Dirac Cohomology of Complex Simple Lie Groups

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Dirac Operator - basic settings

- The idea of Dirac operator is first introduced by Paul Dirac in the 1930s, which can be seen as a square root of the Laplacian operator $\sum_i \frac{\partial^2}{\partial e_i^2}$ in $\mathbb{R}^n$.
- It is defined as $D := \sum_i e_i \frac{\partial}{\partial e_i}$, where e_i are seen as elements in the Clifford algebra $C(\mathbb{R}^n)$ satisfying the condition:

$$uv + vu = 2\langle u, v\rangle \cdot 1, \qquad u, v \in \mathbb{R}^n.$$

- A generalization: Let G be a connected real reductive Lie group, and K be a maximal compact subgroup.
- Write $\mathfrak{g}_0$, $\mathfrak{k}_0$ as the Lie algebra of G and K respectively, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the corresponding Cartan decomposition on the **complexified** Lie algebra.
- Fix an orthonormal basis $Z_1, \cdots, Z_n$ of $\mathfrak{p}_0$ with respect to the inner product induced by the Killing form $B(\cdot,\cdot)$.
- Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$, and $C(\mathfrak{p})$ be the Clifford algebra of $\mathfrak{p}$ with respect to $B(\cdot,\cdot)$.
- The **Dirac operator** $D \in U(\mathfrak{g}) \otimes C(\mathfrak{p})$ is defined as $D = \sum_{i=1}^n Z_i \otimes Z_i$. ($D$ is independent of the choice of orthonormal basis Z_i)

Dirac operator - further properties

- The formula of D^2 is given by

$$D^2 = -\Omega_{\mathfrak{g}} \otimes 1 + \Omega_{\mathfrak{k}_\Delta} + (\|\rho_c\|^2 - \|\rho_{\mathfrak{g}}\|^2) 1 \otimes 1,$$

  where:

- $\Omega_{\mathfrak{g}}$ (resp. $\Omega_{\mathfrak{k}}$) is the Casimir operator of $\mathfrak{g}$ (resp. $\mathfrak{k}$). (Casimir operator $\approx$ Laplacian operator).
- $\Omega_{\mathfrak{k}_\Delta}$ be the image of $\Omega_{\mathfrak{k}}$ under $\Delta : \mathfrak{k} \to U(\mathfrak{g}) \otimes C(\mathfrak{p})$ be given by $\Delta(X) = X \otimes 1 + 1 \otimes \alpha(X)$, where α is the action map $\mathfrak{k} \to \mathfrak{so}(\mathfrak{p})$ followed by the usual identifications $\mathfrak{so}(\mathfrak{p}) \cong \wedge^2(\mathfrak{p}) \hookrightarrow C(\mathfrak{p})$.
- $\rho_{\mathfrak{g}}$ and ρ_c are the corresponding half sums of positive roots of $\mathfrak{g}$ and $\mathfrak{k}$.

As an operator, D acts on vector spaces given by the following:

- Let π be an admissible $(\mathfrak{g}, K)$-module, and S_G be a spin module for $C(\mathfrak{p})$.
- Consider $\pi \otimes S_G$. It has a natural $U(\mathfrak{g}) \otimes C(\mathfrak{p})$-module structure. So D gives an operator

$$D : \pi \otimes S_G \to \pi \otimes S_G$$

Dirac cohomology

- More precisely, $\pi \otimes S_G$ is is an admissible $(U(\mathfrak{g}) \otimes C(\mathfrak{p}), \widetilde{K})$ module, where $\widetilde{K}$ is the spin double cover of K given by the pullback of the map:

$$\begin{array}{ccc} \widetilde{K} & \longrightarrow & Spin(\mathfrak{p}_0) \\ \downarrow & & \downarrow pr \\ K & \xrightarrow[Ad]{} & SO(\mathfrak{p}_0) \end{array},$$

  and $(k, s) \in \widetilde{K} \subset K \times Spin(\mathfrak{p}_0)$ acts on $\pi \otimes S_G$ by the diagonal action.
- By the independence of the choice of Z_i in the definition of D, D commutes with every element in $\widetilde{K}$.
- The Dirac cohomology of π is defined as the $\widetilde{K}$-module

$$H_D(\pi) := \ker\ D/(\ker\ D \cap \mathrm{im}\ D)$$

An application of Dirac cohomology

- If π is a unitary, irreducible representation, then the formula of D^2 obtained earlier gives the **Parthasarathy's inequality**

$$||\kappa + \rho_c||^2 - ||\chi||^2 \geq 0$$

  for all $\widetilde{K}$-types V_κ appearing in $\pi \otimes S_G$ with highest weight κ, where χ is infinitesimal character of π.
- The equality holds iff $V_\kappa \in \ker D^2$ and $\kappa + \rho_c = \chi$.
- As a result, $\ker D^2$**, if** $\neq 0$**, determines the infinitesimal character of** π.
- More generally, for any irreducible admissible $(\mathfrak{g}, K)$-modules (regardless of unitarity), we have:

Theorem (Huang-Pandžić)

Let π be an admissible, irreducible $(\mathfrak{g}, K)$-module with infinitesimal character χ. Assume that $H_D(\pi) \neq 0$, and that it contains the $\widetilde{K}$-type with highest weight $\tau \in \mathfrak{t}^* \subset \mathfrak{h}^*$. Then χ is conjugate to $\tau + \rho_c$ under $W(\mathfrak{g}, \mathfrak{h})$.

- We will focus one the case when π is unitary. In such a case, $H_D(\pi) = \ker(D^2) = \ker(D)$.

Some questions

Given a reductive group G, one may ask the following:

- **(Classification)** Given π is unitary, when is $H_D(\pi) \neq 0$? (Important note: the infinitesimal character of such π must be $\rho_c +$ highest weight of a K-type)
- **(Multiplicity-one)** Suppose $H_D(\pi) \neq 0$, is there a unique $K-$type of π contributing to $H_D(\pi)$?
- **(Determination of spin-LKT)** More generally, can one determine which $K-$type(s) of π contributes to $H_D(\pi)$?

Complex Groups

Here is a conjecture on the classification and multiplicity-one problem for complex groups:

Conjecture (Barbasch-Pandžić; Dong-Huang)

Let G be a complex simple Lie group. Then all irreducible unitary π with $H_D(\pi) \neq 0$ are real parabolically induced from a unipotent representation with nonzero Dirac cohomology tensored with some unitary characters. Moreover, there is a unique K-type of π contributing to $H_D(\pi)$.

Theorem (Barbasch-Dong-W; Ding-Dong; Dong-W)

The conjecture holds for complex simple Lie groups G except E_8.

Note that the conjecture does not say anything about the third problem - Given the structure of π, it is not obvious **which** $K-$type of π contributes to $H_D(\pi)$.

Some ingredients

Here are some ingredients of the proof:

- Huang-Pandžić's Theorem gives a necessary condition of the infinitesimal character of π in order for $H_D(\pi) \neq 0$ (we call it **HP-condition** from now on). This cuts off a lot of unitary representations (such as the principal series representation $Ind_B^G(triv)$).
- We (re)prove the unitary dual for classical groups under HP-condition in [BDW], and obtain the **non-unitary certificates** of these groups.
- More precisely, all such representations are real parabolically induced from a unitary character tensored with a **small** unipotent representation π_u. Here 'small' means they can be obtained by a single $\theta-$lift (or double $\theta-$lift for Type D) from the trivial representation for classical groups.
- The classification and multiplicity-one theorem for classical groups are proved simultaneously by carefully keeping track of the Dirac cohomologies of π_u and $Ind_P^G(\pi_u \otimes \chi)$.
- For exceptional groups, we do not have the classification of unitary dual. But there is only a finite number of groups, so one can use Dong's finiteness theorem.

Dong's approach

To study $H_D(\pi)$ for all real reductive Lie groups (rather than just complex groups), Dong proved the following:

Theorem (Dong)

For all reductive Lie groups G, there is a finite number of unitary $(\mathfrak{g}, K)-$modules with non-zero Dirac cohomology called **scattered representations**, *such that all unitary $(\mathfrak{g}, K)-$modules π with non-zero Dirac cohomology are either*

- *a scattered representation; or*
- *cohomologically induced from a scattered $(\mathfrak{l}, K \cap L)-$module in the weakly good range.*

Moreover, for any fixed G, there is a finite algorithm computing the scattered representations and their Dirac cohomologies.

The behavior of Dirac cohomology before and after cohomological induction in weakly good range is well-studied by [Dong-Huang]. Therefore, all three problems of Dirac cohomology can be solved for exceptional groups. This is carried out for complex exceptional groups except E_8 – `atlas` cannot determine unitarity of the trivial representation of E_8!

Barbasch-Pandžić v.s. Dong

- In Barbasch-Pandžić's approach, we have a uniform description of the unitary representations with non-zero Dirac cohomology using **real parabolic induction**. But the spin-LKT problem is not clear.
- In Dong's approach, one uses **cohomological induction**, where the spin-LKT problem can be solved. But one needs to study scattered representations for each G. There are infinitely many classical groups to study.

Question

How to understand scattered representations using Barbasch-Pandžić's perspective? That is, how to realize scattered representations as real parabolically induced module (but not cohomologically induced module in weakly good range)? Given such knowledge on scattered representations, can one obtain the unique $K-$*type contributing to Dirac cohomology?*

Some Combinatorics

We begin with $G = GL(n, \mathbb{C})$. Let's start with some combinatorics:

Definition

Let $p \geq q$ be two positive integers with $p - q \in 2\mathbb{N}$. We call

$$\mathcal{C} = [p, p-2, \ldots, q+2, q]_A$$

an **A-string** of integers.

Definition

Two disjoint A-strings $\mathcal{C} = [p, \ldots, q]_A$ and $\mathcal{C}' = [p', \ldots, q']_A$ are **linked** if $p > p' > q$ or $p' > p > q'$.

The union of chains $\bigcup_{i=0}^{m} \mathcal{C}_i$ are **interlaced** if the $\mathcal{C}_i$'s are pairwise disjoint, and for all $i \neq j$, there is a sequence $i = n_0, n_1 \ldots, n_p = j$ such that $\mathcal{C}_{n_q}$ and $\mathcal{C}_{n_{q+1}}$ are linked for all $0 \leq q \leq p-1$ (we also decree that a single chain $\mathcal{C}$ is interlaced).

For example, $\begin{matrix} [10 & & 8]_A & & [6]_A & & [4]_A & & \\ & [9 & & 7 & & 5 & & 3 & 1]_A \end{matrix}$ are interlaced.

Unitary dual of $GL(n, \mathbb{C})$

Now we relate the chains with unitary representations of $GL(n, \mathbb{C})$ with integral infinitesimal character:

Proposition

All irreducible, unitary representations π of $GL(n, \mathbb{C})$ with integral infinitesimal character χ are in one-one correspondence with the collection of union of A-strings $\pi \longleftrightarrow \bigcup_{i=0}^{m} \mathcal{C}_i$ of total length n. If χ is further assumed to satisfy HP-condition, then all $\mathcal{C}_i$'s are pairwise disjoint.

- The chain $[p, \ldots, q]_A$ corresponds to $(\det)^{p+q}$ of $GL(\frac{p-q+2}{2})$.
- Under the correspondence in the proposition, π is induced from the $(\det)^{p_i+q_i}$'s corresponding to each $\mathcal{C}_i$.
- The entries of $\mathcal{C}_i$ give the infinitesimal character of π under the above correspondence.
- If we study $SL(n, \mathbb{C})$ instead of $GL(n, \mathbb{C})$, one can apply some normalization on the infinitesimal characters, so that one can fix the smallest integer among all $\mathcal{C}_i$'s to be 1.

Scattered Representations of $SL(n, \mathbb{C})$

Example

In $SL(5, \mathbb{C})$, consider $[6, 4, 2]_A \cup [3, 1]_A$. The correspondence gives

$$Ind_{S(GL(3)\times GL(2))}^{SL(5)}(\det^a \otimes \det^b) \longleftrightarrow [6, 4, 2]_A \cup [3, 1]_A$$

with a, b satisfying $3a + 2b = 0$ and $a - b = 2$. Its infinitesimal character equals

$$2\lambda = (6, 4, 2, 3, 1) - (\tfrac{16}{5}, \tfrac{16}{5}, \tfrac{16}{5}, \tfrac{16}{5}, \tfrac{16}{5}).$$

Now we look at which $\bigcup_{i=0}^m \mathcal{C}_i$ corresponds to a scattered representation in $SL(n)$:

Theorem (Dong-W)

Let π be a unitary representation of $SL(n, \mathbb{C})$. Then $\pi \longleftrightarrow \cup_{i=0}^m \mathcal{C}_i$ is scattered if and only if $\bigcup_{i=0}^m \mathcal{C}_i$ is interlaced.

The idea is, the 'good range' condition in cohomological induction corresponds to some positivity condition on the infinitesimal character 2λ, which in turn is related to coordinates of $\mathcal{C}_i$. In the above example, the induction is not in 'good range', since $\langle (6, 4, \mathbf{2}, \mathbf{3}, 1), e_3 - e_4 \rangle < 0$.

Number of Scattered Representations

Example

Let us start from $SL(2,\mathbb{C})$. The only interlaced A-string is $[3 \quad 1]_A$, which corresponds to the trivial representation.
The interlaced A-strings of $SL(3,\mathbb{C})$ are

$$[5 \quad 3 \quad 1]_A \quad \text{and} \quad \begin{array}{ccc} [3 & & 1]_A \\ & [2]_A & \end{array}.$$

The interlaced A-strings of $SL(4,\mathbb{C})$ are

$$[7 \quad 5 \quad 3 \quad 1]_A, \quad \begin{array}{cccc} [5 & & 3 & 1]_A \\ & [4]_A & & \end{array}, \quad \begin{array}{cccc} [5 & 3 & & 1]_A \\ & & [2]_A & \end{array}, \quad \text{and} \quad \begin{array}{cccc} & [3 & & 1]_A \\ [4 & & 2]_A & \end{array}.$$

Theorem (Dong-W)

The number of scattered representations of $SL(n,\mathbb{C})$ is equal to 2^{n-2}. Moreover, one can calculate the (unique) $K-$type contributing to all each scattered representation (using Littlewood-Richardson Rule).

Scattered representation for Classical Types

For other classical types, recall in Barbasch-Pandžić's conjecture there are some small unipotent representations π_u other than the unitary characters.

Theorem (Dong-W)

Let π be a unitary representation of classical type. Then all unitary representations with non-zero Dirac cohomology are in bijection with the union of pairwise disjoint A-strings **and (possibly) a U-string**, *i.e.* $\pi \longleftrightarrow \bigcup_{i=0}^{m} \mathcal{C}_i \cup \mathcal{U}$. *Moreover, π is scattered if and only if the strings $\bigcup_{i=0}^{m} \mathcal{C}_i \cup \mathcal{U}$ are interlaced.*

Here a U-string looks like $\mathcal{U} = [u \dots v]_X$, whose coordinates correspond to the infinitesimal character of a unipotent representation of Type $X = B, C$ or D. For example,

$$\pi \quad \longleftrightarrow \quad \begin{matrix} & [10 & & 8 & & 6]_A & & [4]_A & & & \\ [11 & & 9]_A & & [7 & & 5 & & 3 & 2 & 1]_B \end{matrix}$$

where $\pi = Ind_{GL(2)\times GL(3)\times GL(1)\times SO(11)}^{SO(23)} \left((\det)^{10} \boxtimes (\det)^{8} \boxtimes (\det)^{4} \boxtimes \pi_{min} \right)$. Here π_{min} is the unipotent representation of $SO(11)$ with associated variety equal to the cloure of the minimal orbit, which can be obtained from the θ-lift of the trivial representation of $Sp(2)$.

Conclusions

Here are some conclusions, current progress and some open problems:

- The number of scattered representations of Type A_n, B_n, C_n, D_n in terms of n is known.
- Using Littlewood-Richardson Rule (for Type A) or Blattner-type formula (for classical types), one can obtain the (unique) spin-LKT for the scattered representations.
- Similar techniques can be applied to $GL(n, \mathbb{R})$ (where A-chains correspond to Speh representations instead of unitary characters), and $Spin(n, \mathbb{C})$ [Zhang].
- For other real reductive groups, the multiplicity-one statement is false. There can be more than one spin-LKTs contributing to $H_D(\pi)$. However, these spin-LKTs seem to appear in π with multiplicity one.
- One can calculate the **Dirac index** of π, which is easier to calculate but contains (possibly) less information than $H_D(\pi)$. This is carried out for $U(p, q)$, $Sp(2n, \mathbb{R})$ and $SO^*(2n)$. **Question:** Does the Dirac index contain **all** information on $H_D(\pi)$ when G is Hermitian symmetric?

Thank you!