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THE NONEXISTENCE OF EXPANSIVE POLYCYCLIC GROUP ACTIONS ON THE CIRCLE $\mathbb{S}^1$

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Abstract. We show that the circle $\mathbb{S}^1$ admits no expansive polycyclic group actions.

1. Introduction

Expansivity is closely related to the structural stability in differential dynamical systems. Which spaces can admit an expansive homeomorphism has been intensively studied. It is known that the Cantor set, the 2-adic solenoid, and the tori $\mathbb{T}^n$ with $n \geq 2$ admit expansive homeomorphisms. O'Brien and Reddy showed that every compact orientable surface of positive genus admits an expansive homeomorphism [20]. However, the circle admits no expansive homeomorphisms [7]. It was proved by Kato and Mouron that several classes of one-dimensional continua admit no expansive homeomorphisms [8, 9, 16, 17]. Hiraide obtained the nonexistence of expansive homeomorphisms on the sphere $\mathbb{S}^2$ (see [5]). Mañé showed that no infinite dimensional compact metric space admits an expansive homeomorphism, and no compact metric space with positive dimension admits a minimal expansive homeomorphism [13]; the latter result of Mañé was extended to the case of pointwise recurrence by Shi, Xu, and Yu very recently [23].

T. Ward once asked whether the circle $\mathbb{S}^1$ can admit an expansive nilpotent group action. This question was answered in negative by Connell, Furman, and Hurder using the technique of semi-ping-pong in an unpublished paper [3] and by Inaba and Tsuchiya in more general situation [6], which is also implied by Margulis' work when the action is minimal [14]. Mai, Shi, and Wang showed the nonexistence of expansive actions by commutative or nilpotent groups on Peano continua with a free dendrite [12, 25]. Recently, Liang, Shi, Xie, and Xu showed that if the group G is of subexponential growth and X is a Suslinian continuum, then G cannot act on X expansively [10]. Contrary to the case of $\mathbb{Z}$ action, Shi and Zhou constructed an expansive $\mathbb{Z}^2$ action on an infinite dimensional continuum [26]; Meyerovitch and Tsukamoto constructed a minimal expansive $\mathbb{Z}^2$ action on a compact metric space X with $\dim(X) > 0$ (see [15]). Mouron constructed for each positive integer n, a

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continuum X which admits an expansive $\mathbb{Z}^{n+1}$ action but admits no expansive $\mathbb{Z}^n$ actions [18]. These examples indicate that there are essential differences between expansive $\mathbb{Z}$ actions and expansive $\mathbb{Z}^n$ actions with $n > 1$. One may consult [1,2,11] for some interesting studies around expansive $\mathbb{Z}^n$ actions and consult [24] for an application of expansive $\mathbb{Z}^2$ actions to Ramsey theory.

The aim of the paper is to continue the study of the existence of expansive group actions on continua. Explicitly, we obtain the following theorem.

Theorem 1.1. *The circle $\mathbb{S}^1$ admits no expansive polycyclic group actions.*

Here we give some remarks on the condition in the main theorem. It is known that a polycyclic group may have exponential growth. In fact, Wolf showed that a polycyclic group is either virtually nilpotent or has exponential growth [27]. Thus Theorem 1.1 is not implied by the main theorem in [10]. In addition, Rosenblatt proved that every finitely generated solvable group either is virtually nilpotent or contains a free non-abelian subsemigroup [22]. So a non-virtually-nilpotent polycyclic group must contain a free non-abelian subsemigroup; and thus the argument of using semi-ping-pong technique does not work in this case. The proof of Theorem 1.1 relies on a detailed description of the structure of any expansive subgroup of Homeo($[0,1]$) and the existence of quasi-invariant Radon measure for any polycyclic subgroup of Homeo($\mathbb{R}$) established by Plante [21]. At last, we should note that there does exist an expansive solvable group action on $[0,1]$ (and so does on $\mathbb{S}^1$) (see e.g. [26]).

2. Existence of minimal open intervals

Let us first recall some definitions around group actions. Given a group G and a topological space X. Let Homeo(X) be the homeomorphism group of X. A group homomorphism $\phi : G \to \text{Homeo}(X)$ is called a *continuous action* of G on X; we use the symbol (X, G, ϕ) to denote this action and also call it a *dynamical system*. For brevity, we usually use gx or $g(x)$ instead of $\phi(g)(x)$ and use (X, G) instead of (X, G, ϕ) if no confusion occurs. For $x \in X$, the set $Gx := \{gx : g \in G\}$ is called the *orbit* of x under the action of G; if $Gx = \{x\}$, then x is called a *fixed point* of G; if Gx is finite, then x is called a *periodic point* of G; if Gx is dense in X, then the action (X, G) is called *topologically transitive* and x is called a topologically transitive point of (X, G); if every point of X is transitive, then (X, G) is called *minimal*. A subset E of X is said to be *G-invariant* if $Gx \subset E$ for every $x \in E$; thus if E is G-invariant, we naturally get a *restriction action* $(E, G|_E)$ of G on E; $(E, G|_E)$ is called a *subaction* or a *subsystem* of (X, G). It is well known that if X is a Polish space and G is countable, then (X, G) is topologically transitive if and only if for every nonempty open sets U, V in X, there is some $g \in G$ such that $gU \cap V \neq \emptyset$; if X is a compact metric space, (X, G) is minimal if and only if it contains no proper closed subsystem. If X is a compact metric space with metric d, then the action (X, G) is called *expansive* if there is some $c > 0$ such that for every $x \neq y \in X$, there is some $g \in G$ such that $d(gx, gy) > c$; such c is called an *expansivity constant* of (X, G).

Lemma 2.1. *Let G be a countable group acting continuously on the closed interval $[0,1]$. If the action is expansive, then there is a G-invariant nonempty open set U in $[0,1]$ such that the subsystem $(U, G|_U)$ is topologically transitive.*

Proof. Let $c > 0$ be an expansivity constant for the action $([0,1], G)$. Take $0 = x_1 < x_2 < \cdots < x_{n-1} < x_n = 1$ such that $x_{i+1} - x_i < c$ for each i. Set $A = \cup_{i=1}^{n} \overline{Gx_i}$. Then A is a G-invariant closed subset in $[0,1]$. If for each i, $\overline{Gx_i}$ is nowhere dense, then A is nowhere dense. Thus $[0,1] \setminus A$ is nonempty and open. Choose $x \neq y$ in the same connected component of $[0,1] \setminus A$. Then $|gx - gy| < c$ for each $g \in G$, which contradicts the expansivity of $([0,1], G)$. So there exists some i_0 such that the interior U of $\overline{Gx_{i_0}}$ is nonempty. Clearly, U is G-invariant and $(U, G|_U)$ is topologically transitive. □

The following lemma is well known and easy to be checked.

Lemma 2.2. *Let X be a compact metric space and let G be a group acting continuously on X. Suppose H is a finite index subgroup of G. If (X, G) is expansive, then so is the subgroup action (X, H).*

Lemma 2.3. *Let X be a compact metric space and let G be a group acting on X expansively. Let Y be a subset of X and $H = \{g \in G : g(Y) = Y\}$. Suppose that the subgroup H has infinite index in G and let $G = g_1H \cup g_2H \cup \cdots$ be a coset decomposition of G with respect to H. If $\lim_{n\to\infty} \operatorname{diam}(g_nY) = 0$, then the restriction action (Y, H) is also expansive.*

Proof. To the contrary, assume that (Y, H) is not expansive. Let $c > 0$ be an expansivity constant for (X, G). Since $\lim_{n\to\infty} \operatorname{diam}(g_nY) = 0$, there is some positive integer N such that for each $n > N$, we have $\operatorname{diam}(g_nY) < c$. By the uniform continuity, there is some $\delta > 0$ such that $d(g_nx, g_ny) \leq c$ for each $n = 1, \cdots, N$ whenever $d(x, y) \leq \delta$. By the assumption that (Y, H) is not expansive, there are $x' \neq y' \in Y$ with $d(hx', hy') \leq \delta$. Thus $d(gx', gy') \leq c$ for each $g \in G$. This contradicts the expansivity of (X, G). □

Proposition 2.4. *Let G be a group acting continuously on the closed interval $[0,1]$. If the action is expansive, then there is a subgroup H of G, and an H-invariant open interval (a, b) in $[0,1]$ such that the restriction action $((a,b), H|_{(a,b)})$ is minimal and $([a,b], H|_{[a,b]})$ is expansive.*

Proof. Fix an expansivity constant $c > 0$ for the action $([0,1], G)$. By Lemma 2.1, we can take a G-invariant nonempty open set U in $[0,1]$ with $(U, G|_U)$ being topologically transitive. Let $NT(U)$ be the set of all nontransitive points of $(U, G|_U)$.

Claim A. $U \setminus \overline{NT(U)}$ is nonempty. Otherwise, we can take $x_1 < x_2 < \cdots < x_n$ in $NT(U)$ such that each connected component of $U \setminus \{x_1, x_2, \cdots, x_n\}$ has diameter less than c. Since each x_i is a nontransitive point, the closure $\overline{Gx_i}$ contains no interior point. Thus $U \setminus \cup_{i=1}^{n} \overline{Gx_i}$ is nonempty and G-invariant. Take a connected component A of $U \setminus \cup_{i=1}^{n} \overline{Gx_i}$. Then the diameter $\operatorname{diam}(gA) < c$ for each $g \in G$. This contradicts the expansivity of $([0,1], G)$. Thus Claim A holds.

From Claim A, we can take a maximal open interval (a, b) in $U \setminus \overline{NT(U)}$. Let $H = \{g \in G : g(a, b) = (a, b)\}$.

Claim B. $((a,b), H|_{(a,b)})$ is minimal. In fact, from the maximality of (a, b), we have $g(a, b) \cap (a, b) = \emptyset$ for every $g \in G \setminus H$. This together with the topological transitivity of $(U, G|_U)$ implies Claim B.

Claim C. $([a,b], H|_{[a,b]})$ is expansive. This is clear if H has finite index in G by Lemma 2.2. So we may assume that the index $[G : H] = \infty$. Let $G = \cup_{i=1}^{\infty} g_iH$ be

a coset decomposition of G with respect to H. Then the sets $g_i([a,b])(i=1,2,\cdots)$ are pairwise disjoint and hence $\lim_{i\to\infty}\operatorname{diam}(g_i([a,b]))=0$. Thus, by Lemma 2.3, $([a,b],H|_{[a,b]})$ is expansive.

We complete the proof from Claim B and Claim C. □

3. Minimal polycyclic subgroups of Homeo($\mathbb{R}$)

Recall that a group G is *polycyclic* if it has a subnormal decreasing series $G=N_0\rhd N_1\rhd\cdots\rhd N_n\rhd N_{n+1}=\{e\}$ such that N_i/N_{i+1} is cyclic for each $i\geq 0$. It is known that every finitely generated nilpotent group is polycyclic and every polycyclic group is solvable; every subgroup and every quotient group of a polycyclic group is polycyclic. One may consult [4] for a detailed introduction to polycyclic groups.

The following proposition can be seen in [27].

Proposition 3.1. *A solvable group G is polycyclic if and only if every subgroup of G is finitely generated.*

Let $\mathbb{R}$ be the real line. For $a\in\mathbb{R}\setminus\{0\}$ and $b\in\mathbb{R}$, define the affine transformation $A_{a,b}:\mathbb{R}\to\mathbb{R}$ by $A_{a,b}(x)=ax+b$ for all $x\in\mathbb{R}$. Set $\mathrm{Aff}(\mathbb{R})=\{A_{a,b}:a\in\mathbb{R}\setminus\{0\},b\in\mathbb{R}\}$. Then $\mathrm{Aff}(\mathbb{R})$ is a solvable group and is called the *affine transformation group* on $\mathbb{R}$. It is known that $\mathrm{Aff}(\mathbb{R})$ consists of the homeomorphisms f on $\mathbb{R}$ satisfying that $|f(x)-f(y)|=c|x-y|$ for some $c=c(f)>0$ and for all $x,y\in\mathbb{R}$.

Lemma 3.2. *Let G be a subgroup of $\mathrm{Aff}(\mathbb{R})$. If G contains an $A_{a,c}$ with $|a|\neq 1$, $c\in\mathbb{R}$ and an $A_{1,b}$ with $b\neq 0$, then G cannot be polycyclic.*

Proof. WLOG, we may assume that G contains an element of the form $A_{a,0}$; otherwise, we need only consider a conjugation of G by the translation $L:=A_{1,\frac{c}{a-1}}$ on $\mathbb{R}$ ($LA_{a,c}L^{-1}=A_{a,0}$ and $LA_{1,b}L^{-1}=A_{1,b}$). It is easy to check that $A_{a,0}A_{1,b}A_{a,0}^{-1}=A_{1,ab}$. From this we see that the set $S:=\{A_{1,a^mb}:m\in\mathbb{Z}\}$ is contained in G. Since the subgroup $\langle S\rangle$ generated by S is not finitely generated, G is not polycyclic by Proposition 3.1. □

Let X be a topological space and let G be a group acting on X. A Borel measure μ on X is called a *Radon measure* if it is finite on every compact subset of X; it is called *quasi-invariant* if for every $g\in G$ there is some $c(g)>0$ such that $\mu(g^{-1}A)=c(g)\mu(A)$ for every Borel subset A in X; it is called *invariant* if $c(g)=1$ for every $g\in G$. Clearly, if μ is quasi-invariant, then $c(g_1g_2)=c(g_1)c(g_2)$ for every $g_1,g_2\in G$. We remark here our definition of quasi-invariant measure is stronger than the usual meaning that just requires every group element preserves the measure class.

The following theorem is due to Plante [21].

Theorem 3.3. *Let G be a polycyclic group acting continuously on the real line $\mathbb{R}$. Then there is a nontrivial G-quasi-invariant Radon measure μ on $\mathbb{R}$. (Here, "nontrivial" means $\mu(A)>0$ for some Borel set A.)*

Two actions (X,G,ϕ) and (Y,G,ψ) are said to be *topologically conjugate* if there is a homeomorphism $h:X\to Y$ such that $h(\phi(g)(x))=\psi(g)(h(x))$ for every $x\in X$ and $g\in G$. The following proposition clarifies the structure of minimal actions on $\mathbb{R}$ by polycyclic groups.

Proposition 3.4. *Let G be a polycyclic group and let $\phi : G \to \mathrm{Homeo}(\mathbb{R})$ be a continuous action. If $(\mathbb{R}, G, \phi)$ is minimal, then $(\mathbb{R}, G, \phi)$ is topologically conjugate to an action $(\mathbb{R}, G, \psi)$ with each element of $\psi(G)$ being isometric.*

Proof. From Theorem 3.3, we can take a nontrivial G-quasi-invariant Radon measure μ on $\mathbb{R}$. Now we define a map $h : \mathbb{R} \to \mathbb{R}$ as does in [21]:

$$h(x) = \begin{cases} \mu([0, x]), & x \geq 0; \\ -\mu([x, 0]), & x < 0. \end{cases}$$

Since $(\mathbb{R}, G, \phi)$ is minimal, the support $\mathrm{supp}(\mu) = \mathbb{R}$. Furthermore, μ contains no atoms by the minimality of G and quasi-invariance of μ (see the appendix). These imply that h is a homeomorphism. From the quasi-invariance of μ, we see that for each $g \in G$, hgh^{-1} is an affine transformation on $\mathbb{R}$. Now define an action $\psi : G \to \mathrm{Homeo}(\mathbb{R})$ by $\psi(g) = hgh^{-1}$ for each $g \in G$.

By the definition of polycyclic group, there is a subnormal decreasing series $G = N_0 \rhd N_1 \rhd \cdots \rhd N_n \rhd N_{n+1} = \{e\}$ such that N_i/N_{i+1} is cyclic for each $i \geq 0$. Take $g_i \in N_i \setminus N_{i+1}$ such that $N_i/N_{i+1} = \langle g_i N_{i+1} \rangle$. Thus for each $i \geq 0$, $N_i = \langle g_i, g_{i+1}, \cdots, g_n \rangle$. Let $f_i = \psi(g_i) = A_{a_i, b_i}$ for each $i \geq 0$ and for some $a_i, b_i \in \mathbb{R}$.

Assume to the contrary that there is some $i_0 \in \{0, 1, \cdots, n\}$ such that f_{i_0} is not isometric and for each $i_0 < i \leq n$, f_i is isometric; that is $|a_{i_0}| \neq 1$ and $|a_i| = 1$ for $i > i_0$. Since $\psi(G)$ is polycyclic, from Lemma 3.2, either $f_i = A_{-1, b_i}$ or $f_i = A_{1,0}$ for each $i > i_0$.

Case 1. There is $i_1 > i_0$ such that $f_{i_1} = A_{-1, b_{i_1}}$ and $f_i = A_{1,0}$ for all $i > i_1$. Then f_{i_1} has a unique fixed point $c = b_{i_1}/2$, which is also a fixed point of f_i with $i < i_1$ by the normality of N_{i_1}. Thus c is a fixed point of $\psi(G)$. This contradicts the minimality of the action $(\mathbb{R}, G, \psi)$.

Case 2. For each $i > i_0$, $f_i = A_{1,0}$. Thus the unique fixed point $x = b_{i_0}/(1 - a_{i_0})$ of f_{i_0} is also a fixed point of f_i with $i < i_0$ by the normality of N_{i_0}. Thus x is the fixed point of $\psi(G)$, which contradicts the minimality of $(\mathbb{R}, G, \psi)$ again.

So the assumption is false and each f_i is isometric. Hence each element of $\psi(G)$ is isometric. □

4. Proof of the main theorem

In this section, we will prove the main theorem. We first establish the nonexistence of expansive polycyclic group actions on the interval $[0, 1]$ in the following proposition.

Proposition 4.1. *The closed interval $[0, 1]$ admits no expansive polycyclic group actions.*

Proof. Assume to the contrary that there is an expansive action $([0, 1], G, \phi)$ by a polycyclic group G. By Proposition 2.4, there is an open interval (a, b) in $[0, 1]$ and a subgroup H of G such that (a, b) is H-invariant and the restriction action $((a, b), H|_{(a,b)})$ is minimal and $([a, b], H|_{[a,b]})$ is expansive. Let $c > 0$ be an expansivity constant for the action $([a, b], H|_{[a,b]})$. Notice that H is still polycyclic. Applying Proposition 3.4, there is an orientation-preserving homeomorphism $\xi : (0, 1) \to \mathbb{R}$ such that for every $g \in H$, $\xi\phi(g)\xi^{-1}$ is an isometric transformation on $\mathbb{R}$. Take

$a < a' < b' < b$ such that $a' - a < c$ and $b - b' < c$. Let $A' = \xi(a')$ and $B' = \xi(b')$. By uniform continuity, there is $\delta > 0$, such that for every $x, y \in [A' - 1, B' + 1]$ with $|x - y| < \delta$, we always have $|\xi^{-1}(x) - \xi^{-1}(y)| < c$. Take $u, v \in \mathbb{R}$ with $|u - v| < \min\{1, \delta\}$. Then for every $g \in H$, by the isometry of $\xi\phi(g)\xi^{-1}$, we have

$$\begin{aligned} &|\phi(g)(\xi^{-1}(u)) - \phi(g)(\xi^{-1}(v))| \\ &= |\xi^{-1}(\xi\phi(g)\xi^{-1})(u) - \xi^{-1}(\xi\phi(g)\xi^{-1})(v)| < c. \end{aligned}$$

This contradicts the expansivity of $([a, b], H|_{[a,b]})$. □

The following proposition can be seen in [19].

Proposition 4.2. *Let a group G act continuously on the circle $\mathbb{S}^1$ and let Λ be a minimal closed subset of $(\mathbb{S}^1, G)$. Then there are three case: (a) $\Lambda = \mathbb{S}^1$; (b) Λ is a Cantor set; (c) Λ is finite.*

Proof of Theorem 1.1. Assume to the contrary that there is an expansive action on $\mathbb{S}^1$ by a polycyclic group G. Let Λ be a minimal closed set for the action. From Proposition 4.2, we discuss into three cases.

Case 1. $\Lambda = \mathbb{S}^1$. By the amenability of polycyclic groups, there is a G-invariant probability Borel measure μ on $\mathbb{S}^1$. Since $(\mathbb{S}^1, G)$ is minimal, the support $\operatorname{supp}\mu = \mathbb{S}^1$ and μ has no atoms. Recall a classical result that any subgroup of $\mathrm{Homeo}_+(\mathbb{S}^1)$ preserving a probability measure with full support and without atoms is topologically conjugate to a subgroup consisting of rotations (see [19, Proposition 1.1.1]), where $\mathrm{Homeo}_+(\mathbb{S}^1)$ is the group of orientation preserving homeomorphisms of $\mathbb{S}^1$. Since $\mathrm{Homeo}_+(\mathbb{S}^1)$ has index two in $\mathrm{Homeo}(\mathbb{S}^1)$, we conclude that $(\mathbb{S}^1, G)$ is not expansive in this case. This is a contradiction.

Case 2. Λ is a Cantor set. Take a maximal interval (α, β) in $\mathbb{S}^1 \setminus \Lambda$. Let $H = \{g \in G : g(\alpha, \beta) = (\alpha, \beta)\}$. Then H is a subgroup of G with index $[G : H] = \infty$. Let $G = g_1 H \cup g_2 H \cup \cdots$ be a coset decomposition of G with respect to H. Then the sets $g_i([\alpha, \beta])$ are mutually disjoint and hence $\lim_{i\to\infty} \operatorname{diam}(g_i([\alpha, \beta])) = 0$. Now it follows from Lemma 2.3 that $([\alpha, \beta], H|_{[\alpha,\beta]})$ is expansive. But this contradicts Proposition 4.1.

Case 3. Λ is finite. Fix a maximal interval (α, β) in $\mathbb{S}^1 \setminus \Lambda$. Let $H = \{g \in G : g(\alpha, \beta) = (\alpha, \beta)\}$. Then $[G : H] < \infty$. If the cardinality of Λ is greater than 1, then $([\alpha, \beta], H|_{[\alpha,\beta]})$ is expansive by Lemma 2.2. This contradicts Proposition 4.1. If Λ consists of only one point, say O. We view $\mathbb{S}^1$ as the quotient of $[0, 1]$ by collapsing the two endpoints $\{0, 1\}$ to one point O. Then the action of H on $\mathbb{S}^1$ can be naturally lifted to an action on $[0, 1]$. Clearly, the lifted action is still expansive. This contradicts Proposition 4.1 again.

All together, we see that the assumption is false and thus complete the proof. □

5. Appendix

In this section, we will explain why the measure μ appearing in Proposition 3.4 has no atoms. Though this fact is well known to experts, it is hard to find the proof from literatures. So we give a detailed proof here.

Lemma 5.1. *Let G be a subgroup of $\mathrm{Homeo}_+(\mathbb{R})$ admiting a quasi-invariant Radon measure μ. If the action of G is minimal, then μ is atomless.*

Proof. By the quasi-invariance, there is a group homomorphism $c : G \to (0, +\infty)$ from G to the multiplication group of positive real numbers such that $g_*\mu = c(g)\mu$ for each $g \in G$. Clearly, μ is G-invariant if and only if $c(g) = 1$ for all $g \in G$.

If μ is G-invariant, then it has no atoms by the minimality of the action and the local finiteness of μ. Now suppose that $x \in \mathbb{R}$ is an atom of μ. Then μ is not G-invariant and hence there is some $f \in G$ with $c(f) \neq 1$.

Claim 1. For each $g \in G \setminus \{e\}$, g has at most one fixed point.

Otherwise, there is a maximal finite open interval (a, b) of $\mathbb{R} \setminus \mathrm{Fix}(g)$. By the minimality, there is some $h \in G$ such that $hx \in (a, b)$. Then $g^n hx \in (a, b)$ for each $n \in \mathbb{Z}$. Thus

$$\mu([a, b]) \geq \sum_{n\in\mathbb{Z}} \mu(\{g^n hx\}) = \sum_{n\in\mathbb{Z}} c(g)^n c(h^{-1})\mu(\{x\}) = \infty,$$

which contradicts the local finiteness of μ.

Claim 2. For each $g \in G \setminus \{\mathrm{id}\}$, $c(g) \neq 1$.

To the contrary, assume $c(g) = 1$ for some nontrivial element $g \in G$.

(1) If g has no fixed point, then we may assume that $g(y) > y$ for any $y \in \mathbb{R}$. Pick $a \in \mathbb{R}$ and we have $\mathbb{R} = \bigcup_{n\in\mathbb{Z}}[g^n a, g^{n+1}a)$. Since $c(g) = 1$, the measure of $[g^n a, g^{n+1}a)$ is independent of n, saying a positive constant $\lambda > 0$. Recall that $c(f) \neq 1$. We may assume that $c(f) > 1$ by replacing f with f^{-1}. For sufficiently large N, we have

$$\mu(\{f^{-N}x\}) = (f_*^N\mu)(\{x\}) = c(f)^N\mu(\{x\}) > \lambda.$$

This is absurd since $f^{-N}x$ must locate in some interval $[f^n a, f^{n+1}a)$ whose measure is λ.

(2) If the fixed point set of g is nonempty then g has a unique fixed point p by Claim 1. Without loss of generality, we assume that $g(y) > y$ for any $y \in (-\infty, p)$. By the very arguments used in (1), we also obtain a contradiction.

Combining (1) and (2), we complete the proof of Claim 2.

By Claim 2, we conclude that $c : G \to (0, +\infty)$ is an injective homomorphism. Thus G is commutative.

Claim 3. For each $g \in G \setminus \{\mathrm{id}\}$, we have $\mathrm{Fix}(g) = \emptyset$.

Otherwise, there is some nontrivial $g \in G$ with $\mathrm{Fix}(g) = \emptyset$. By Claim 1, g has a unique fixed point. Since G is commutative, $\mathrm{Fix}(g)$ is G-invariant. Hence G has a global fixed point which contradicts the minimality.

By Claim 3, we can choose an element $g \in G$ satisfying $g(y) > y$ for any $y \in \mathbb{R}$. Set

$$H = \{h \in G : hx \in (g^{-1}x, x)\}.$$

Claim 4. For each $h \in H$, we have $h^{-1}g^{-1}(x) > g^{-1}(x)$.

Otherwise, we have $h^{-1}g^{-1}(x) \leq g^{-1}(x)$ and hence $h(g^{-1}x) \geq g^{-1}x$. Note that $h(x) < x$. Thus there must be a fixed point of h in $[g^{-1}x, x)$, which contradicts Claim 3.

Finally, by Claim 4, we have $h^{-1}g^{-1}(x) \in (g^{-1}x, x)$ and hence

$$\begin{aligned}\mu([g^{-1}x, x]) &\geq \frac{1}{2}\sum_{h\in H}(\mu(\{hx\}) + \mu(\{h^{-1}g^{-1}x\})) \\ &\geq \frac{1}{2}\sum_{h\in H}\big(c(h) + c(h^{-1})\big)\min\big(\mu(\{x\}), \mu(\{g^{-1}x\})\big) \\ &= \infty.\end{aligned}$$

This contradicts the local finiteness of μ. Thus μ has no atoms. □

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