$$\begin{aligned} \|D(f^{sn}g^{tn})_x u\| \le & l\|Dg_x^{tn}u\|\exp sn(\lambda_i(g^{tn}x,f)+\varepsilon) \\ \le & l^2\|u\|\exp[tn\lambda_j(x,g)+sn\lambda_i(x,f)+(s+t)n\varepsilon] \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} & \|D(g^{tn}f^{sn}h^{stn^2})_x u\| \\ & \quad \ge l^{-1}\|D(f^{sn}h^{stn^2})_x u\|\exp(tn\lambda_j(x,g)-tn\varepsilon) \\ & \quad \ge l^{-2}\|Dh^{stn^2}u\|\exp[sn\lambda_i(x,f)+tn\lambda_j(x,g)-(t+s)n\varepsilon]. \end{aligned} \tag{2.6}$$

Since $f^{sn}g^{tn} = g^{tn}f^{sn}h^{stn^2}$, it follows from (2.5) and (2.6) that

$$\|Dh^{stn^2}u\| \le l^4\|u\|\exp[2(t+s)n\varepsilon] \quad \forall n\ge 0,\ \forall x\in B_n,\ \forall 0\ne u\in E_{ij}(x).$$

Let

$$C_n = h^{-stn^2}(B_n)\cap B_n.$$

Then $\mu(C_n) > 1-\frac{10}{26}\delta$. For all $x\in C_n$ and all $0\ne u\in E_{ij}(x)$, we have

$$\begin{aligned} \|Dh_x^{2stn^2}u\| = & \|Dh^{stn^2}Dh_x^{stn^2}u\| \\ \le & l^4\|Dh_x^{stn^2}u\|e^{2(s+t)n\varepsilon} \\ \le & l^8\|u\|e^{4(s+t)n\varepsilon}. \end{aligned}$$

Let

$$D_n = h^{-(2r-st)n}g^{-2tn}f^{-2sn}(C_n)\cap h^{-(2r-st)n}g^{-2tn}(A_f^l)\cap h^{-(2r-st)n}(A_g^l)\cap A_h^l.$$

Then

$$\mu(D_n) > 1 - \frac{10}{26}\delta - \frac{3}{26}\delta = 1 - \frac{\delta}{2} > 1 - \delta,$$

and so

$$\mu(D_n \cap A_{\varepsilon,C}) > 0.$$

For any $x \in D_n \cap A_{\varepsilon,C}$ and any $0 \neq u \in E_{ij}(x)$, we have

$$\begin{aligned} &\|D(f^s g^t h^r)_x^{2n} u\| \\ &=\|Dh^{2stn^2} Df^{2sn} Dg^{2tn} Dh_x^{(2r-st)n} u\| \\ &\leq l^8 e^{4(s+t)n\varepsilon} l e^{2sn\lambda_i(x,f)+2sn\varepsilon} l e^{2tn\lambda_j(x,g)+2tn\varepsilon} l e^{|2r-st|n\varepsilon} \|u\| \\ &=l^{11}\|u\| \exp n[2s\lambda_i(x,f) + 2t\lambda_j(x,g) + \lambda\varepsilon]), \end{aligned} \tag{2.7}$$

where $\lambda = 6s + 6t + |2r - st|$. In addition, from the definition of $A_{\varepsilon,C}$ in (2.4) we get that for all $x \in A_{\varepsilon,C}$, there is $0 \neq u \in E_{ij}(x)$ such that

$$\begin{aligned} &\|D(f^s g^t h^r)_x^{2n} u\| \\ &>C^{-1}\|u\| \exp 2n[s\lambda_i(x,f) + t\lambda_j(x,g) + |\lambda|\varepsilon] \\ &=C^{-1} \exp(n|\lambda|\varepsilon)\|u\| \exp n[2s\lambda_i(x,f) + 2t\lambda_j(x,g) + |\lambda|\varepsilon]. \end{aligned} \tag{2.8}$$

Clearly, (2.7) and (2.8) contradict each other if $n > (\log(l^{11}C))/|\lambda|\varepsilon$. Hence we must have $\mu(A_\varepsilon^+) = 0$ for all $\mu \in \mathcal{M}(M,\alpha)$ and all $\varepsilon > 0$. This completes the proof of Theorem A. ∎

Proof of Corollary A.1. By taking $s = t = 0$ and $r = 1$ in (1.3) we know that the Lyapunov exponent of any $0 \neq u \in E_{ij}(x)$ is equal to 0 with respect to h, and so is that of any $0 \neq u \in T_xM$.

Let p be a periodic point of h of period n, that is, $h^n(p) = p$. Since $fh = hf$ and $gh = hg$, we get that both $f^n(p)$ and $g^n(p)$ are periodic orbits of h with period n. Since there are only finitely many periodic points of h of period n, we get that

$$\{f^s g^t h^r(p) : s,t,r \in \mathbb{Z}\}$$

is a finite set. Hence we can define a measure $\mu \in \mathcal{M}(M,\alpha)$ supported on the set. By finiteness and invariance we know that $\mu(\{p\}) > 0$, i.e., p is a generic point of μ. The fact that all Lyapunov exponents of h at p equal zero gives that the modulus of all eigenvalues of $Dh^n(p)$ are equal to one. ∎

Proof of Corollary A.2. Assume that $h_{\rm top}(h) > 0$. Then there is some h invariant probability measure ν on M such that the metric entropy $h_\nu(h) > 0$ by the variational principle. Consider the probability measure sequence

$$\nu_n \equiv \frac{1}{4n^2} \sum_{|i|,|j|\le n} (f^i g^j)_* \nu.$$

Passing to a subsequence if necessary, suppose ν_n converges to a probability measure μ in the weak-$*$ topology. Then μ is α invariant. Hence by Corollary A.1 all Lyapunov exponents of h are zero with respect to μ. Hence $h_\mu(h) = 0$ by Ledrappier–Young's formula.

On the other hand, since h commutes with every element in $\alpha(\mathcal{H})$, we have $h_{(f^i g^j)_*\nu}(h) = h_\nu(h)$ for all $i, j \in \mathbb{Z}$. Since the entropy map $\nu \to h_\nu(h)$ is affine, we have $h_{\nu_n}(h) = h_\nu(h)$. As the action α is C^∞, it follows from [NP] that $0 = h_\mu(h) \ge \lim_{n\to\infty} h_{\nu_n}(h) = h_\nu(h) > 0$. This is a contradiction. ∎

3. Faithfulness: Proof of Theorem B and its corollaries

Let T be a diffeomorphism on a manifold M with a hyperbolic set Λ. For any $x \in \Lambda$, the **stable manifold** of x for T is defined by

$$W^s(x,T) = \{y \in M : d(T^n x, T^n y) \to 0 \text{ as } n \to \infty\},$$

which is T-invariant. For any $\varepsilon > 0$, the **local stable manifold** $W^s_\varepsilon(x,T)$ is the set

$$\{y \in M : d(T^n x, T^n y) \le \varepsilon \text{ for all } n \ge 0\}.$$

It is well known that $W^s_\varepsilon(x,T) \subset W^s(x,T)$ and

$$W^s(x,T) = \bigcup_{n\ge 0} T^{-n} W^s_\varepsilon(T^n x, T).$$

Lemma 3.1: *Suppose p is a common fixed point of f, g and h, and f is Anosov and has simple eigenvalues on stable direction with $\lambda_- > \lambda_+^2$ at p. Then either $h = \mathrm{id}$ or $h^2 = \mathrm{id}$ on $W^s(p,f)$.*

Proof. Note that the eigenvalue of Dh_p restricted to each stable eigenspace is ± 1 by Corollary A.1. We may assume it is 1, otherwise use h^2 instead of h. Since f has simple eigenvalues on stable direction, and h commutes with f, we must have $Dh_p|_{E^s(p,f)} = \mathrm{id}$, where

$$E^s(p,f) = \{v \in T_p(M) : \|Df_p(v)\| < \|v\|\}.$$

Denote $r' = \min\{r, 2\}$. Take $\varepsilon > 0$ small enough such that $\lambda_- - \varepsilon > (\lambda_+ + \varepsilon)^{r'}$ and such that for any $x, y \in W^s_\varepsilon(p, f)$ and $n \in \mathbb{N}$,

$$C_1(\lambda_- - \varepsilon)^n d(x, y) < d(f^n(x), f^n(y)) < C_2(\lambda_+ + \varepsilon)^n d(x, y) \tag{3.1}$$

for some fixed constants $C_1, C_2 > 0$. It is clear that $hW^s(p, f) = W^s(p, f)$ by $hf = fh$. So there is $\varepsilon' \leq \varepsilon$ such that $hW^s_{\varepsilon'}(p, f) \subset W^s_\varepsilon(p, f)$.

Suppose $h(x) \neq x$ for some $x \in W^s_{\varepsilon'}(p, f)$. Let $x_n = f^n(x)$. Then by (3.1) we have

$$\frac{d(x_n, h(x_n))}{d(x_n, p)} = \frac{d(f^n(x), f^n(h(x)))}{d(f^n(x), p)} \geq \frac{C_1}{C_2} \frac{(\lambda_- - \varepsilon)^n d(x, h(x))}{(\lambda_+ + \varepsilon)^n d(x, p)}. \tag{3.2}$$

Note that $W^s_{\varepsilon'}(p, f)$ is a C^r submanifold tangent to $E^s(p, f)$ at p. Take a local coordinate system on $W^s(p)$ at p. We have

$$\begin{aligned} h(x_n) - p &= \int_0^1 Dh_{p+t(x_n-p)}(x_n - p)dt \\ &= \left(\mathrm{id} + \left(\int_0^1 Dh_{p+t(x_n-p)}dt - \mathrm{id}\right)\right)(x_n - p). \end{aligned}$$

Since h is a C^r diffeomorphism and $Dh_p|_{E^s(p,f)} = \mathrm{id}$, the equation gives

$$\left|\int_0^1 Dh_{p+t(x_n-p)}dt - \mathrm{id}\right| \leq C_3|x_n - p|^{r'-1}$$

for some $C_3 > 0$. Hence we get

$$|h(x_n) - x_n| \leq C_3|x_n - p|^{r'}.$$

Note that $|h(x_n) - x_n| = d(x_n, h(x_n))$ and $|x_n - p| = d(x_n, p)$. So by (3.1)

$$\frac{d(x_n, h(x_n))}{d(x_n, p)} \leq C_3 d(x_n, p)^{r'-1} \leq C_3C_2(\lambda_+ + \varepsilon)^{n(r'-1)}d(x, p)^{r'-1}$$

for all $n > 0$, contradicting (3.2) and the fact $\lambda_- - \varepsilon > (\lambda_+ + \varepsilon)^{r'}$.

Then we must have $h(x) = x$ for any $x \in W^s_{\varepsilon'}(p, f)$, and then $h = \mathrm{id}$ on $W^s(p, f)$ by using the facts $W^s(p, f) = \bigcup_{n\geq 0} f^{-n}W^s_{\varepsilon'}(p, f)$ and $fh = hf$. ∎

Lemma 3.2: *Suppose p is a periodic point of f with period n and f has only finitely many periodic points of period n. Then there are $m, k \in \mathbb{N}$ such that p is a common fixed point of f^n, g^m and h^k.*

In particular, if p is the unique fixed point of f, then $h(p) = p = g(p)$.

Proof. Since $fh = hf$, $h(p)$ is a periodic point of f with period n. By the finiteness of the n-periodic point set of f, there is some k such that $h^k(p) = p$. Then we have

$$f^n g^k(p) = g^k f^n h^{kn}(p) = g^k(p),$$

that is, $g^k(p)$ is also a periodic point of f with period n. This implies that $g^{kl}(p) = g^k(p)$ for some $l \in \mathbb{N}$. Taking $m = lk - k$, we then complete the proof.

The second part of the lemma is now obvious. ∎

Proof of Theoren B. Without loss of generality, we may suppose f is an Anosov element that has simple eigenvalues on stable direction with $\lambda_- > \lambda_+^{\min\{r,2\}}$. By spectral decomposition, f has basic sets $\Omega_1, \ldots, \Omega_t$ (see [Bo]). On each basic set Ω_i, we take a periodic point $p_i \in \Omega_i$. Assume $f^{n_i}(p_i) = p_i$ for some $n_i \in \mathbb{N}$. Then there are m_i and k_i such that p_i is a common fixed point of f^{n_i}, g^{m_i} and h^{k_i} by Lemma 3.2. Applying Lemma 3.1 to $f^{n_i}, g^{m_i k_i}$ and $h^{n_i m_i k_i}$, we get $h^{2n_i m_i k_i} = \mathrm{id}$ on $W^s(p_i, f^{n_i})$. Since $M = \bigcup_{i=1}^t \overline{W^s(p_i, f^{n_i})}$, we get $h^{2k} = \mathrm{id}$, where

$$k = \prod_{i=1}^{t} n_i m_i k_i. \qquad ∎$$

Proof of Corollary B.2. Since f has only one fixed point, we have

$$f(p) = g(p) = h(p) = p$$

by Lemma 3.2. Since $\dim E_p^s(f) = 1$, restricted to $E_p^s(f)$, Df_p and Dg_p commute. Hence, $Df_p \cdot Dg_p = Dg_p \cdot Df_p \cdot Dh_p$ implies $Dh_p|_{E_p^s(f)} = \mathrm{id}$. By the proof of Lemma 3.1, we have that h is identity on $W^s(p, f)$. From [Ne], we know that f is transitive. So $W^s(p, f)$ is dense in M, and h is identity on M. ∎

4. Affine Anosov action on tori: Proof of Theorems C and D

Before the proof of Theorem C, let us recall two classical results.

Theorem 4.1 (Adler–Palais [AP]): *If $R, S \in \mathrm{Aff}(\mathbb{T}^n)$ with R being ergodic, then any homeomorphism Φ of $\mathbb{T}^n$ with $\Phi R = S\Phi$ is in $\mathrm{Aff}(\mathbb{T}^n)$.*

Theorem 4.2 (Franks–Manning [Fr, Ma]): *Any Anosov diffeomorphism of $\mathbb{T}^n$ is topologically conjugate to a hyperbolic toral automorphism.*

Proof of Theoren C. Suppose $k=f^rg^sh^t$ is Anosov for some $r,s,t\in\mathbb{Z}$. Then by Theorem 4.2, there is a homeomorphism Φ of $\mathbb{T}^n$ such that $K=\Phi^{-1}k\Phi\in\mathrm{Aff}(\mathbb{T}^n)$. Since K is topologically transitive and affine, K is ergodic.

Denote $F=\Phi^{-1}f\Phi$, $G=\Phi^{-1}g\Phi$ and $H=\Phi^{-1}h\Phi$. Then we have $FH=HF$, $GH=HG$, $FG=GFH$, and $K=F^rG^sH^t$.

Since $H^{-1}KH=K$ and $K\in\mathrm{Aff}(\mathbb{T}^n)$, $H\in\mathrm{Aff}(\mathbb{T}^n)$ by Theorem 4.1. Similarly, since $F^{-1}KF=KH^{-s}$ and $K,KH^{-s}\in\mathrm{Aff}(\mathbb{T}^n)$, $F\in\mathrm{Aff}(\mathbb{T}^n)$; and since $G^{-1}KG=KH^r$ and $K,KH^r\in\mathrm{Aff}(\mathbb{T}^n)$, $G\in\mathrm{Aff}(\mathbb{T}^n)$. ∎

Lemma 4.3: *Let $A,B,\mathcal{C}\in\mathrm{GL}(n,\mathbb{R})$ such that $AB=BA\mathcal{C}, A\mathcal{C}=\mathcal{C}A$ and $B\mathcal{C}=\mathcal{C}B$. Suppose A is hyperbolic with stable linear space $E^s\subset\mathbb{R}^n$. If the modular of each eigenvalue of $\mathcal{C}$ is equal to 1, then E^s is B and $\mathcal{C}$ invariant.*

Proof. For any $v\in E^s$, we have

$$\lim_{n\to\infty}A^n\mathcal{C}v=\lim_{n\to\infty}\mathcal{C}A^nv=0$$

by $A\mathcal{C}=\mathcal{C}A$, so E^s is $\mathcal{C}$ invariant. Since the modular of each eigenvalue of $\mathcal{C}$ is equal to 1, the increasing rate of matrix norm $\|\mathcal{C}^n\|$ is bounded by a polynomial in n by an easy calculation. Thus we have

$$\lim_{n\to\infty}A^nBv=\lim_{n\to\infty}BA^n\mathcal{C}^nv=\lim_{n\to\infty}B\mathcal{C}^nA^nv=0$$

by $A\mathcal{C}=\mathcal{C}A$ and $AB=BA\mathcal{C}$. So E^s is B invariant. ∎

Lemma 4.4: *Let $A,B,\mathcal{C}\in\mathrm{GL}(1,\mathbb{R})$ such that $AB=BA\mathcal{C}$. Then $\mathcal{C}=\mathrm{Id}$.*

Proof. Since $\mathrm{GL}(1,\mathbb{R})$ is commutative, we have $AB=BA\mathcal{C}=AB\mathcal{C}$, which means that $\mathcal{C}$ is identity. ∎

Lemma 4.5: *Let $A,B,\mathcal{C}\in\mathrm{GL}(2,\mathbb{R})$ such that $AB=BA\mathcal{C}, A\mathcal{C}=\mathcal{C}A$ and $B\mathcal{C}=\mathcal{C}B$. If the modular of each eigenvalue of $\mathcal{C}$ is equal to 1, then $\mathcal{C}^2=\mathrm{Id}$.*

Proof. Consider $A,B,\mathcal{C}$ as matrices in $\mathrm{GL}(2,\mathbb{C})$.

Claim 1. The eigenvalues of $\mathcal{C}$ are 1 or -1. In fact, assume that $\mathcal{C}$ has an eigenvalue λ with $\lambda^n\neq 1$ for $n=1,2$. By $A\mathcal{C}=\mathcal{C}A$, we can take a nonzero vector $v\in\mathbb{C}^2$ such that $\mathcal{C}v=\lambda v$ and $Av=\gamma v$ for some $\gamma\neq 0$. Then we have

$$ABv=BA\mathcal{C}v=\lambda\gamma Bv\quad\text{and}\quad AB^2v=B^2A\mathcal{C}^2v=\lambda^2\gamma Bv.$$

So v,Bv and B^2v are three eigenvectors of A with pairwise different eigenvalues, which is a contradiction. Hence $\lambda=1$ or $\lambda^2=1$, which means $\lambda=1$ or -1.

Claim 2. If -1 is an eigenvalue of $\mathcal{C}$, then $\mathcal{C}^2 = \mathrm{id}$. In fact, we can take $v \in \mathbb{C}^2$ such that $\mathcal{C}v = -v$ and v, Bv are two eigenvectors of A with different eigenvalues as shown in Claim 1. So, under the base $\{v, Bv\}$, $\mathcal{C}$ has the form $\mathcal{C} = \left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$, and $\mathcal{C}^2 = \mathrm{Id}$.

Claim 3. If all eigenvalues of $\mathcal{C}$ are 1, then $\mathcal{C} = \mathrm{id}$. In fact, if A has two different eigenvalues, then $\mathcal{C}$ is diagonal by $A\mathcal{C} = \mathcal{C}A$. So $\mathcal{C}$ can only be identity in this case. Similarly, if B has only simple eigenvalues, then $\mathcal{C} = \mathrm{id}$. Thus we may suppose $A, B, \mathcal{C}$ have only eigenvalues $\lambda, \gamma, 1$, respectively. If A is diagonal, then A and B are commutative, and then $\mathcal{C}$ is identity by $AB = BA\mathcal{C}$. So we may suppose the eigenspace V_λ of A corresponding to λ is of dimension 1. Fix a nonzero vector $v \in V_\lambda$. Then $\mathcal{C}V_\lambda = V_\lambda$ by $A\mathcal{C} = \mathcal{C}A$. Therefore $ABv = BA\mathcal{C}v = \lambda Bv$ and we get $BV_\lambda = V_\lambda$. So $Bv = \gamma v$. Take a vector w linearly independent of v. Then, under the base $\{v, w\}$, $A, B, \mathcal{C}$ have the forms

$$A = \begin{bmatrix} \lambda & x \\ 0 & \lambda \end{bmatrix}, \quad B = \begin{bmatrix} \gamma & y \\ 0 & \gamma \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix},$$

for some $x, y, z \in \mathbb{C}$, which implies $\mathcal{C} = \mathrm{Id}$ by $AB = BA\mathcal{C}$. ∎

Proof of Theorem D. From Theorem B, there is a homeomorphism Φ on $\mathbb{T}^n$ such that

$$\Phi^{-1} f \Phi([x]) = [Ax + a], \quad \Phi^{-1} g \Phi([x]) = [Bx + b] \quad \text{and} \quad \Phi^{-1} h \Phi([x]) = [\mathcal{C}x + c]$$

for any $x \in \mathbb{R}^n$, where $A, B, \mathcal{C} \in \mathrm{GL}(n, \mathbb{Z})$ and $a, b, c \in \mathbb{R}^n$. It is directly checked that $AB = BA\mathcal{C}$, $A\mathcal{C} = \mathcal{C}A$ and $B\mathcal{C} = \mathcal{C}B$. Clearly A is hyperbolic. Let $E^s \subset \mathbb{R}^n$ be the stable linear subspace of A. We assume $\dim(E^s) = 1$ or 2.

By Lemma 4.3, E^s is A, B and $\mathcal{C}$ invariant. Since $0 \in \mathbb{T}^n$ is a common fixed point of A, B and $\mathcal{C}$, we get that the modular of each eigenvalue of $\mathcal{C}$ is 1 by Corollary A.1. Applying Lemmas 4.4 and 4.5 to $A|_{E^s}$, $B|_{E^s}$ and $\mathcal{C}|_{E^s}$, we know that $\mathcal{C}|_{E^s} = \mathrm{Id}$ if $\dim(E^s) = 1$ and $\mathcal{C}|_{E^s} = \pm\mathrm{Id}$ if $\dim(E^s) = 2$. It follows that $\mathcal{C}$ or $-\mathcal{C}$, as automorphism of $\mathbb{T}^n$, is identity on $\mathbb{T}^n$ by the density of $[E^s]$ in $\mathbb{T}^n$. Hence $\mathcal{C}$ or $-\mathcal{C}$ is identity as matrix in $\mathrm{GL}(n, \mathbb{Z})$. Thus

$$\Phi^{-1} h \Phi([x]) = [\pm x + c].$$

So h is conjugate to either a translation or an affine transformation $T_{-\mathrm{Id}, c}$ for some $c \in \mathbb{T}^n$, and the formal case occurs if f is a codimensional 1 Anosov diffeomorphism.

Clearly, if h is conjugate to $T_{-\mathrm{Id},c}$, then $h^2 = \mathrm{id}$. If h is conjugate to a translation $T_{\mathrm{Id},c}$, then we can get $[x + kc] = \Phi^{-1}h^k\Phi([x]) = [x]$ for some $k > 0$ and for any $[x] \in \mathbb{T}^n$ by using the fact that h sends a fixed point of f to a fixed point of f, and f has only a finite number of fixed points. ∎

Remark 4.6: From the proof we see that the integer k can be chosen as a factor of the number of the fixed points of f.

5. Smooth Rigidity: Proof of Theorem E

5.1. Setting up the problem and the KAM scheme. Before proceeding to specifics we will show how the general KAM scheme described in [DK, Section 3.3] and [DK1, Section 1.1] is adapted to the $\mathcal{H}$ action α.

Step 1. Setting up the linearized equation.

Let $\widetilde{\alpha}$ be a small perturbation of α. To prove the existence of a C^∞ map H such that $\widetilde{\alpha} \circ H = H \circ \alpha$, we need to solve the nonlinear conjugacy problem

$$\alpha \circ \Omega - \Omega \circ \alpha = -R \circ (I + \Omega)$$

where the lift of $\widetilde{\alpha}$ is $\alpha + R$ and the lift of H is $I + \Omega$; and the corresponding linearized conjugacy equation is

$$\alpha \circ \Omega - \Omega \circ \alpha = -R \tag{5.1}$$

for small Ω and R.

Lemma 5.1 shows that obtaining a C^∞ conjugacy for one ergodic generator suffices for the proof of Theorem E. Hence we just need to solve equation (5.1) for one ergodic generator.

Step 2. Solving the linearized conjugacy equation for a particular element.

We classify the obstructions for solving the linearized equation (5.1) for an individual generator (see Lemmas 5.5 and 5.6) and obtain tame estimates for the solution. This means a finite loss of regularity in the chosen collection of norms in the Fréchet spaces, such as C^r or Sobolev norms.

Step 3. Constructing projection of the perturbation to the twisted cocycle space.

First note that R is a twisted cocycle not over α but over $\widetilde{\alpha}$ (see Lemma 3.3 of [DK]), thus (5.1) is not a twisted coboundary equation over the linear action α, just an approximation. Second, note that even if (5.1) is a twisted coboundary over α, it is impossible to produce a C^∞ conjugacy for a single ergodic generator of the action. Therefore, we consider three generators, and reduce the problem of solving the linearized equation (5.1) to solving simultaneously the following system:

$$\begin{aligned} A\circ\Omega-\Omega\circ A&=-R_A,\\ B\circ\Omega-\Omega\circ B&=-R_B,\\ \mathcal{C}\circ\Omega-\Omega\circ\mathcal{C}&=-R_{\mathcal{C}}, \end{aligned} \tag{5.2}$$

where A and B are ergodic generators and $\mathcal{C}$ is the center:

$$A:=\alpha(g_1),\qquad B:=\alpha(g_2),\qquad \mathcal{C}:=\alpha(g_3)$$

and

$$R_A:=R(g_1),\quad R_B:=R(g_2),\quad R_{\mathcal{C}}:=R(g_3).$$

As mentioned above, R does not satisfy this twisted cocycle condition:

$$\begin{aligned} L(R_A,R_{\mathcal{C}})&\overset{\text{def}}{=}\mathcal{C}R_A-R_A\circ\mathcal{C}-(AR_{\mathcal{C}}-R_{\mathcal{C}}\circ A)=0,\\ L(R_B,R_{\mathcal{C}})&\overset{\text{def}}{=}\mathcal{C}R_B-R_B\circ\mathcal{C}-(BR_{\mathcal{C}}-R_{\mathcal{C}}\circ B)=0,\\ L(R_A,R_B)&\overset{\text{def}}{=}R_A\circ B+AR_B-R_B\circ A\mathcal{C}-BR_{\mathcal{C}}\circ A-B\mathcal{C}R_A=0. \end{aligned} \tag{5.3}$$

However, the difference

$$L(R_A,R_B),\quad L(R_B,R_{\mathcal{C}})\quad\text{and}\quad L(R_A,R_B)$$

is quadratically small with respect to R (see Lemma 5.8). More precisely, the perturbation R can be split into two terms,

$$R=\mathcal{P}R+\mathcal{E}(R),$$

so that $\mathcal{P}R$ is in the space of twisted cocycles and the error $\mathcal{E}(R)$ is bounded by the size of L with the fixed loss of regularity (see Lemma 5.7). More precisely, the system

$$\begin{aligned} -\mathcal{P}R_A&=-(R_A-\mathcal{E}(R_A))=A\Omega-\Omega\circ A,\\ -\mathcal{P}R_B&=-(R_B-\mathcal{E}(R_B))=B\Omega-\Omega\circ B,\\ -\mathcal{P}R_{\mathcal{C}}&=-(R_{\mathcal{C}}-\mathcal{E}(R_{\mathcal{C}}))=\mathcal{C}\Omega-\Omega\circ\mathcal{C} \end{aligned} \tag{5.4}$$

has a common solution Ω after subtracting a part quadratically small to R.

Step 4. Conjugacy transforms the perturbed action into an action quadratically close to the target.

The common approximate solution Ω to equations (5.2) above provides a new perturbation

$$\widetilde{\alpha}^{(1)} \stackrel{\text{def}}{=} H^{-1} \circ \widetilde{\alpha} \circ H,$$

where

$$H = I + \Omega,$$

much closer to α than $\widetilde{\alpha}$; i.e., the new error

$$R^{(1)} \stackrel{\text{def}}{=} \widetilde{\alpha}^{(1)} - \alpha$$

is expected to be small with respect to the old error R.

Step 5. The process is iterated and the conjugacy is obtained.

The iteration process is set and is carried out, producing a C^∞ conjugacy which works for the action generated by the three generators A, B and $\mathcal{C}$. Ergodicity assures that it works for all the other elements of the action α.

What is described above highlights the essential features of the KAM scheme for the $\mathcal{H}$ action on the torus. The last two steps can follow Sections 5.2–5.4 in [DK] word by word without modification. Hence completeness of Steps 2 and 3 admits the conclusion of Theorem E.

At the end of this section, we prove a simple lemma which shows that obtaining a C^∞ conjugacy for one ergodic generator suffices for the proof of Theorem E.

Lemma 5.1: *Let α be a Heisenberg group $\mathcal{H}$ action by automorphisms of $\mathbb{T}^N$ such that for some $a \in \mathcal{H}$ the automorphism $\alpha(a)$ is ergodic. Let $\widetilde{\alpha}$ be a C^1 small perturbation of α such that there exists a C^∞ map $H : \mathbb{T}^N \to \mathbb{T}^N$ which is C^1 close to identity and satisfies*

$$\widetilde{\alpha}(a) \circ H = H \circ \alpha(a).$$

Then H conjugates the corresponding maps for all the other elements of the action; i.e., for all $b \in \mathcal{H}$ we have

$$\widetilde{\alpha}(b) \circ H = H \circ \alpha(b). \tag{5.5}$$

Proof. Let b be any element in $\mathcal{H}$ other than a. If $ba = ab$, it follows from (5.5) and commutativity that

$$\alpha(a) \circ \tilde{b} = \tilde{b} \circ \alpha(a)$$

where $\tilde{b} = \alpha(b) \circ H^{-1} \circ \widetilde{\alpha}(b)^{-1} \circ H$.

If $ba = abc$, where c is in the center of $\mathcal{H}$, then similarly we obtain

$$\begin{aligned}\alpha(a)\circ(\alpha(b)\circ H^{-1}\circ\widetilde{\alpha}(b)^{-1}\circ H) &= \alpha(b)\circ(\alpha(ac^{-1})\circ H^{-1})\circ\widetilde{\alpha}(b)^{-1}\circ H\\ &\overset{(1)}{=} \alpha(b)\circ H^{-1}(\widetilde{\alpha}(ac^{-1})\circ\widetilde{\alpha}(b)^{-1})\circ H\\ &= \alpha(b)\circ H^{-1}\widetilde{\alpha}(b^{-1})\circ(\widetilde{\alpha}(a)\circ H)\\ &= (\alpha(b)\circ H^{-1}\widetilde{\alpha}(b^{-1})\circ H)\circ\alpha(a).\end{aligned}$$

Here (1) comes from the fact that H also conjugates $\alpha(c)$ and $\widetilde{\alpha}(c)$ which is from previous analysis.

Then the conclusion follows immediately from the following fact (see Lemma 3.2 of [DK]): for any C^1 small enough map $F : \mathbb{T}^N \to \mathbb{T}^N$, if $AF = F\circ A$, where $A \in GL(N,\mathbb{Z})$ and is ergodic, then $F = 0$. ∎

5.2. Some notation and basic facts.

(1) A result of Kronecker [Kr] states that an integer matrix with all eigenvalues on the unit circle has to have all eigenvalues roots of unity. Then there exists $n \in \mathbb{N}$ such that all eigenvalues of $\mathcal{C}^n$ are 1. Using the relation $AB = BA\mathcal{C}$, we obtain $AB^n = B^nA\mathcal{C}^n$. Hence we can assume that all eigenvalues of $\mathcal{C}$ are 1, otherwise we just turn to A, B^n and $\mathcal{C}^n$ instead of A, B and $\mathcal{C}$.

(2) We denote the dual map of A on $\mathbb{Z}^N$ by A^*, which induces a decomposition of $\mathbb{R}^N$ into expanding, neutral and contracting subspaces. We denote these subspaces by $V_1(A)$ (the expanding subspace), $V_2(A)$ (the neutral subspace) and $V_3(A)$ (the contracting subspace), respectively. Then we have the splitting

$$\mathbb{R}^N = V_1(A)\bigoplus V_2(A)\bigoplus V_3(A).$$

It is clear that all three subspaces $V_i(A)$, $i = 1, 2, 3$ are A invariant and

$$\begin{aligned}&\|A^iv\| \ge C\rho^i\|v\|, &&\rho>1,\quad i\ge 0, &&v\in V_1(A),\\ &\|A^iv\| \ge C\rho^{-i}\|v\|, &&\rho>1,\quad i\le 0, &&v\in V_3(A),.\\ &\|A^iv\| \ge C|i|^{-N}\|v\|, &&\rho>1,\quad i\ne 0, &&v\in V_2(A)\end{aligned}\tag{5.6}$$

(3) For $v \in \mathbb{Z}^N$,

$$|v| \overset{\text{def}}{=} \max\{\|\pi_1(v)\|, \|\pi_2(v)\|, \|\pi_3(v)\|\}$$

where $\|\cdot\|$ is the Euclidean norm and $\pi_i(v)$ are projections of v to subspaces V_i ($i = 1, 2, 3$) from (5.6), that is, to the expanding, neutral, and contracting subspaces of $\mathbb{R}^N$ for A (or B), respectively. It is clear that the norm $|\cdot|$ and the Euclidean norm are equivalent.

(4) For $v \in \mathbb{Z}^N$ we say v is mostly in $i(A)$ for $i = 1, 2, 3$ and write $v \hookrightarrow i(A)$ if the projection $\pi_i(v)$ of v to the subspace V_i corresponding to A is sufficiently large:

$$|v| = \|\pi_i(v)\|.$$

The notation $v \hookrightarrow 1, 2(A)$ will be used for v which is mostly in $1(A)$ or mostly in $2(A)$.

(5) Call $n \in \mathbb{Z}^N$ minimal if n is the lowest point on its A orbit in the sense that $n \hookrightarrow 3(A)$ and $An \hookrightarrow 1, 2(A)$. There is one such minimal point on each nontrivial dual A orbit; we choose one on each dual A orbit and denote it by $n_{\min}$. Then $n_{\min}$ is substantially large both in $1, 2(A)$ and in $3(A)$.

(6) In what follows, C will denote any constant that depends only on the given linear action α with chosen generators A, B and $\mathcal{C}$ and on the dimension of the torus; $C_{x,y,z,\dots}$ will denote any constant that in addition to the above dependence also depends on parameters $x, y, z, \dots$.

(7) Let θ be a C^∞ function. We then can write

$$\theta = \sum_{n \in \mathbb{Z}^N} \widehat{\theta}_n e_n$$

where $e_n = e^{2\pi i n \cdot x}$ are the characters. Then:

(i) $\|\theta\|_a \stackrel{\text{def}}{=} \sup_n |\widehat{\theta}_n| |n|^a$, $a > 0$.

(ii) The following relations hold (see, for example, Section 3.1 of [Ll]):

$$\|\theta\|_r \leq C \|\theta\|_{C^r}, \quad \|\theta\|_{C^r} \leq C \|\theta\|_{r+\sigma},$$

where $\sigma > N + 1$, and $r \in \mathbb{N}$.

(iii) For any $F \in SL(N, \mathbb{Z})$, $(\widehat{\theta \circ F})_n = \widehat{\theta}_{(F^\tau)^{-1} n}$ where F^τ denotes the transpose of F. We call $(F^\tau)^{-1}$ the dual map on $\mathbb{Z}^N$. To simplify the notation in the rest of the paper, whenever there is no confusion as to which map we refer to, we will denote the dual map by the same symbol F.

(8) For a map $\mathcal{F}$ with coordinate functions f_i $(i = 1, \dots, k)$ define

$$\|\mathcal{F}\|_a \stackrel{\text{def}}{=} \max_{1 \leq i \leq k} \|f_i\|_a.$$

For two maps $\mathcal{F}$ and $\mathcal{G}$ define

$$\|\mathcal{F},\mathcal{G}\|_a \stackrel{\text{def}}{=} \{\|\mathcal{F}\|_a, \|\mathcal{G}\|_a\};$$

$\|\mathcal{F}\|_{C^r}$ and $\|\mathcal{F},\mathcal{G}\|_{C^r}$ are defined similarly. For any $n \in \mathbb{Z}^N$,

$$\widehat{\mathcal{F}}_n \stackrel{\text{def}}{=} ((\widehat{f}_1)_n, \ldots, (\widehat{f}_k)_n). \tag{5.7}$$

5.3. Orbit growth for the dual action. In this section the crucial estimates for the exponential growth along individual orbits of the dual action are obtained. The following follows directly from the proof of Lemma 4.3 in [DK]:

Lemma 5.2: *Let Q_i be ergodic matrices in $\mathcal{SL}(N,\mathbb{Z})$, $1 \le i \le m$. Suppose there exist constants C, $\tau > 0$ such that for every nonzero vector $n \in \mathbb{Z}^N$ and for any $k = (k_1, \ldots, k_m) \in \mathbb{Z}^m$*

$$\|Q_1^{k_1} \cdots Q_m^{k_m} n\| \ge C \exp\{\tau \|k\|\} \|n\|^{-N}. \tag{5.8}$$

Then the following are satisfied.

(a) *For any C^∞ function φ on the torus $\mathbb{T}^N$ and any $y \in \mathbb{C}$, the following sums*

$$S_K(\varphi, n, y, Q) \stackrel{\text{def}}{=} \sum_{k=(k_1,\ldots,k_m)\in K} y^{\|k\|} \widehat{\varphi}_{Q_1^{k_1}\cdots Q_m^{k_m} n}$$

converge absolutely for any $K \subset \mathbb{Z}^m$, where Q stands for $Q_1, \ldots, Q_m$.

(b) *Assume, in addition to the assumptions in (a), that for a vector $n \in \mathbb{Z}^N$ and for every $k = (k_1, \ldots, k_m) \in K = K(n) \subset \mathbb{Z}^m$ we have*

$$p_1(\|k\|)\|Q_1^{k_1} \cdots Q_m^{k_m} n\| \ge \|n\|, \tag{5.9}$$

where p_1 is a polynomial. Then

$$|S_K(\varphi, n, y, Q)| \le C_{a,y,\delta} \|\varphi\|_a \|n\|^{-a+\kappa_y+\delta}$$

for any $a > \kappa_{y,Q} \stackrel{\text{def}}{=} \frac{N+1}{\tau} |\log|y||$.

(c) *If the assumption (5.9) is also satisfied for every $n \in \mathbb{Z}^N$, then the function*

$$S(\varphi, y, Q) \stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}^N} S_{K(n)}(\varphi, n, y, Q) e_n$$

is a C^∞ function if φ is. Moreover, the following norm comparison holds:

$$\|S(\varphi, y, Q)\|_{C^r} \le C_{r,y} \|\varphi\|_{C^{r+\sigma}}$$

for any $r \geq 0$ *and* $\sigma > N + 2 + [\kappa_{y,Q}]$.

Corollary 5.3: *Suppose* Q_i, $P_i \in \mathcal{SL}(N, \mathbb{Z})$, $1 \leq i \leq m$, *and suppose*

$$K = K(n) \subseteq \mathbb{Z}^m.$$

If condition (5.8) is satisfied for any $n \in \mathbb{Z}^m$ *and any* $k \in K(n)$, *then for any* C^∞ *map* φ *defined on the torus* $\mathbb{T}^N$ *we obtain:*

(1) *The following sums*

$$S_K(\varphi, n, P; Q) \stackrel{\mathrm{def}}{=} \sum_{k=(k_1,\ldots,k_m)\in K(n)} P_1^{k_m} \cdots P_m^{k_1} \widehat{\varphi}_{Q_1^{k_1}\cdots Q_m^{k_m} n}$$

converge absolutely (we recall (5.7) for the definition of $\widehat{\varphi}_n$, $n \in \mathbb{Z}^N$*), where* P *stands for* $P_1, \ldots, P_m$ *and* Q *stands for* $Q_1, \ldots, Q_m$.

(2) *Assume in addition that for a vector* $n \in \mathbb{Z}^N$ *and for every* $k = (k_1, \ldots, k_m) \in K = K(n) \subset \mathbb{Z}^m$ *we have*

$$p_1(\|k\|)\|Q_1^{k_1} \cdots Q_m^{k_m} n\| \geq \|n\|,$$

where p_1 *is a polynomial. Then*

$$|S_K(\varphi, n, P, Q)| \leq C_{a,P,\delta}\|\varphi\|_a \|n\|^{-a+\kappa_1+\delta}$$

for any $a > \kappa_{P,Q} \stackrel{\mathrm{def}}{=} \frac{N+1}{\tau}|\log\|P\||$, *where*

$$\|P\| = \max\{\|P_i\| : 1 \leq i \leq m\}.$$

(3) *If the assumption (5.9) is satisfied for every* $n \in \mathbb{Z}^N$, *then the function*

$$S(\varphi, P, Q) \stackrel{\mathrm{def}}{=} \sum_{n\in\mathbb{Z}^N} S_{K(n)}(\varphi, n, P, Q) e_n$$

is a C^∞ *function if* φ *is. Moreover, the following norm comparison holds:*

$$\|S(\varphi, P, Q)\|_{C^r} \leq C_{r,P}\|\varphi\|_{C^{r+\sigma}}$$

for any $r \geq 0$ *and* $\sigma > N + 2 + [\kappa_{P,Q}]$.

Proof. Since

$$\sum_{k=(k_1,\ldots,k_m)\in K(n)} \|P_1^{k_m} \cdots P_m^{k_1} \widehat{\varphi}_{Q_1^{k_1}\cdots Q_m^{k_m} n}\| \leq \sum_{k=(k_1,\ldots,k_m)\in K(n)} \|P\|^{\|k\|} \|\widehat{\varphi}_{Q_1^{k_1}\cdots Q_m^{k_m} n}\|,$$

we get the conclusion immediately from the above lemma. ∎

In the subsequent part we prove the exponential growth along individual orbits of ergodic elements. It may be viewed as a generalization of Lemma 4.3 in [DK] to higher rank non-abelian actions by toral automorphisms.

LEMMA 5.4: *There exist constants $C, \tau > 0$ such that for every nonzero vector $v \in \mathbb{Z}^N$ and for any $k = (k_1, k_2) \in \mathbb{Z}^2 \backslash \{0\}$,*

$$\|A^{k_1} B^{k_2} v\| \geq C \exp\{\tau(|k_1| + |k_2|)\} \|v\|^{-N}.$$

Proof. From the Lyapunov space decomposition in Theorem A, we see that the proof of Lemma 4.3 in [DK] also applies to this case word by word. At first, we can show that there exists $\tau > 0$ such that, for any $k = (k_1, k_2) \in \mathbb{Z}^2 \backslash \{0\}$, there exists a Lyapunov space in which the Lyapunov exponent of $A^{k_1} B^{k_2}$ is greater than $\tau(|k_1| + |k_2|)$. Note that $A^{k_1} B^{k_2}$ is ergodic, which implies that the projection of v to this space is greater than $\gamma \|v\|^{-N}$, where γ is a constant only dependent on the decomposition in Theorem A. Then we get the conclusion. ∎

5.4. TWISTED COBOUNDARY EQUATION OVER A MAP ON THE TORUS. Obstructions to solving a one-cohomology equation for a function over an ergodic toral automorphism in C^∞ category are sums of Fourier coefficients of the given function along a dual orbit of the automorphism. This is the content of the Lemma 4.2 in [DK]. The same characterization holds however for a one-cohomology equation for a map over ergodic toral automorphisms as well due to the estimate in Corollary 5.3. The proofs of the two lemmas below follow closely the proof of Lemma 4.2 in [DK] for solving a one-cohomology equation for functions.

LEMMA 5.5: *Let P and Q be matrices in $\mathcal{SL}(N, \mathbb{Z})$ and Q be ergodic. For a map θ defined on $\mathbb{T}^N$, if there exists a C^∞ map ω which is C^0 small enough on $\mathbb{T}^N$ such that*

$$P\omega - \omega \circ Q = \theta, \tag{5.10}$$

then the following sums along all nonzero dual orbits are zero, i.e.,

$$\sum_{i=-\infty}^{\infty} P^{-(i+1)} \hat{\theta}_{Q^i n} = 0 \quad \forall n \neq 0.$$

Proof. Since ω is C^0 small enough, equation (5.10) in the dual space has the form

$$P\widehat{\omega}_n - \widehat{\omega}_{Qn} = \widehat{\theta}_n \quad \forall n \in \mathbb{Z}^N. \tag{5.11}$$

Replacing n by $Q^i n$ and applying $P^{-(i+1)}$ to the equation, we get that for any m, $\ell > 0$,

$$\sum_{i=-m}^{\ell} P^{-i}\widehat{\omega}_{Q^i n} - \sum_{i=-m}^{\ell} P^{-(i+1)}\widehat{\omega}_{Q^{i+1}n} = \sum_{i=-m}^{\ell} P^{-(i+1)}\widehat{\theta}_{Q^i n},$$

which simplifies to

$$P^m\widehat{\omega}_{Q^{-m}n} - P^{-(\ell+1)}\widehat{\omega}_{Q^{\ell+1}n} = \sum_{i=-m}^{\ell} P^{-(i+1)}\widehat{\theta}_{Q^i n}.$$

Then the conclusion follows immediately if

$$\lim_{m\to\infty} P^m\widehat{\theta}_{Q^{-m}n} = 0 \quad \forall n \neq 0,$$

which is a direct consequence of part (1) of Corollary 5.3 with $K = K(n) = \mathbb{Z}$ for any $n \neq 0$. ∎

Note that $\kappa_{P^{-1},Q}$ is defined in part (2) of Corollary 5.3.

Lemma 5.6: *Let P and Q be ergodic matrices in $\mathcal{SL}(N,\mathbb{Z})$. Let θ be a C^∞ map on the torus which is C^σ small enough, where $\sigma > N + 2 + \kappa_{P^{-1},Q}$. If for all nonzero $n \in \mathbb{Z}$ the following sums along the dual orbits are zero, i.e.,*

$$\sum_{i=-\infty}^{\infty} P^{-(i+1)}\widehat{\theta}_{Q^i n} = 0 \quad \forall n \neq 0, \tag{5.12}$$

then the equation

$$P\omega - \omega \circ Q = \theta \tag{5.13}$$

has a C^∞ solution ω, and the estimate

$$\|\omega\|_{C^r} \leq C_r\|\theta\|_{C^{r+\sigma}} \tag{5.14}$$

holds for any $r \geq 0$, where C_r is a number only dependent on Lyapunov exponents of P.

Proof. Suppose ω is a C^∞ solution C^0 small enough to (5.13). Then equation (5.13) in the dual space has the form

$$P\widehat{\omega}_n - \widehat{\omega}_{Qn} = \widehat{\theta}_n \quad \forall n \in \mathbb{Z}^N. \tag{5.15}$$

For $n = 0$, since P is ergodic, we can immediately calculate

$$\widehat{\omega}_0 = (P - I)^{-1}\widehat{\theta}_0.$$

For $n \neq 0$ the dual equation has two solutions,

$$\widehat{\omega}_n^{\pm} = \pm \sum_{\substack{i \geq 0 \\ i \leq -1}} P^{-(i+1)}\widehat{\theta}_{Q^i n}, \quad n \neq 0.$$

Each sum converges absolutely by part (1) of Corollary 5.3. By assumption (5.12), $\widehat{\omega}_n^+ = \widehat{\omega}_n^- \stackrel{\text{def}}{=} \widehat{\omega}_n$. This gives a formal solution

$$\omega = \sum \widehat{\omega}_n^+ e_n = \sum \widehat{\omega}_n^- e_n.$$

We estimate each $\widehat{\omega}_n$ using both of the forms in order to show that ω is C^∞. In the notation of Corollary 5.3 we can write

$$\widehat{\omega}_n^+ = S_{K^+}(P^{-1}\theta, n, P^{-1}, Q) \quad \text{and} \quad \widehat{\omega}_n^- = -S_{K^-}(P^{-1}\theta, n, P^{-1}, Q).$$

Here

$$K^+ = \{i \in \mathbb{Z} : i \geq 0\}$$

and

$$K^- = \{i \in \mathbb{Z} : i \leq -1\}.$$

If n is mostly contracting, i.e., if $n \hookrightarrow 3(A)$, then

$$\|A^i n\| \geq C\rho^{-i}\|n\| \quad \forall i \leq -1. \tag{5.16}$$

If n is not mostly contracting, i.e., if $n \hookrightarrow 1, 2(A)$, then

$$\|A^i n\| \geq Ci^{-N}\|n\| \quad \forall i \geq 0. \tag{5.17}$$

Thus the polynomial estimate needed for the application of part (2) of Corollary 5.3 is satisfied in either K^+ or K^- for any $n \in \mathbb{Z}^N$. This estimate implies that (5.14) holds. Finally, this also implies that smallness of C^σ norm of θ guarantees C^0 smallness of ω. ∎

5.5. Construction of the projection.

Lemma 5.7: *Fix $\sigma = N+3+\kappa_{A^{-1},A}$. There exists $\delta > 0$ such that for any C^∞ maps θ, ψ, ω on $\mathbb{T}^N$ that are C^σ small enough, it is possible to split θ, ψ and ω as*

$$\theta = \Delta_A\Omega + \mathcal{R}\theta, \quad \psi = \Delta_B\Omega + \mathcal{R}\psi,$$
$$\omega = \Delta_{\mathcal{C}}\Omega + \mathcal{R}\omega$$

for a C^∞ map Ω, so that

$$\|\mathcal{R}\theta, \mathcal{R}\psi, \mathcal{R}\omega\|_{C^r} \leq C_r\|R_1, R_2, R_3\|_{C^{r+\delta}}$$

and

$$\|\Omega\|_{C^r} \leq C_r\|\theta, \omega, \psi\|_{C^{r+\sigma}}$$

for any $r \geq 0$, where

$$R_1 \stackrel{\text{def}}{=} \Delta_{\mathcal{C}}\theta - \Delta_A\omega, \tag{5.18}$$

$$R_2 \stackrel{\text{def}}{=} \Delta_{\mathcal{C}}\psi - \Delta_B\omega \tag{5.19}$$

and

$$R_3 \stackrel{\text{def}}{=} \theta\circ B + A\psi - \psi\circ A\mathcal{C} - B\omega\circ A - B\mathcal{C}\theta. \tag{5.20}$$

Proof. (1) Construction of Ω and $\mathcal{R}\theta$.

Let $\mathcal{R}\theta = \sum_n \widehat{\mathcal{R}\theta}_n e_n$ where

$$\widehat{\mathcal{R}\theta}_n \stackrel{\text{def}}{=} \begin{cases} \sum_{i\in\mathbb{Z}} A^{-i}\widehat{\theta}_{A^in}, & n = n_{\min}, \\ 0, & \text{otherwise}, \end{cases}$$

for $n \neq 0$ and $\widehat{\mathcal{R}\theta}_0 \stackrel{\text{def}}{=} 0$.

Note that $n_{\min}$ is substantially large both in the expanding and in the contracting direction for A; then both (5.16) and (5.17) hold if $n = n_{\min}$. The following estimate is obtained from (3) of Corollary 5.3:

$$\|\mathcal{R}\theta\|_{C^r} \leq C_r\|\theta\|_{C^{r+\sigma}}, \quad \forall r \geq 0. \tag{5.21}$$

Since $\theta - \mathcal{R}\theta$ satisfies the solvable condition in Lemma 5.6, by using Lemma 5.6 there is a C^∞ function Ω such that

$$\Delta_A\Omega = \theta - \mathcal{R}\theta \tag{5.22}$$

with estimates

$$\|\Omega\|_{C^r} \le C_r\|\theta - \mathcal{R}\theta\|_{C^{r+\sigma}} \le C_r\|\theta\|_{C^{r+2\sigma}}, \quad \forall r \ge 0.$$

(2) Estimates for $\mathcal{R}\theta$.

Rewrite (5.20) to get

$$A\psi - \psi \circ A\mathcal{C} = B\omega \circ A + B\mathcal{C}\theta - \theta \circ B + R_3.$$

Lemma 5.5 shows that the obstructions for $B\omega \circ A + B\mathcal{C}\theta - \theta \circ B + R_3$ with respect to $A\mathcal{C}$ vanish; therefore for any $n \neq 0$ we get

$$\begin{aligned}&\sum_i A^{-(i+1)}\widehat{\theta}_{B(A\mathcal{C})^i n}\\ &\quad = \sum_i A^{-(i+1)}B\mathcal{C}\widehat{\theta}_{(A\mathcal{C})^i n} + \sum_i A^{-(i+1)}B\widehat{\omega}_{A(A\mathcal{C})^i n} + \sum_i A^{-(i+1)}(\widehat{R_3})_{(A\mathcal{C})^i n}\end{aligned}$$

since all the sums involved converge absolutely by (1) of Corollary 5.3. Furthermore, by using the relation

$$B(A\mathcal{C})^i = A^i B, \quad \forall i \in \mathbb{Z}, \tag{5.23}$$

we obtain from the above relation

$$\begin{aligned}&\sum_i A^{-(i+1)}\widehat{\theta}_{A^iBn} - \sum_i BA^{-(i+1)}\widehat{\theta}_{A^in}\\ &\quad = \Big(\sum_i A^{-(i+1)}B\mathcal{C}\widehat{\theta}_{(A\mathcal{C})^i n} - \sum_i BA^{-(i+1)}\widehat{\theta}_{A^i n}\Big)\\ &\qquad + \sum_i A^{-(i+1)}B\widehat{\omega}_{A(A\mathcal{C})^i n} + \sum_i A^{-(i+1)}(\widehat{R_3})_{(A\mathcal{C})^i n}.\end{aligned} \tag{5.24}$$

Next, we compute the sum $\sum_i A^{-(i+1)}B\mathcal{C}\widehat{\theta}_{(A\mathcal{C})^i n} - \sum_i BA^{-(i+1)}\widehat{\theta}_{A^i n}$. To do so, we split it into two sums, $\sum_i = \sum_{i\ge 0} + \sum_{i \le -1}$, and then use relation (5.18) to simplify each one. Set

$$\Lambda = \Delta_A \omega.$$

Then for any $n \neq 0$, we obtain from the proof of Lemma 5.6

$$\sum_{i\ge 1} A^{-(i+1)}\widehat{\Lambda}_{A^i n} = \widehat{\omega}_n - A^{-1}\widehat{\Lambda}_n = A^{-1}\widehat{\omega}_{An} \tag{5.25}$$

and

$$-\sum_{i\le -1} A^{-(i+1)}\widehat{\Lambda}_{A^i n} = \widehat{\omega}_n. \tag{5.26}$$

Using relation (5.18) we get

$$\text{(5.27)}\quad \widehat{\theta}_{A^in}-\mathcal{C}^{-i}\widehat{\theta}_{\mathcal{C}^iA^in}=\begin{cases}\sum_{0\le j\le i-1}\mathcal{C}^{-(j+1)}(\widehat{\Lambda}_{\mathcal{C}^jA^in}-(\widehat{R_1})_{\mathcal{C}^jA^in}), & i\ge 1,\\ -\sum_{i\le j\le -1}\mathcal{C}^{-(j+1)}(\widehat{\Lambda}_{\mathcal{C}^jA^in}-(\widehat{R_1})_{\mathcal{C}^jA^in}), & i\le -1.\end{cases}$$

We note that in the two right-side sums the middle terms satisfy the condition $|j|\le|i|$.

Using (5.27) for the case of $i\ge 1$ we obtain

$$\begin{aligned}\sum_{i\ge 0}&A^{-(i+1)}B\mathcal{C}\widehat{\theta}_{(A\mathcal{C})^in}-\sum_{i\ge 0}BA^{-(i+1)}\widehat{\theta}_{A^in}\\ &=\sum_{i\ge 1}A^{-(i+1)}B\mathcal{C}\widehat{\theta}_{(A\mathcal{C})^in}-\sum_{i\ge 1}BA^{-(i+1)}\widehat{\theta}_{A^in}\\ &\overset{(1)}{=}\sum_{i\ge 1}BA^{-(i+1)}(\mathcal{C}^{-i}\widehat{\theta}_{(A\mathcal{C})^in}-\widehat{\theta}_{A^in})\\ &=-\sum_{i\ge 1}\sum_{j=0}^{i-1}BA^{-(i+1)}\mathcal{C}^{-(j+1)}(\widehat{\Lambda}_{\mathcal{C}^jA^in}-(\widehat{R_1})_{\mathcal{C}^jA^in})\\ &=-\sum_{j\ge 0}\sum_{i\ge j+1}BA^{-(i+1)}\mathcal{C}^{-(j+1)}(\widehat{\Lambda}_{\mathcal{C}^jA^in}-(\widehat{R_1})_{\mathcal{C}^jA^in}).\end{aligned}$$

Here (1) is from relation (5.23). Of course, to justify the change of order of summation in the last equality, we must prove the absolute convergence of the sum. Using the notation in Corollary 5.3 we can write

$$\begin{aligned}\sum_{i\ge 0}&A^{-(i+1)}B\mathcal{C}\widehat{\theta}_{(A\mathcal{C})^in}-\sum_{i\ge 0}BA^{-(i+1)}\widehat{\theta}_{A^in}\\ &=BA\mathcal{C}S_K((\Lambda-R_1),n,\{A,\mathcal{C}\},\{\mathcal{C},A\}),\end{aligned}$$

where $K=\{(j,i)\in\mathbb{Z}^2: i-1\ge j\ge 0\}$. For any i, j with $|j|\le|i|$, (5.8) in Lemma 5.2 shows that

$$\begin{aligned}|\mathcal{C}^jA^in|&\ge C|j|^{-N}|A^in|\ge C_1|i|^{-N}\exp(\tau_A|i|)|n|^{-N}\\ &\ge C_2\exp\{\tau_A(|i|+|j|)/4\}|n|^{-N},\end{aligned}$$

where C, C_1 and C_2 are fixed numbers only dependent on A and $\mathcal{C}$; and τ is given in Lemma 5.4. This justifies applying part (1) of Corollary 5.3 to show the absolute convergence of the sum.

Furthermore, we have

$$\sum_{j\geq 0}\sum_{i\geq j+1} B\mathcal{C}^{-(j+1)}A^{-(i+1)}\widehat{\Lambda}_{\mathcal{C}^j A^i n} = \sum_{j\geq 0} B\mathcal{C}^{-(j+1)}A^{-j}\Big(\sum_{k\geq 1}A^{-(k+1)}\widehat{\Lambda}_{(A\mathcal{C})^j A^k n}\Big)$$
$$\overset{(1)}{=} \sum_{j\geq 0} B\mathcal{C}^{-(j+1)}A^{-(j+1)}\widehat{\omega}_{A(A\mathcal{C})^j n}$$
$$\overset{(2)}{=} \sum_{j\geq 0} A^{-(j+1)}B\widehat{\omega}_{A(A\mathcal{C})^j n}.$$

Here (1) follows from (5.25) and (2) uses relation (5.23) again.

Hence we obtain

$$\begin{aligned}&\sum_{i\geq 0}A^{-(i+1)}B\mathcal{C}\widehat{\theta}_{(A\mathcal{C})^i n} - \sum_{i\geq 0} BA^{-(i+1)}\widehat{\theta}_{A^i n}\\ &= -\sum_{j\geq 0} A^{-(j+1)}B\widehat{\omega}_{A(A\mathcal{C})^j n} + \sum_{j\geq 0}\sum_{i\geq j+1} BA^{-(i+1)}\mathcal{C}^{-(j+1)}(\widehat{R_1})_{\mathcal{C}^j A^i n}.\end{aligned} \tag{5.28}$$

To compute the sum $\sum_{i\leq -1}$ we use (5.27) for the case of $i\leq -1$:

$$\sum_{i\leq -1} A^{-(i+1)}B\mathcal{C}\widehat{\theta}_{(A\mathcal{C})^i n} - \sum_{i\leq -1} BA^{-(i+1)}\widehat{\theta}_{A^i n}$$
$$= \sum_{i\leq -1} BA^{-(i+1)}(\mathcal{C}^{-i}\widehat{\theta}_{(A\mathcal{C})^i n} - \widehat{\theta}_{A^i n})$$
$$= \sum_{i\leq -1}\sum_{j=i}^{-1} BA^{-(i+1)}\mathcal{C}^{-(j+1)}(\widehat{\Lambda}_{\mathcal{C}^j A^i n} - (\widehat{R_1})_{\mathcal{C}^j A^i n})$$
$$= \sum_{j\leq -1}\sum_{i\leq j} BA^{-(i+1)}\mathcal{C}^{-(j+1)}(\widehat{\Lambda}_{\mathcal{C}^j A^i n} - (\widehat{R_1})_{\mathcal{C}^j A^i n}).$$

Again we need to show the absolute convergence. We can also write

$$\sum_{i\leq -1} A^{-(i+1)}B\mathcal{C}\widehat{\theta}_{(A\mathcal{C})^i n} - \sum_{i\leq -1} BA^{-(i+1)}\widehat{\theta}_{A^i n}$$
$$= BA\mathcal{C}S_{K'}((\Lambda - R_1), n, \{A,\mathcal{C}\}, \{\mathcal{C}, A\}),$$

where $K' = \{(j,i)\in\mathbb{Z}^2 : i\leq j\leq -1\}$. Then (5.28) shows that the absolute convergence follows from the same reasoning as in the previous part.

Furthermore, by using (5.26) and relation (5.23) again we obtain

$$\begin{aligned}\sum_{j\le -1}\sum_{i\le j} B\mathcal{C}^{-(j+1)}A^{-(i+1)}\widehat{\Lambda}_{\mathcal{C}^jA^in} &= \sum_{j\le -1} B\mathcal{C}^{-(j+1)}A^{-(j+1)}\Big(\sum_{k\le -1}A^{-(k+1)}\widehat{\Lambda}_{A(A\mathcal{C})^jA^kn}\Big)\\ &= -\sum_{j\le -1} B\mathcal{C}^{-(j+1)}A^{-(j+1)}\widehat{\omega}_{A(A\mathcal{C})^jn}\\ &= -\sum_{j\le -1} A^{-(j+1)}B\widehat{\omega}_{A(A\mathcal{C})^jn}.\end{aligned}$$

Hence we obtain

$$\begin{aligned}&\sum_{i\le -1}A^{-(i+1)}B\mathcal{C}\widehat{\theta}_{(A\mathcal{C})^in}-\sum_{i\le -1}BA^{-(i+1)}\widehat{\theta}_{A^in}\\ &\quad=-\sum_{j\le -1}A^{-(j+1)}B\widehat{\omega}_{A(A\mathcal{C})^jn}-\sum_{j\le -1}\sum_{i\le j}BA^{-(i+1)}\mathcal{C}^{-(j+1)}(\widehat{R_1})_{\mathcal{C}^jA^in}.\end{aligned}\tag{5.29}$$

By using (5.24), (5.28) and (5.29) for any $n\neq 0$ we obtain

$$\begin{aligned}&\sum_i A^{-(i+1)}\widehat{\theta}_{A^iBn}-\sum_i BA^{-(i+1)}\widehat{\theta}_{A^in}\\ &\quad=\sum_{j\ge 0}\sum_{i\ge j+1}BA^{-(i+1)}\mathcal{C}^{-(j+1)}(\widehat{R_1})_{\mathcal{C}^jA^in}\\ &\qquad-\sum_{j\le -1}\sum_{i\le j}BA^{-(i+1)}\mathcal{C}^{-(j+1)}(\widehat{R_1})_{\mathcal{C}^jA^in}+\sum_i A^{-(i+1)}(\widehat{R_3})_{(A\mathcal{C})^in}.\end{aligned}$$

Iterating this equation with respect to B we obtain

$$\begin{aligned}&\sum_i A^{-(i+1)}\widehat{\theta}_{A^in}-\lim_{\ell\to\infty}\sum_i B^{-\ell}A^{-(i+1)}\widehat{\theta}_{A^iB^\ell n}\\ &\quad=-\sum_{k\ge 0}\sum_{j\ge 0}\sum_{i\ge j+1}B^{-k}A^{-(i+1)}\mathcal{C}^{-(j+1)}(\widehat{R_1})_{\mathcal{C}^jA^iB^kn}\\ &\qquad+\sum_{k\ge 0}\sum_{j\le -1}\sum_{i\le j}B^{-k}A^{-(i+1)}\mathcal{C}^{-(j+1)}(\widehat{R_1})_{\mathcal{C}^jA^iB^kn}\\ &\qquad-\sum_{k\ge 0}\sum_i B^{-(k+1)}A^{-(i+1)}(\widehat{R_3})_{(A\mathcal{C})^iB^kn}.\end{aligned}$$

Condition (5.8) in Corollary 5.3 is satisfied by Lemma 5.4. Hence the limit above is 0 from part (1) of Corollary 5.3; and the absolute convergence of the sum involving $(A\mathcal{C})^iB^k$ is justified by the same reasoning. To show the absolute convergence of the other two sums involving $\mathcal{C}^j A^i B^k$ where $|j| \le |i|$, by the same reason the following inequality is sufficient:

$$\begin{aligned}\|\mathcal{C}^j A^i B^k n\| &\ge C|j|^{-N}\|A^i B^k n\| \ge C|i|^{-N}\exp\{\tau_{A,B}(|i|+|k|)\}\|n\|^{-N}\\ &\ge C\exp\{\frac{1}{2}\tau_{A,B}(|i|+|k|)\}\|n\|^{-N}\\ &\ge C\exp\{\frac{1}{4}\tau_{A,B}(|i|+|j|+|k|)\}\|n\|^{-N}.\end{aligned}$$

Hence by the notation of Corollary 5.3 we obtain

$$\begin{aligned}\sum_i A^{-(i+1)}\widehat{\theta}_{A^i n} =& - S_{K_1}((A\mathcal{C})^{-1}R_1, n, \{B^{-1}, A^{-1}, \mathcal{C}^{-1}\}, \{\mathcal{C}, A, B\})\\ &+ S_{K_2}((A\mathcal{C})^{-1}R_1, n, \{B^{-1}, A^{-1}, \mathcal{C}^{-1}\}, \{\mathcal{C}, A, B\})\\ &- B^{-1}S_{K_3}(A^{-1}R_3, n, \{B^{-1}, A^{-1}\}, \{A\mathcal{C}, B\}),\end{aligned}$$

where

$$\begin{aligned}K_1 =&\{(k_1,k_2,k_3)\in\mathbb{Z}^3 : k_1\ge 0,\ k_2\ge k_1+1,\ k_3\ge 0\},\\ K_2 =&\{(k_1,k_2,k_3)\in\mathbb{Z}^3 : k_1\le -1,\ k_2\le k_1,\ k_3\ge 0\}\end{aligned}$$

and

$$K_3 =\{(k_1,k_2)\in\mathbb{Z}^2 : k_2\ge 0\}.$$

By iterating backwards and applying the same reasoning, we obtain

$$\begin{aligned}\sum_i A^{-(i+1)}\widehat{\theta}_{A^i n} =& S_{K_1'}((A\mathcal{C})^{-1}R_1, n, \{B^{-1}, A^{-1}, \mathcal{C}^{-1}\}, \{\mathcal{C}, A, B\})\\ &- S_{K_2'}((A\mathcal{C})^{-1}R_1, n, \{B^{-1}, A^{-1}, \mathcal{C}^{-1}\}, \{\mathcal{C}, A, B\})\\ &+ B^{-1}S_{K_3'}(A^{-1}R_3, n, \{B^{-1}, A^{-1}\}, \{A\mathcal{C}, B\}),\end{aligned}$$

where

$$\begin{aligned}K_1' =&\{(k_1,k_2,k_3)\in\mathbb{Z}^3 : k_1\ge 0,\ k_2\ge k_1+1,\ k_3\le -1\},\\ K_2' =&\{(k_1,k_2,k_3)\in\mathbb{Z}^3 : k_1\le -1,\ k_2\le k_1,\ k_3\le -1\}\end{aligned}$$

and

$$K_3' =\{(k_1,k_2)\in\mathbb{Z}^2 : k_2\le -1\}.$$

Then according to part (3) of Corollary 5.3, the needed estimate for $\mathcal{R}\theta$ with respect to R follows if, in at least one of the union of half-spaces $K^+ = \bigcup_{i=1}^3 K_i$ and $K^- = \bigcup_{i=1}^3 K_i'$, the dual action satisfies some polynomial lower bound for every $n = n_{\min}$.

Write A and B in block diagonal forms as stated in the proof of Corollary 5.3. In case $An \hookrightarrow 2(A)$, let n_1 be the largest projection of n to some neutral block J' for A. Then

$$\|n_1\| \geq C\|n\|.$$

Let the Lyapunov exponent of B on this block be ν. For all j, i, k the Lyapunov exponent of $\mathcal{C}^j A^i B^k$ on J' is $k\nu$. Then if $\nu \geq 0$, on the half-space K^+ we obtain

$$\|\mathcal{C}^j A^i B^k n\| \geq C|j|^{-N}|i|^{-N}|k|^{-N}\|n_1\| \geq C_1|ijk|^{-N}\|n\| \quad \forall k \geq 0$$

and

$$\|\mathcal{C}^j A^i B^k n\| \geq C|j|^{-N}|i|^{-N}\exp(k\nu/2)\|n_1\| \geq C_1|ij|^{-N}\|n\|$$

on K^- for $k < 0$ if $\nu < 0$. Thus the polynomial estimate needed for the application of part (3) of Corollary 5.3 is satisfied for such n.

In case $An \hookrightarrow 1(A)$, let n_1 and n_2 be the largest projections of n to some blocks J_1 and J_2 with positive Lyapunov exponent λ_1 and negative Lyapunov exponent λ_2, respectively. Let ν_1 and ν_2 be corresponding Lyapunov exponents of B on the two blocks. Then

$$\|n_1\| \geq C\|n\|, \quad \|n_2\| \geq C\|n\|.$$

For all j, i, k the Lyapunov exponent of $\mathcal{C}^j A^i B^k$ on J_1 is $\chi(j,i,k)^+ = i\lambda_1 + k\nu_1$ and is $\chi(j,i,k)^- = i\lambda_2 + k\nu_2$ on J_2. Next, we need to show that

$$\{(j,i,k) : \chi(j,i,k)^+ \geq 0\} \cup \{(j,i,k) : \chi(j,i,k)^- \geq 0\} \tag{5.30}$$

covers either K^+ or K^-. This boils down to requiring $k(\frac{\nu_1}{\lambda_1} - \frac{\nu_2}{\lambda_2}) \geq 0$. Namely, for any $(j, i) \in \mathbb{Z}^2$, (j, i, k) belongs to the union in (5.30) if $k(\frac{\nu_1}{\lambda_1} - \frac{\nu_2}{\lambda_2}) \geq 0$ and this is true for $k \geq 0$ or for $k \leq 0$ depending on the sign of $\frac{\nu_1}{\lambda_1} - \frac{\nu_2}{\lambda_2}$. Therefore we obtain

$$\|\mathcal{C}^j A^i B^k n\| \geq C|ijk|^{-N}\|n\| \tag{5.31}$$

in K^+ or in K^-.

Now choose the half-space in which the estimate (5.31) holds, that is, choose one of the sums $-S_{K_1} + S_{K_2} - S_{K_3}$ or $S_{K_1'} - S_{K_2'} + S_{K_3'}$. Then the assumptions of (3) of Corollary 5.3 are satisfied for one of the sums above, and therefore the estimate for

$$\|\mathcal{R}\theta\|_{C^r} \leq C_r\|R_1, R_2, R_3\|_{r+\sigma_1} \tag{5.32}$$

for any $r \geq 0$ and $\sigma_1 > N + 2 + [\kappa_{A,B,\mathcal{C}}]$, where $\kappa_{A,B,\mathcal{C}} = \frac{N+1}{\tau}|\log\|A, B, A\mathcal{C}\||$.

(3) Estimates for $\mathcal{R}\omega$.

By (5.18) and (5.22) we obtain

$$\Delta_A(\omega - \Delta_{\mathcal{C}}\Omega) = \Delta_{\mathcal{C}}\mathcal{R}\theta - R_1. \tag{5.33}$$

Define

$$\mathcal{R}\omega \overset{\text{def}}{=} \omega - \Delta_{\mathcal{C}}\Omega \quad \text{and} \quad \mathcal{R}\psi \overset{\text{def}}{=} \psi - \Delta_B\Omega.$$

By applying Lemma 5.6 to equation (5.33) we obtain

$$\|\mathcal{R}\omega\|_{C^r} \le C_r\|\Delta_{\mathcal{C}}\mathcal{R}\theta - R_1\|_{C^{r+\sigma}} \le C_r\|R_1,\, R_2,\, R_3\|_{C^{r+\sigma+\sigma_1}} \tag{5.34}$$

for any $r \ge 0$.

(4) Estimates for $\mathcal{R}\psi$ in (5.20).

Substituting ω by $\Delta_{\mathcal{C}}\Omega + \mathcal{R}\omega$ and ψ by $\Delta_B\Omega + \mathcal{R}\psi$ we get

$$A\mathcal{R}\psi - \mathcal{R}\psi \circ A\mathcal{C} = R_3 + B\mathcal{C}\mathcal{R}\theta - \mathcal{R}\theta \circ B + B\mathcal{R}\omega \circ A.$$

Again, Lemma 5.6 implies that

$$\begin{aligned} \|\mathcal{R}\psi\|_{C^r} &\le C_r\|R_3 + \mathcal{R}\theta \circ B + B\mathcal{C}\mathcal{R}\theta + B\mathcal{R}\omega \circ A\|_{r+\sigma_3} \\ &\le C_r\|R_1, R_2, R_3\|_{C^{r+\sigma+\sigma_1+\sigma_3}} \end{aligned} \tag{5.35}$$

where $\sigma_3 > N + 2 + [\kappa_{A^{-1},A\mathcal{C}}]$.

Let $\delta = \sigma + \sigma_1 + \sigma_3$. Then the conclusion follows from (5.21), (5.32), (5.34) and (5.35). ∎

In fact θ, ψ and ω play the roles of R_A, R_B and $R_{\mathcal{C}}$ in (5.2). The following lemma shows that R_1, R_2, R_3 cannot be large if $\alpha + R$ is a Heisenberg group action. It is in fact quadratically small with respect to R.

Lemma 5.8: *If $\widetilde{\alpha} = \alpha + R$ is a C^∞ Heisenberg group action on $\mathbb{T}^N$, then for any $r \ge 0$*

$$\|R_1, R_2, R_3\|_{C^r} \le C_r\|\theta, \psi, \omega\|_{C^r}\|\theta, \psi, \omega\|_{C^{r+1}}. \tag{5.36}$$

Proof. The estimates for R_1 and R_2 follow the same way as in the proof of Lemma 4.7 in [DK]. We just need to show the estimate for R_3. Note that

$$\begin{gathered} \widetilde{\alpha}_A \circ \widetilde{\alpha}_B = \widetilde{\alpha}_B \circ \widetilde{\alpha}_{\mathcal{C}} \circ \widetilde{\alpha}_A, \\ (A+\theta) \circ (B+\psi) = (B+\psi) \circ (\mathcal{C}+\omega) \circ (A+\theta). \end{gathered}$$

Then

$$\begin{aligned}&\theta\circ B+A\psi\\&\quad=\psi\circ(\omega\circ(A+\theta)+\mathcal{C}A+\mathcal{C}\theta)+B\omega\circ(A+\theta)+B\mathcal{C}\theta+\theta\circ B-\theta\circ(B+\psi).\end{aligned}$$

Therefore,

$$\begin{aligned}R_3=&\theta\circ B+A\psi-\psi\circ A\mathcal{C}-B\omega\circ A-B\mathcal{C}\theta\\=&\psi\circ(\omega\circ(A+\theta)+\mathcal{C}A+\mathcal{C}\theta)-\psi\circ A\mathcal{C}\\&+\theta\circ B-\theta\circ(B+\psi)+B\omega\circ(A+\theta)-B\omega\circ A.\end{aligned}$$

The estimate (5.36) for C^r norms follows similarly (see, for example, [La, Appendix II]):

$$\begin{aligned}\|R_3\|_{C^r}\le&C_r\|\psi,\omega\circ(A+\theta)+\mathcal{C}\theta\|_{C^r}\|\psi,\omega\circ(A+\theta)+\mathcal{C}\theta\|_{C^{r+1}}\\&+C_r\|\psi,\theta\|_{C^r}\|\psi,\theta\|_{C^{r+1}}+C_r\|\omega,\theta\|_{C^r}\|\omega,\theta\|_{C^{r+1}}\\\le&C_r\|\theta,\psi,\omega\|_{C^r}\|\theta,\psi,\omega\|_{C^{r+1}}.\qquad\blacksquare\end{aligned}$$

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