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The structure of periodic point free distal homeomorphisms on the annulus

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Abstract

Let A be an annulus in the plane $\mathbb{R}^2$ and $g : A \to A$ be a boundary components preserving homeomorphism which is distal and has no periodic points. Then there is a continuous decomposition of A into g-invariant circles such that all the restrictions of g on them share a common irrational rotation number and all these circles are linearly ordered by the inclusion relation on the sets of bounded components of their complements in $\mathbb{R}^2$. Finally, we show that g is conjugate to an irrational rotation if and only if there exists a transversal and examples without transversals are also constructed.

1 Introduction

Recurrence is one of the most fundamental notions in the theory of dynamical system. There are various definitions to describe the recurrence behaviors of a point in a system, such as periodic point, almost periodic point, distal point, recurrent point, regularly recurrent point, and so on. There has been a considerable progress in studying the structures of the dynamical systems all points of which possess some kind of recurrence.

Montgomery [28] proved that every pointwise periodic homeomorphism on a connected manifold is periodic. For an infinite compact minimal metric system each point of which is regularly recurrent, Block and Keesling [5] proved that it is topologically conjugate to an adding machine. Shi et al. [37] showed that every pointwise recurrent expansive homeomorphism is topologically conjugate to a subshift of some symbol system, which extends a classical result of Mañé [26] for minimal expansive homeomorphisms. The structure of

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pointwise recurrent maps having the pseudo orbit tracing property is completely determined by Mai and Ye [25].

There are also many interesting results around the structures of recurrent maps on low-dimensional spaces. Mai [24] showed that a pointwise recurrent graph map is either topologically conjugate to an irrational rotation on the circle or of finite order. Naghmouchi [31] and Blokh [6] characterized the structures of pointwise recurrent maps on uniquely arcwise connected curves. Kolev and Pérouème [20] showed recurrent homeomorphisms on compact surfaces with negative Euler characteristic are of finite order. Foland [14] proved that any equicontinuous homeomorphism on a closed 2-cell is topologically conjugate either to a reflection of a disk in a diameter or to a rotation of a disk about its center. Ritter [35] further determined the structure of equicontinuous homeomorphisms on the 2-sphere and annulus. Oversteegen and Tymchatyn [33] proved that recurrent homeomorphisms on the plane are periodic.

The notion of distality was introduced by Hilbert for better understanding equicontinuity [12]. The study of minimal distal systems culminates in the beautiful structure theorem of Furstenberg [16], which describes completely the relations between distality and equicontinuity for minimal systems. Considering minimal distal actions on compact manifolds, Rees [34] proved a sharpening of Furstenberg's structure theorem.

The aim of the paper is to study the structure of distal homeomorphisms on annulus without periodic points. One may consult [3, 7, 15, 17] for many interesting related investigations.

We obtain the following theorem.

Theorem 1.1 *Let A be an annulus in the plane $\mathbb{R}^2$ and $g : A \to A$ be a boundary components preserving homeomorphism which is distal and has no periodic points. Then there is a continuous decomposition of A into g-invariant circles such that all the restrictions of g on them share a common irrational rotation number and all these circles are linearly ordered by the inclusion relation on the sets of bounded components of their complements in $\mathbb{R}^2$.*

In [3], the authors show that some conjugacies of an irrational pseudo-rotation on the annulus can approximate to an irrational rotation. Clearly, the hypotheses about the homeomorphism in our setting are much stronger than in [3]. However, even under these stronger hypotheses we cannot hope the homeomorphism in Theorem 1.1 can be conjugate to an irrational rotation. This is the so-called linearization. We will show this is equivalent to the existence of a transversal. Here, a transversal is an arc in the annulus that intersects each minimal circle exactly once. It is clear that if g is conjugate to an irrational rotation then there is a transversal. The following theorem shows that the other direction holds as well.

Theorem 1.2 *Let A be an annulus in the plane $\mathbb{R}^2$ and $g : A \to A$ be a boundary components preserving homeomorphism which is distal and has no periodic points. If there is a transversal, then g is conjugate to an irrational rotation.*

Finally, we will construct examples that cannot be linearized.

Theorem 1.3 *For each irrational number $\alpha \in (0, 1)$, there is a distal homeomorphism g as in Theorem 1.1 which has rotation number α and cannot be linearized.*

The paper is organized as follows. In Sect. 2, we will introduce some concepts and facts in the theories of dynamical system and topology. Specially, we will give the definitions of solenoid and adding machine from the viewpoint of topological groups and recall some results around distal homeomorphisms. In Sect. 3, we will show that there exists no adding machine contained in the boundary of an f-invariant open disk under some appropriate assumptions.

Based on this result, we show in Sect. 4 the existence of an f-invariant circle in the boundary just mentioned. In Sect. 5, we show further that there are sufficiently many f-invariant circles in the annulus. Relying on all these results, we show in Sect. 6 the existence of the expected decompositions. In Sect. 7, we show that g can be linearized assuming the existence of a transversal. In Sect. 8, examples that are not linearized are constructed.

2 Preliminaries

In this section, we will recall some notions, notations, and elementary facts in the theories of dynamical system and topology.

2.1 Recurrence, minimal sets, and factors

By a *dynamical system* we mean a pair (X, f), where X is a metric space and $f : X \to X$ is a homeomorphism. For $x \in X$, the *orbit* of x is the set $O(x, f) \equiv \{f^i(x) : i \in \mathbb{Z}\}$. If there is some $n > 0$ such that $f^n(x) = x$, then x is called a *periodic point* of f and the minimal such n is called the *period* of x. A periodic point x of period 1 is called a *fixed point*, that is $f(x) = x$. A subset A of $\mathbb{Z}$ is *syndetic* if there is $l > 0$ with $A \cap \{p, p+1, \ldots, p+l\} \neq \emptyset$ for any $p \in \mathbb{Z}$. We call x an *almost periodic point* if for any open neighborhood U of x, the set $N(x, U) \equiv \{i \in \mathbb{Z} : f^i(x) \in U\}$ is syndetic. If there is a sequence of positive integers $n_1 < n_2 < \cdots$ such that $f^{n_i}(x) \to x$, then we call x a *recurrent point*; and if for any open neighborhood U of x, there always exists $n > 0$ such that $f^n(U) \cap U \neq \emptyset$, then we call x a *nonwandering point*. Clearly, an almost periodic point is recurrent and a recurrent point is nonwandering. If each point of X is nonwandering, then we call f *nonwandering*.

A subset S of X is *f-invariant* if $f(S) = S$; we use $f|_S$ to denote the restriction of f to S. If S is an f-invariant nonempty closed subset of X and contains no proper f-invariant closed subset, then we call S a *minimal set* of f. If X is a minimal set, we call the system (X, f) is *minimal*. It is clear from the definition that S is minimal if and only if for each $x \in S$, $O(x, f)$ is dense in S. By an argument of Zorn's lemma, we have that if X is compact, then there always exists a minimal set of f. We have known that each point of a compact minimal set is almost periodic (see e.g. [1, Chap.1-Theorem 1]).

For any two dynamical systems (X, f) and (Y, g), if there is a continuous surjection $\phi : X \to Y$ such that $\phi \circ f = g \circ \phi$, then we say that (Y, g) is a *factor* of (X, f) and (X, f) is an *extension* of (Y, g); we call ϕ a *factor map* or a *semiconjugation* between (X, f) and (Y, g); if ϕ is a homeomorphism, then we call (X, f) and (Y, g) are *topologically conjugate*. Clearly, if M is a minimal set of f, then $\phi(M)$ is a minimal set of g. It is well known that if $\mathbb{S}^1$ is the unit circle and $f : \mathbb{S}^1 \to \mathbb{S}^1$ is an orientation preserving homeomorphism without periodic points, then $(\mathbb{S}^1, f)$ is semiconjugate to a rigid minimal rotation on $\mathbb{S}^1$ (see e.g. [38, Theorem 6.18]).

A topological space U is called an *open disk* if it is homeomorphic to the unit open disk in the plane $\mathbb{R}^2$. By the Riemann mapping theorem, we know that a connected open subset U of $\mathbb{C}$ is an open disk if and only if it is simply connected.

The following theorem is implied by Brouwer's lemma (see e.g. [13, 23]).

Theorem 2.1 *If U is an open disk and $f : U \to U$ is an orientation-preserving nonwandering homeomorphism, then f has a fixed point in U.*

2.2 Continuous and semi-continuous decompositions

Let $(X, \mathscr{T})$ be a topological space. A *partition* of X is a collection $\mathscr{D}$ of nonempty, mutually disjoint subsets of X such that $\cup\mathscr{D} = X$. Define $\pi : X \to \mathscr{D}$ by letting $\pi(x)$ be the unique $D \in \mathscr{D}$ such that $x \in D$ for each $x \in X$. We endow $\mathscr{D}$ with the largest topology so that π is continuous, that is $\mathscr{U} \subset \mathscr{D}$ is open iff $\cup\mathscr{U} \in \mathscr{T}$. The topological space $\mathscr{D}$ so defined is called the *decomposition* of X. We also call $\mathscr{D}$ the *quotient space* of X by identifying each element of $\mathscr{D}$ into a point and call π the *quotient map*. The partition $\mathscr{D}$ is called *upper semi-continuous* provided that whenever $D \in \mathscr{D}$, $U \in \mathscr{T}$, and $D \subset U$, there exists $V \in \mathscr{T}$ with $D \subset V$ such that if $A \in \mathscr{D}$ and $A \cap V \neq \emptyset$, then $A \subset U$.

Now suppose X is a compact metric space with metric d. Let 2^X be the collection of all nonempty closed subsets of X and let $C(X) = \{A \in 2^X : A \text{ is connected}\}$. The *Hausdorff metric* H_d on 2^X is defined by $H_d(A, B) = \inf\{\epsilon : A \subset B_d(B, \epsilon) \text{ and } B \subset B_d(A, \epsilon)\}$ for each $A, B \in 2^X$. Then 2^X and $C(X)$ are both compact metric spaces with respect to H_d, called the *hyperspaces* of X (see e.g. [30, Theorems 4.13 and 4.17]). Let $\mathscr{D}$ be a partition of X such that each element of $\mathscr{D}$ is closed. The partition $\mathscr{D}$ is called *continuous* if the quotient map $\pi : X \to \mathscr{D}$, thought of as a map from X into 2^X, is continuous.

If $\mathscr{D}$ is a partition of X, then it induces an equivalence relation $R \subset X \times X$ by defining $(x, y) \in R$ if $\{x, y\} \subset A$ for some $A \in \mathscr{D}$. The relation R is called a *closed relation* if it is a closed subset of $X \times X$.

The following proposition can be seen in [18, Proposition 2.2].

Proposition 2.2 *Let X be a compact metric space. Let $\mathscr{D}$ be a partition of X and R be the equivalence relation induced by $\mathscr{D}$. If R is closed, then $\mathscr{D}$ is upper semi-continuous.*

Proposition 2.2 together with the compactness of hyperspaces implies the following proposition.

Proposition 2.3 *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism and $M \subset \mathbb{R}^2$ be a compact minimal set of f. Let $\mathscr{M}$ be the set of all components of M and let $\mathscr{M}' = \{\{x\} : x \in \mathbb{R}^2 \setminus M\}$. Then $\mathscr{M} \cup \mathscr{M}'$ is an upper semi-continuous decomposition of $\mathbb{R}^2$.*

A *continuum* is a connected compact metric space. If X is a continuum contained in the plane $\mathbb{R}^2$ such that $\mathbb{R}^2 \setminus X$ is connected, then we call that X *does not separate the plane*. The following theorem is due to R. L. Moore (see e.g. [22, p.533, Theorem 8] for a slightly more general form).

Theorem 2.4 *The space of an upper semi-continuous decomposition of $\mathbb{R}^2$ into continua, which do not separate $\mathbb{R}^2$, is homeomorphic to $\mathbb{R}^2$.*

2.3 Topological dimension

In this paper, the dimension of a metric space is referred to the covering dimension. We recall the definition here and the readers can find more details in [32].

Let X be a metric space. For a finite open covering $\mathscr{U} = \{U_1, \ldots, U_n\}$ of X, the order of $\mathscr{U}$ is defined by

$$\operatorname{ord}(\mathscr{U}) = \max\{k : \exists 1 \leq i_1 < i_2 < \cdots < i_k \leq n \ s.t.\ U_{i_1} \cap \cdots \cap U_{i_k} \neq \emptyset\}.$$

Now the *covering dimension* of X, or simply the dimension of X, denoted by $\dim X$, is at most n if every finite open covering of X can be refined by a finite open covering whose order is at most $n+1$. If $\dim X \leq n$ and the statement $\dim X \leq n-1$ is false, then we say $\dim X = n$.

We list some known results used later.

Lemma 2.5 [36, Theorem 19] *Let X be a compact subset of $\mathbb{R}^n$. Then* $\dim X < n$ *if and only if the interior of X in $\mathbb{R}^n$ is empty.*

Lemma 2.6 [32, Remark 27-9] *Let $X = \varprojlim (X_\alpha)_{\alpha \in I}$ be an inverse limit of compact metric space. Then*

$$\dim X \leq \limsup_{\alpha \in I} \dim X_\alpha.$$

Lemma 2.7 [27, Corollary 2] *Let G be a locally compact group and H is closed subgroup of G. Then*

$$\dim G = \dim H + \dim G/H.$$

2.4 Solenoids and adding machines

Let X be a compact metric space with metric d and $f : X \to X$ be a homeomorphism. We say (X, f) is *equicontinuous* if for each $\epsilon > 0$, there is a $\delta > 0$ such that $d(f^i(x), f^i(y)) < \epsilon$ for any $i \in \mathbb{Z}$, whenever $d(x, y) < \delta$. Let K be a compact abelian metric group and $a \in K$. The *rotation* $\rho_a : K \to K$ is defined by $\rho_a(x) = ax$ for any $x \in K$. Clearly, if $\{a^n : n \in \mathbb{Z}\}$ is dense in K, then (K, ρ_a) is minimal and equicontinuous.

The following theorem shows that minimal rotations on compact abelian metric groups are the only equicontinuous minimal systems (see [38, Theorem 5.18]).

Theorem 2.8 (Halmos-von Neumann) *Let X be a compact metric space and $f : X \to X$ be a minimal and equicontinuous homeomorphism. Then (X, f) is topologically conjugate to a minimal rotation on a compact abelian metric group.*

For each positive integer i, let K_i be a compact metric group and let $f_i : K_{i+1} \to K_i$ be a surjective continuous group homomorphism. The *inverse limit* of $\{K_i, f_i\}$ is

$$\varprojlim \{K_i, f_i\} \equiv \left\{ (x_i) \in \prod_{i=1}^{\infty} K_i : x_i = f_i(x_{i+1}) \right\},$$

which is a compact metric group under the multiplication "$\cdot$" defined by $(x_i) \cdot (y_i) = (x_i y_i)$. If each K_i is the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ and $\varprojlim\{K_i, f_i\}$ is not the circle, then we call $\varprojlim\{K_i, f_i\}$ a *solenoid*; if each K_i is a finite cyclic group and $\varprojlim\{K_i, f_i\}$ is not finite, then we call $\varprojlim\{K_i, f_i\}$ an *adding machine*. As topological spaces, a solenoid is a homogeneous indecomposable circle-like continuum and an adding machine is a Cantor set. We also call a minimal rotation on an adding machine an adding machine.

Since every compact metric group is an inverse limit of compact Lie groups (see e.g. [29, Chap. 4.6-4.7]), and the only connected Lie groups of dimension 1 is the circle group and the only Lie groups of dimension 0 are finite groups, the following proposition is clear.

Proposition 2.9 *If K is a connected compact metric group of dimension* 1*, then it is either a circle or a solenoid; if K is a compact metric group of dimension* 0 *and has a dense cyclic subgroup, then it is either a finite cyclic group or an adding machine.*

Proof Let K be a compact metric group. By the above arguments, we can write K as an inverse limit of compact Lie groups, i.e., $K = \varprojlim L_i$. Then each L_i is a quotient of K. By Lemma 2.7, one has $\dim L_i \le \dim K$. On the other hand, it follows from Lemma 2.6 that $\dim K \le \limsup \dim L_i$. WLOG, we may assume that $\dim L_i = \dim K$ for each i.

Now assume that K is a compact connected metric group of dimension 1. Then each L_i is a compact connected Lie group of dimension 1. Thus each L_i is just the circle. This shows that in this case K is either a circle or a solenoid.

Assume that K is a compact metric group of dimension 0 and has a dense cyclic subgroup. Then each L_i is a finite Lie group and has a dense cyclic subgroup. Thus each L_i is a finite cyclic group. This implies that K is either a finite cyclic group or an adding machine. □

A *curve* is an 1-dimensional continuum. The following corollaries are immediate from Theorem 2.8 and Proposition 2.9.

Corollary 2.10 *Let X be a curve and $f : X \to X$ be a minimal equicontinuous homeomorphism. Then (X, f) is topologically conjugate to a minimal rotation either on the circle or on a solenoid.*

Corollary 2.11 *Let X be a compact metric space of dimension 0 and $f : X \to X$ be a minimal equicontinuous homeomorphism. Then (X, f) is either a periodic orbit or topologically conjugate to an adding machine.*

Proof By Theorem 2.8, (X, f) is conjugate to a minimal rotation on some compact abelian metric group. Thus we may assume that X is a compact metric group of dimension 0. Since (X, f) is minimal, X has a dense cyclic subgroup. Then X satisfies the second condition in Proposition 2.9 and the assertion is followed. □

The following proposition is shown by Bing [4], which is also implied by the main result in [19].

Proposition 2.12 *Solenoids are not planar continua.*

We call x in a system (X, f) *regularly recurrent* if for any open neighborhood U of x, there is a positive integer n such that $f^{kn}(x) \in U$ for each $k = 0, 1, \ldots$.

The following proposition is implied by the definition (see [5] for a characterization of adding machine using regular recurrence).

Proposition 2.13 *If (X, f) is an adding machine, then each point x of X is regularly recurrent.*

2.5 Structures of distal homeomorphisms

Let X be a compact metric space with metric d and let $f : X \to X$ be a homeomorphism. We call that (X, f) is *distal* if for any $x \ne y \in X$, $\inf_{i\in\mathbb{Z}}\{d(f^i(x), f^i(y))\} > 0$. We suggest the readers to consult [1] for the proofs of the following well known facts: (1) If (X, f) is distal, then X is a disjoint union of minimal sets; (2) If (X, f) is minimal and distal and (Y, g) is a factor of (X, f), then (Y, g) is also minimal and distal; (3) Let (X, f) be a minimal distal system. Then it has a maximal equicontinuous factor.

Lemma 2.14 [34, §6] *Let (X, f) be a minimal distal system and $\pi : (X, f) \to (Y, g)$ be a factor map. Then the covering dimension of the fibers $\pi^{-1}(y)$, $y \in Y$, is constant and*

$$\dim(X) = \dim(Y) + \dim \pi^{-1}(y).$$

Lemma 2.15 [8, p.192, Theorem 3·17.13] *Let (X, f) be a minimal distal system and $\pi : (X, f) \to (Y, g)$ be a factor map. If (Y, g) is equicontinuous and there is some $y \in Y$ with $\dim \pi^{-1}(y) = 0$, then (X, f) is also equicontinuous.*

Clearly, Lemma 2.15 implies a distal minimal system of zero dimension is equicontinuous. In fact, this is also true for non-minimal distal systems of zero dimension (see [2, Corollary 1.9]).

Proposition 2.16 *Let X be a compact connected metric space and $f : X \to X$ be a minimal distal homeomorphism. If $\dim(X) = 1$, then (X, f) is equicontinuous.*

Proof Suppose that (X, f) is not equicontinuous. Then the maximal equicontinuous factor of (X, f) is nontrivial. Let $\pi : (X, f) \to (Y, g)$ be the factor map to its maximal equicontinuous factor. In particular, Y is connected. By Lemma 2.14, we have $\dim(Y) = 1$ and for each $y \in Y$, $\dim \pi^{-1}(y) = 0$. Then it follows from Lemma 2.15 that (X, f) is equicontinuous. This contradiction shows that (X, f) is equicontinuous. □

3 Nonexistence of an adding machine in the boundary of an open disk

We call a topological space X is an *arc* (resp. *open arc*) if it is homeomorphic to the closed interval $[0, 1]$ (resp. the open interval $(0, 1)$). We call X a *circle* if it is homeomorphic to the unit circle in the plane, that is X is a simple closed curve.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation preserving homeomorphism and $U \subset \mathbb{R}^2$ be a bounded f-invariant open disk. A *cross-cut* of U is an open arc γ in U with $\overline{\gamma}$ being an arc joining two points of ∂U. *Cross-sections* of U are connected components of $U \setminus \gamma$ for some cross cut γ of U. A *chain* for U is a sequence of sections $\mathscr{C} = (D_i)_{i=1}^{\infty}$ such that $D_1 \supset D_2 \supset \cdots$ and $\overline{\partial_U D_i} \cap \overline{\partial_U D_i} = \emptyset$ for all $i \neq j$. Two chains $(D_i)_{i=1}^{\infty}$ and $(D'_i)_{i=1}^{\infty}$ are called *equivalent* if for any $i > 0$ there is $j > i$ such that $D_j \subset D'_i$ and $D'_j \subset D_i$. A chain $(D_i)_{i=1}^{\infty}$ is called a *prime chain* if $\mathrm{diam}(\partial_U D_i) \to 0$. An equivalence class of prime chains is called a *prime end* of U. We use $b_{\mathscr{E}}(U)$ to denote the set of all prime ends of U. Let $\hat{U} = U \cup b_{\mathscr{E}}(U)$.

Now we topologize $\hat{U}$ as follows. For a cross-section D of U and for a prime chain (U_i) representing $p \in b_{\mathscr{E}}(U)$, if $U_i \subset D$ for sufficiently large i, then we call p *divides* D. Set $\mathscr{E}(D) = \{p \in b_{\mathscr{E}}(U) : p \text{ divides } D\}$. Consider the family $\mathscr{B}$ consisting of all sets of the form $D \cup \mathscr{E}(D)$ for some cross-section D, together with all open subsets of U. Then $\mathscr{B}$ is a topological basis on $\hat{U}$. We endow $\hat{U}$ with the topology generated by $\mathscr{B}$.

The following theorem is known as the Carathéodory's prime ends compactification theorem (see [9, 10]).

Theorem 3.1 (Prime ends compactification) *$\hat{U}$ is homeomorphic to the unit closed disk and $b_{\mathscr{E}}(U)$ is homeomorphic to the unite circle $\mathbb{S}^1$.*

It is well known that the homeomorphism $f|_U$ can be extended to a homeomorphism $\hat{f} : \hat{U} \to \hat{U}$. We call the rotation number of $\hat{f}|_{\mathbb{S}^1}$ the *prime ends rotation number* of $f|_{\overline{U}}$.

The following theorem is due to Cartwright and Littlewood [11]. One may consult [21] for the proof of the converse direction under more general settings.

Theorem 3.2 *If f is nonwandering and has no periodic point in ∂U, then the prime ends rotation number of $f|_{\overline{U}}$ is irrational.*

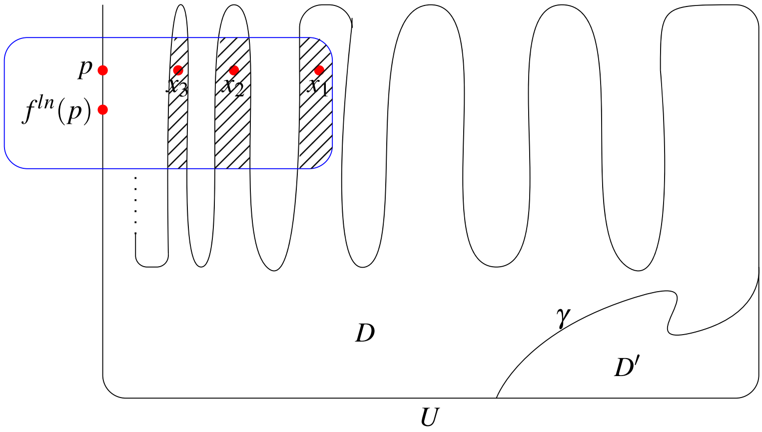

Fig. 1 Regularly recurrent point in ∂U

Now we use Theorem 3.2 to prove a key result. We remark here that in this section we mainly shows that the boundary of an invariant disc is essentially an irrational circle rotation. First note that by Lemma 2.5 that dim $\partial U \leq 1$. Further, dim $\partial U = 1$ since it is connected and nondegenerate. Also, every minimal set of ∂U is of dimension 0 or 1.

Proposition 3.3 *If the prime ends rotation number of $f|_{\overline{U}}$ is irrational, then no minimal set in ∂U is an adding machine.*

Proof Assume to the contrary that there is a minimal set $K \subset \partial U$, which is an adding machine. Fix $p \in K$. Since the rotation number of $\hat{f}|_{b_{\mathscr{E}}(U)}$ is irrational, $\hat{f}|_{b_{\mathscr{E}}(U)}$ is semi-conjugate to an irrational rotation on the unit circle. Then we take a cross-cut γ of U such that for each cross-section D of γ, $\mathscr{E}(D)$ contains the closure of a wandering interval (if any) of $\hat{f}|_{b_{\mathscr{E}}(U)}$ and such that p is not an endpoint of γ in $\overline{U}$. Take a sufficiently small $\epsilon > 0$ such that $V \equiv B(p, \epsilon) \cap U$ is contained in a cross-section D of γ. Let D' be the cross-section of γ other than D. By Proposition 2.13, there is some positive integer n such that

$$f^{kn}(p) \in B(p, \epsilon) \tag{3.1}$$

for all $k \geq 0$. Take a sequence (x_i) in V such that $x_i \to p$. Passing to a subsequence if necessary, we suppose $x_i \to q \in b_{\mathscr{E}}(U) \subset \hat{U}$. So, there is some $l > 0$ such that

$$\hat{f}^{ln}(q) \in \mathscr{E}\left(D'\right). \tag{3.2}$$

Then for sufficiently large i, by Eqs. (3.1) and (3.2), and by the continuity, we have both $f^{ln}(x_i) \in V \subset D$ and $f^{ln}(x_i) = \hat{f}^{ln}(x_i) \in D'$ (see Figs. 1 and 2). This is a contradiction. □

4 Existence of an *f*-invariant circle in the boundary of an open disk

Proposition 4.1 *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a nonwandering homeomorphism and let U be an f-invariant open disk. If $f|_{\partial U} : \partial U \to \partial U$ is distal and f has no periodic points except for an only fixed point $O \in U$, then there is an f-invariant circle C in ∂U such that $(C, f|_C)$ is minimal and O belongs to the bounded component of $\mathbb{R}^2 \setminus C$.*

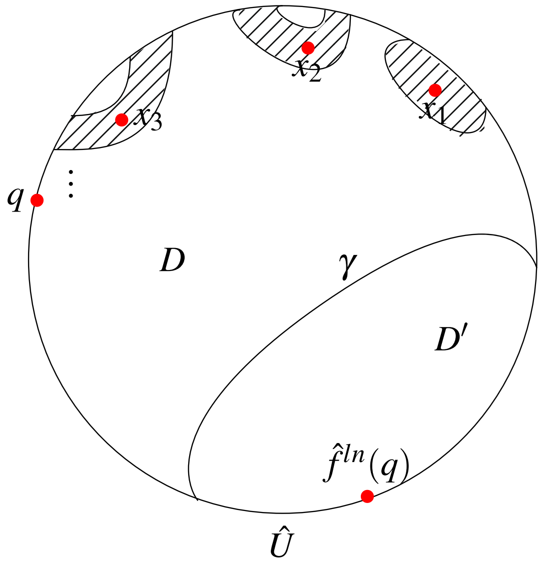

Fig. 2 Regularly recurrent point in the prime end

Proof Fix a minimal set M in ∂U. Then it follows from Lemma 2.5 that $\dim(M) \leq 1$. Noting that f is nonwandering and has no periodic point in ∂U, by Theorem 3.2, we have that the prime end rotation number of $f|_{\overline{U}}$ is irrational. Then it follows from Corollary 2.11 and Proposition 3.3 that M is not an adding machine. So $\dim(M) = 1$. Thus M has a component K with $\dim(K) = 1$.

Now we discuss into several cases:

Case 1. There is some $n \geq 1$ such that $f^n(K) = K$ and $f^i(K) \cap K = \emptyset$ for $1 \leq i < n$. Clearly, (K, f^n) is minimal. Then, from Proposition 2.16, it is equicontinuous. By Corollary 2.10 and Proposition 2.12, K is a circle.

Subcase 1.1. $n = 1$. Let $C = K$ and let D be the bounded component of $\mathbb{R}^2 \setminus C$. Since f has no periodic points in C, so by Brouwer's fixed point theorem, $O \in D$. Thus C satisfies the requirement.

Subcase 1.2. $n > 1$. Let $C_i = f^i(K)$, $i = 0, \ldots, n-1$. Then C_i are pairwise disjoint. Let D_i be the bounded component of $\mathbb{R}^2 \setminus C_i$. If there are $i \neq j$ such that $C_i \subset D_j$, then $f^{i-j}(\overline{D}_j) \subset D_j$. This contradicts the assumption that f is nonwandering. So these D_i are pairwise disjoint. Since each D_i contains a fixed point of f^n by Brouwer's fixed point theorem, this contradicts the assumption that O is the only periodic points of f. So this subcase does not occur.

Case 2. $f^i(K)$, $i \in \mathbb{Z}$, are pairwise disjoint. Write $K_i = f^i(K)$. If K separates the plane, then $\mathbb{R}^2 \setminus K$ has a bounded component, so is each K_i. Similar to the arguments in Subcase 1.2, we have that for any $i \neq j$, K_i is contained in the unbounded component of $\mathbb{R}^2 \setminus K_j$. Thus any bounded component of $\mathbb{R}^2 \setminus K$ is a wandering open set of f. This is a contradiction.

From the above discussions, we get the following claim.

Claim A. Either the conclusion of Proposition 4.1 holds, or M has infinitely many components and any nondegenerate component of M does not separate the plane.

If the conclusion of Proposition 4.1 does not hold, then by Claim A together with Proposition 2.3 and Theorem 2.4, we get a factor $g : \mathbb{R}^2 \to \mathbb{R}^2$ of f by identifying each component of M to a point. Let $\pi : \mathbb{R}^2 \to \mathbb{R}^2$ be the factor map. Then $\pi(U)$ is a g-invariant open disk and g is a nonwandering homeomorphism and has no periodic point in $\partial\pi(U)$. So, by Theorem 3.2, the prime ends rotation number of $g|_{\pi(\overline{U})}$ is irrational. Noting that $g|_{\partial\pi(U)}$ is still distal and $\pi(M)$ is totally disconnected and infinite, by Corollary 2.11, we see that $(\pi(M), g)$ is an adding machine contained in the boundary of $\pi(U)$. This contradicts Proposition 3.3.

All together, we complete the proof. □

5 Existence of an intermediate *f*-invariant circle

Lemma 5.1 *Let X be a compact metric space and let $f : X \to X$ be a distal homeomorphism. If K is an f-invariant proper closed subset of X, then there are $\delta > 0$ and a nonempty open subset U of X such that $f^i(U) \cap B(K, \delta) = \emptyset$ for each $i \in \mathbb{Z}$.*

Proof For each positive integer n, let $V_n = \{x \in X : f^i(x) \in B(K, \frac{1}{n}) \text{ for some } i \in \mathbb{Z}\}$. If the conclusion of Lemma 5.1 does not hold, then V_n is a dense open subset of X for each n. Thus by Baire's Theorem, $G \equiv \cap_{n=0}^{\infty} V_n$ is a dense G_δ-set. Take $x \in G \setminus K$. Then $\overline{O(x, f)} \cap K \neq \emptyset$. This contradicts the minimality of $\overline{O(x, f)}$. □

For any two circles C, C' in the plane $\mathbb{R}^2$, write $C \prec C'$ if C is contained in the bounded component of $\mathbb{R}^2 \setminus C'$.

Proposition 5.2 *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation preserving nonwandering homeomorphism. Let A be an f-invariant annulus with two boundary circles $C_1 \prec C_2$. Suppose $f|_A$ is distal and f has no periodic points except for an only fixed point O in the bounded component of $\mathbb{R}^2 \setminus C_1$. Then there is an f-invariant circle C with $C_1 \prec C \prec C_2$.*

Proof By Lemma 5.1, we can take $\delta > 0$ and a nonempty open set U in A such that $f^i(U) \cap B(C_1 \cup C_2, \delta) = \emptyset$ for each $i \in \mathbb{Z}$. Set $W = \cup_{i \in \mathbb{Z}} f^i(U)$. Let K be the unbounded component of $\mathbb{R}^2 \setminus W$. Then K is f-invariant and $C_2 \subset \overset{\circ}{K}$. Let V be a component of $\mathbb{R}^2 \setminus K$. (See Fig. 3) Then V is an open disk by a direct application of Jordan separation theorem. Since f is nonwandering, there is some $n \geq 0$ with $f^n(V) = V$. Then, by Theorem 2.1, there is a periodic point of f^n in V, so is for f. Thus by the assumption, we have $O \in V$. This implies $C_1 \subset V$. From the above discussions, we see that V is the only component of $\mathbb{R}^2 \setminus K$, and hence $f(V) = V$. Now applying Proposition 4.1, we get the required circle. □

6 A decomposition of the annulus into *f*-invariant circles

In this section, we will complete the proof of the main Theorem. All assumptions are as in Theorem 1.1. WLOG, we may assume the annulus $A = \{z \in \mathbb{C} : 1 \leq |z| \leq 2\}$. Then we extend $g : A \to A$ to $f : \mathbb{C} \to \mathbb{C}$ by defining

$$f(x) = \begin{cases} \frac{|x|}{2} \cdot g\left(\frac{2x}{|x|}\right), & |x| > 2; \\ g(x), & 1 \leq |x| \leq 2; \\ |x| \cdot g\left(\frac{x}{|x|}\right), & 0 < |x| < 1; \\ 0, & |x| = 0. \end{cases}$$

It is clear from the definition that f is an orientation preserving nonwandering homeomorphism on the plane and has no periodic points except for the only fixed point 0.

Write $C_1 = \{z : |z| = 1\}$ and $C_2 = \{z : |z| = 2\}$. For each circle C in the plane, we use $D(C)$ and $OD(C)$ to denote the bounded component and unbounded component of $\mathbb{R}^2 \setminus C$, respectively. Let $\prec$ be the transitive order defined in Section 5; that is, for circles C and C' in the plane, $C \prec C'$ iff $C \subset D(C')$. Let

$$\mathscr{C} = \{C : C \text{ is an } f\text{-invariant circle and } C_1 \prec C \prec C_2\} \cup \{C_1, C_2\}.$$

Let $\mathscr{T}$ be the family of all chains of $\mathscr{C}$ respect to $\prec$. Then $\mathscr{T}$ is a partial set with respect to the inclusion relation on the power set of $\mathbb{R}^2$. Using Zorn's lemma, there is a maximal chain $\mathscr{P}$ in $\mathscr{T}$.
Claim A. $\mathscr{P}$ is a partition of A.

Proof of Claim A Assume to the contrary that there is some $v \in A \setminus \cup\mathscr{P}$. Since $C_1, C_2 \in \mathscr{P}$ by the maximality of $\mathscr{P}$, we have $v \in \overset{\circ}{A}$. Set $\mathscr{P}_1 = \{C \in \mathscr{P} : v \notin D(C)\}$ and $\mathscr{P}_2 = \{C \in \mathscr{P} : v \in D(C)\}$. Then $C_1 \in \mathscr{P}_1$ and $C_2 \in \mathscr{P}_2$.

Now we discuss into several cases.

Case 1. $\mathscr{P}_1$ has no maximal element. Let $U = \cup_{C\in\mathscr{P}_1} D(C)$. Then U is an f-invariant open disk. From Proposition 4.1, we have an f-invariant circle C_3 in ∂U with $0 \in D(C_3)$. Clearly, $C_3 \notin \mathscr{P}$ and $C \prec C_3$ for any $C \in \mathscr{P}_1$. If $C_3 \prec C$ for any $C \in \mathscr{P}_2$, then $\{C_3\} \cup \mathscr{P}$ is a chain, which contradicts the maximality of $\mathscr{P}$. So, there must exit a $C_4 \in \mathscr{P}_2$ such that $C_3 \cap C_4 \neq \emptyset$. Let V be a component of $D(C_4) \setminus \overline{D(C_3)}$. Then V is a component of $\mathbb{R}^2 \setminus (C_3 \cup C_4)$, and hence it is an open disk. Noting that f is nonwandering, we have $f^n(V) = V$ for some $n \geq 0$. Then by Theorem 2.1, f has a periodic point in V. This is a contradiction. So, Case 1 does not happen.

Case 2. $\mathscr{P}_2$ has no minimal element. We consider the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Let $\mathbb{L} = \hat{\mathbb{C}} \setminus \{0\}$. Then $\mathbb{L}$ is a plane. Define $\hat{f} : \mathbb{L} \to \mathbb{L}$ by letting $\hat{f}(\infty) = \infty$ and $\hat{f}(x) = f(x)$ for any $x \in \mathbb{C} \setminus \{0\}$. Then $\hat{f}$ is an orientation preserving nonwandering homeomorphism on the plane $\mathbb{L}$ and has no periodic points except for the only fixed point ∞. Similar to the discussions in Case 1, we see that Case 2 does not happen.

Case 3. $\mathscr{P}_1$ has the maximal element C_5 and $\mathscr{P}_2$ has the minimal element C_6. Then $C_5 \prec C_6$. Applying Proposition 5.2, we get an f-invariant circle C_7 such that $C_5 \prec C_7 \prec C_6$. Then $\{C_7\} \cup \mathscr{P}$ is a chain. This contradicts the maximality of $\mathscr{P}$. Thus Case 3 does not occur.

So $A = \cup\mathscr{P}$; that is $\mathscr{P}$ is a partition of A (see Fig. 3). □

Claim B. $\prec$ is a dense and complete linear order on $\mathscr{P}$.

Proof of Claim B (1) Linearity. For any distinct $C, C' \in \mathscr{P}$, we have $C \cap C' = \emptyset$ and hence either $C \subset D(C')$ or $C' \subset D(C)$. This shows that either $C \prec C'$ or $C' \prec C$. Thus $\prec$ is a linear order.

(2) Density. Let $C, C' \in \mathscr{P}$ be with $C \prec C'$. Then we have $\overline{D(C)} \subsetneq D(C')$. Now for any $x \in D(C') \setminus \overline{D(C)}$, there is some $C'' \in \mathscr{P}$ such that $x \in C''$. It follows from the definition of $\prec$ that $C \prec C'' \prec C'$. This shows that $\prec$ is a dense order.

(3) Completeness. To the contrary, assume that $\prec$ is incomplete. That is there is a Dedekind Gap, which means that there is a partition $\mathscr{P} = \mathscr{L} \cup \mathscr{U}$ such that

- For any $C \in \mathscr{L}$ and $C' \in \mathscr{U}$, $C \prec C'$,
- $\mathscr{L}$ has a maximal element C^* and $\mathscr{U}$ has a minimal element C_*.

Then we have

$$\bigcup_{C\in\mathscr{L}} C = \overline{D(C^*)} \setminus D(C_1) \text{ and } \bigcup_{C\in\mathscr{U}} C = \overline{OD(C_*)} \setminus OD(C_2),$$

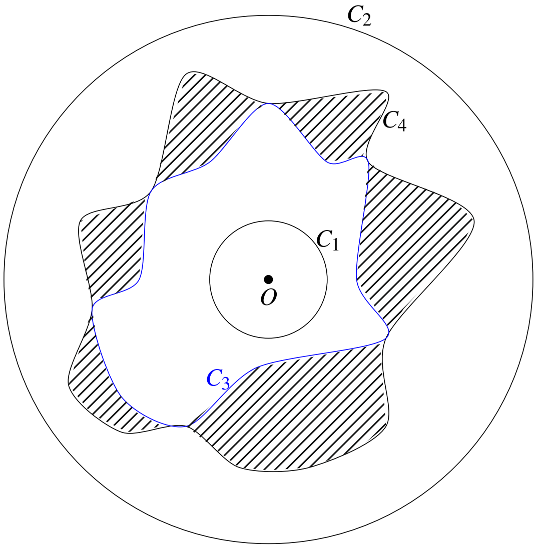

Fig. 3 Description of Case 1 in the proof of Claim A

both of which are closed in A. But this contradicts the connectedness of A. This shows that $\prec$ is complete. □

Now Claim B implies that $(\mathscr{P}, \prec)$ endowed with the ordering topology is homeomorphism to a closed interval. WLOG, we may assume that $(\mathscr{P}, \prec) \cong [1, 2]$ and use C_r to denote elements of $\mathscr{P}$ with $r \in [1, 2]$.

Recall that 2^A is the hyperspace of A endowed with the Hausdorff metric.

Claim C. $\mathscr{P}$ is closed in 2^A. Specially, $\mathscr{P}$ is a continuous decomposition.

Proof of Claim C Let C_{r_n} be a sequence in $\mathscr{P}$ that converges to K in 2^A under Hausdorff metric. Since $(\mathscr{P}, \prec) \cong [1, 2]$, by passing to some subsequence, we may assume that $C_{r_n} \xrightarrow{\prec} C_r$ under the ordering topology for some $r \in [1, 2]$.

Next we will show that $K = C_r$ which implies that $\mathscr{P}$ is closed in 2^A. We may assume as well $C_{r_n} \prec C_r$ for each n. Fix an $x \in K$. Then it follows from the definition of Hausdorff metric that there is a sequence (x_n) with $x_n \in C_{r_n}$ such that $x_n \to x$. Since $x_n \in C_{r_n} \subset D(C_r)$, we have $x \in \overline{D(C_r)}$. To show $x \in C_r$, we assume that $x \in D(C_r)$. Then there is some $s \in [1, r)$ such that $x \in C_s$. Since $r_n \to r$, we have $s < r_n \leq r$ and hence $C_s \prec C_{r_n} \preceq C_r$ for any sufficiently large n. But in this case, x_n cannot converge to x; this is a contradiction. To sum up, we have $x \in C_r$. Hence $K \subset C_r$ as x is arbitrary. On the other hand, for any $y \in C_r$, there is some subsequence (y_{n_i}) with $y_{n_i} \in C_{r_{n_i}}$ such that $y_{n_i} \to y$. Indeed, take any point $z \in C_1$ and let L be the segment connecting z and y in A. Then $L \cap C_{r_n} \neq \emptyset$ for each n and we choose some $y_n \in L \cap C_{r_n}$. It is clear that y is a limit point of (y_n). The above arguments show that $K = C_r$ and the closedness of $\mathscr{P}$ in 2^A is followed. □

For any orientation preserving homeomorphism ϕ on a circle, we use $\rho(\phi)$ to denote the rotation number of ϕ.

Claim D. The rotation numbers of $f|_C$, $C \in \mathscr{P}$, are the same irrational number.

Proof of Claim D Assume to contrary that there are $C' \neq C'' \in \mathscr{P}$ such that the rotation numbers $\rho(f|_{C'}) \neq \rho(f|_{C''})$. Let $\tilde{A}$ be the annulus in A with boundary $C' \cup C''$. Write

$\alpha' = \rho(f|_{C'})$ and $\alpha'' = \rho(f|_{C''})$. Take a homeomorphism $h : \tilde{A} \to \mathbb{S}^1 \times [0, 1]$ such that $h(C') = \mathbb{S}^1 \times \{0\}$ and $h(C'') = \mathbb{S}^1 \times \{1\}$. Let $\tilde{f} = hf|_A h^{-1}$. Then $\rho(\tilde{f}|_{h(C')}) = \alpha'$ and $\rho(\tilde{f}|_{h(C'')}) = \alpha''$. Let $\mathbb{R} \times [0, 1]$ be the universal covering of $\mathbb{S}^1 \times [0, 1]$ and let $F : \mathbb{R} \times [0, 1] \to \mathbb{R} \times [0, 1]$ be a lift of $\tilde{f}$. Noting that $\tilde{f}$ is homotopic to the identity, we have $TF = FT$, where T is the unit translation on $\mathbb{R} \times [0, 1]$ defined by $T(s, t) = (s + 1, t)$. Let $\pi : \mathbb{R} \times [0, 1] \to \mathbb{R}$ be the projection to the first coordinate, that is $\pi(s, t) = s$. For any map $\psi : \mathbb{R} \times [0, 1] \to \mathbb{R} \times [0, 1]$, write $\psi_1 = \pi\psi$. Then there are $m', m'' \in \mathbb{Z}$ such that

$$\lim_{n\to\infty} \frac{F_1^n(x, 0) - x}{n} = \alpha' + m'$$

and

$$\lim_{n\to\infty} \frac{F_1^n(x, 1) - x}{n} = \alpha'' + m''.$$

WLOG, suppose $\alpha' + m' < \alpha'' + m''$. Take a rational number $\frac{p}{q}$ with

$$\alpha' + m' < \frac{p}{q} < \alpha'' + m''.$$

Then we have

$$\lim_{n\to\infty} \frac{\left(T^{-p}F^q\right)_1^n (x, 0) - x}{n} = q\left(\alpha' + m'\right) - p < 0$$

and

$$\lim_{n\to\infty} \frac{\left(T^{-p}F^q\right)_1^n (x, 1) - x}{n} = q\left(\alpha'' + m''\right) - p > 0.$$

Since f is nonwandering, it follows from [15, Theorem 3.3] that $\tilde{f}^q$ has a fixed point in $\mathbb{S}^1 \times [0, 1]$. That is $\tilde{f}$ has a periodic point, so is f. This is a contradiction. So the rotation numbers of $f|_C$, $C \in \mathscr{P}$, are the same, the irrationality of which clearly follows from the Poincaré's classification theorem for circle homeomorphisms. □

All together, we complete the proof of Theorem 1.1.

7 Transversal implies linearization

In this section, we will show that if there is a transversal then g is conjugate to an irrational rotation, i.e., g can be linearized. Recall that $\mathscr{P}$ is the decomposition of the annulus into minimal circles. A transversal is an arc in the annulus that intersects each member in $\mathscr{P}$ exactly once.

Now suppose that γ is a transversal for $\mathscr{P}$. For each $n \in \mathbb{Z}$, let $L_n = g^n(\gamma)$.

Recall that the map $[1, 2] \to \mathscr{P}, \alpha \mapsto C_\alpha$ is continuous. We parametrize $\mathscr{P}$ with C_α : $\alpha \in [1, 2]$. For each $n \in \mathbb{Z}$ and $\alpha \in [1, 2]$, set $\{x_n^\alpha\} = L_n \cap C_\alpha$. Then it is clear that $\{x_n^\alpha\}_{n\in\mathbb{Z}}$ is dense in C_α. WLOG, we may assume that $C_{1.5} = \{z \in \mathbb{C} : |z| = 1.5\}$ and $g|_{C_{1.5}}$ is the rigid rotation.

Claim 1. For each $n \in \mathbb{Z}$, L_n is also a transversal for $\mathscr{P}$ and $L_m \cap L_n = \emptyset$ for any $m \neq n$.

Proof of Claim 1 Since g is a homeomorphism and each $C \in \mathscr{P}$ is g-invariant, we conclude that L_n is also a transversal.

Now suppose that there is some $x \in L_m \cap L_n$. Let $C \in \mathscr{P}$ be such that $x \in C$. Then we have $g^{-m}x, g^{-n}x \in C \cap L_0$. Thus $g^{-m}x = g^{-n}x$, since $|L_0 \cap C| = 1$. But this contradicts the minimality of $g\,|_C$. Thus $L_m \cap L_n = \emptyset$. □

It is easy to see that $\mathscr{P}\,|_{[L_m,L_n]}$ is a partition of $[L_m, L_n]$ for each $m \neq n \in \mathbb{Z}$, where $[L_m, L_n]$ is any region between L_m and L_n that is the closure any one of the components of $A \setminus (L_m \cup L_n)$. Precisely, for each $\alpha \in [1, 2]$, $[L_m, L_n] \cap C_\alpha$ is a curve joining x_m^α and x_n^α. We denote this curve by $C_{m,n}^\alpha$.
Claim 2. For each $m \neq n \in \mathbb{Z}$, $\{C_{m,n}^\alpha\}_{\alpha\in[1,2]}$ is a continuous decomposition of $[L_m, L_n]$.

Proof of Claim 2 It suffices to show that $\{C_{m,n}^\alpha\}_{\alpha\in[1,2]}$ is closed in 2^A. For this, suppose that $C_{m,n}^{\alpha_i} \to K$ in 2^A as $i \to \infty$. By passing to some subsequence, we may assume that $C_{\alpha_i} \overset{\prec}{\to} C_\alpha$. We have shown in previously that $C_{\alpha_i} \to C_\alpha$ under the Hausdorff topology. Thus $K \subset C_\alpha$. What remains to show is $K = C_{m,n}^\alpha$. Since $[L_m, L_n]$ is closed in A, it is clear that $K \subset C_{m,n}^\alpha$. On the other hand, it follows from the continuity of L_m and L_n that $x_m^{\alpha_i} \to x_m^\alpha$ and $x_n^{\alpha_i} \to x_n^\alpha$. Finally, note that K is connected. Thus $K = C_{m,n}^\alpha$ and we complete the proof. □

Claim 3. If $x_{n_i}^\alpha \to x$, then for each $\beta \in [1, 2]$, $x_{n_i}^\beta \to y_\beta$ for some $y_\beta \in C_\beta$.

Proof of Claim 3 Fix a $\beta \in [1, 2]$. To the contrary, assume that there are subsequence $\{k_i\}$ and $\{l_i\}$ of $\{n_i\}$ such that

$$x_{k_i}^\beta \to y', \quad x_{l_i}^\beta \to y'', \quad \text{with} \quad y' \neq y''.$$

There there are $a, b, c, d \in \mathbb{Z}$ such that there is a component (L_a, L_b) of $A \setminus (L_a \cup L_b)$ and a component (L_c, L_d) of $A \setminus (L_c \cup L_d)$ with $y' \in (L_a, L_b)$, $y'' \in (L_c, L_d)$ and $[L_a, L_b] \cap [L_c, L_d] = \emptyset$, where $[L_a, L_b] = \overline{(L_a, L_b)}$ and $[L_c, L_d] = \overline{(L_c, L_d)}$. Then both $[L_a, L_b] \cap \{x_{n_i}^\alpha\}$ and $[L_c, L_d] \cap \{x_{n_i}^\alpha\}$ are infinite. But this contradicts the convergence of $\{x_{n_i}^\alpha\}$. □

Now for each $x \in A$, let C_α be such that $x \in C_\alpha$ and take $x_{n_i}^\alpha \to x$. Then let

$$L_x := \{y : x_{n_i}^\beta \to y, \beta \in [1, 2]\}.$$

Claim 4. L_x is a transversal.

Proof of Claim 4 Clearly, $|L_x \cap C_\beta| = 1$ for any $\beta \in [1, 2]$. It remains to show that L_x is an arc. For this, it suffices to show that the map $[1, 2] \to L_x, \alpha \mapsto x_\alpha$ is continuous, where $\{x_\alpha\} = L_x \cap C_\alpha$.

Fix $\alpha \in (1, 2)$ and a neighborhood U of x_α in A. Further, we can take an open disc V around x_α contained in U. Take $m, n \in \mathbb{Z}$ such that $C_{m,n}^\alpha \subset V$. Then V is a neighborhood of $C_{m,n}^\alpha$. By Claim 2, there are $1 < \beta_1 < \alpha < \beta_2 < 2$ such that $C_{m,n}^\beta \subset V$ for each $\beta \in [\beta_1, \beta_2]$. In particular, $L_x \cap C_\beta \subset V$, for any $\beta \in [\beta_1, \beta_2]$. This shows that $\alpha \mapsto x_\alpha$ is continuous. □

Claim 5. The definition of L_x is independent of the choice of (n_i).

Proof of Claim 5 Suppose that $x_{n_i}^\alpha \to x$ and $x_{m_i}^\alpha \to x$. The we have to show that

$$\lim_{n_i\to\infty} x_{n_i}^\beta = \lim_{m_i\to\infty} x_{m_i}^\beta, \quad \forall\beta \in [1, 2].$$

Let (k_i) be the sequence by putting (n_i) and (m_i) together. Then we have $x_{k_i}^\alpha \to x$. By Claim 3, the sequence $(x_{k_i}^\beta)$ is convergent for each β. This implies our Claim. □

Claim 5 tells us that for each $y \in L_x$, we have $L_x = L_y$. Thus $L_x, x \in C_\alpha$ forms a decomposition of A. Actually, similar to the proof of the continuity of $\mathscr{P}$, we can also show that $L_x, x \in C_\alpha$ is a continuous decomposition.
Claim 6. For each $\alpha \in [1, 2]$, $(x_{n_i}^\alpha)$ converges in C_α if and only if $(n_i\theta)$ converges in $\mathbb{S}^1$.

Proof of Claim 6 This is followed from Claim 3 and our assumption that $C_{1.5} = \{z \in \mathbb{C} : |z| = 1.5\}$ and $g\ |_{C_{1.5}}$ is the rigid rotation. □

Now we are ready to show that g can be linearized. Let θ be the rotation number of g. Take a homeomorphism $\psi : L_0 \to P_0 := [1, 2] \times \{0\}$.
We define the conjugacy $\Psi : A \to A$ by

$$x \mapsto e^{2\pi i n\theta} \psi\left(g^{-n}x\right), \quad \text{for each} \quad x \in L_n, n \in \mathbb{Z},$$

and for $x = \lim x_{n_i}^\alpha$, define

$$\Psi(x) = \lim \Psi\left(x_{n_i}^\alpha\right).$$

Clearly, we have the following claim.
Claim 7. $\Psi g = R_\theta \Psi$.
In addition, it is also clear that Ψ is one-one. Thus it remains to show the continuity of Ψ. For this, suppose that $x_k \to x$ and we assume that $x_k \in C_{\alpha_k}$. Then we have $\alpha_k \to \alpha$ and $x \in C_\alpha$. We claim that $\lim \Psi(x_k) = \Psi(x)$. WLOG, we may assume that $\lim \Psi(x_k) = y$. For any neighborhood V of $\Psi(x)$, there are some $m, n \in \mathbb{Z}$ and $\beta_1 \prec \alpha \prec \beta_2$ such that $\Psi([\beta_1, \beta_2, L_m, L_n]) \subset V$, where $[\beta_1, \beta_2, L_m, L_n]$ is the closure of the component of $A \setminus (\beta_1 \cup \beta_2 \cup L_m \cup L_n)$ containing x. Since $x_k \to x$, $x_k \in [\beta_1, \beta_2, L_m, L_n]$ for all sufficiently large k. This shows that $\Psi(x_k) \to \Psi(x)$. Thus we have shown that g is conjugate to the rigid rotation by Ψ.
This completes the proof of Theorem 1.2.

8 Proof of Theorem 1.3

In this section, we will construct distal homeomorphisms without periodic points that cannot be linearized.

8.1 Some notions

Let $\mathbb{A} = \{z \in \mathbb{C} : 1 \le |z| \le 2\}$ be the standard closed annulus on the complex plane. Let $\widetilde{\mathbb{A}} = \{x+yi \in \mathbb{C} : 1 \le y \le 2\}$ be the universal cover of $\mathbb{A}$ and $\pi : \widetilde{\mathbb{A}} \to \mathbb{A} : x+yi \mapsto ye^{2\pi i x}$ be the covering map. Further, for each $r \in [1, 2]$, we denote

$$C_r = \{z \in \mathbb{C} : |z| = r\}.$$

Now for each point $z \in \mathbb{A}$, we use the polar coordinate (θ, r) to represent it with $\theta \in [0, 2\pi)$ and $1 \le r \le 2$. For each $\beta \in \mathbb{R}$, we define the rigid rotation on $\mathbb{A}$ with angle $2\pi\beta$ by

$$R_\beta : \mathbb{A} \to \mathbb{A}, z \mapsto ze^{2\pi i\beta}.$$

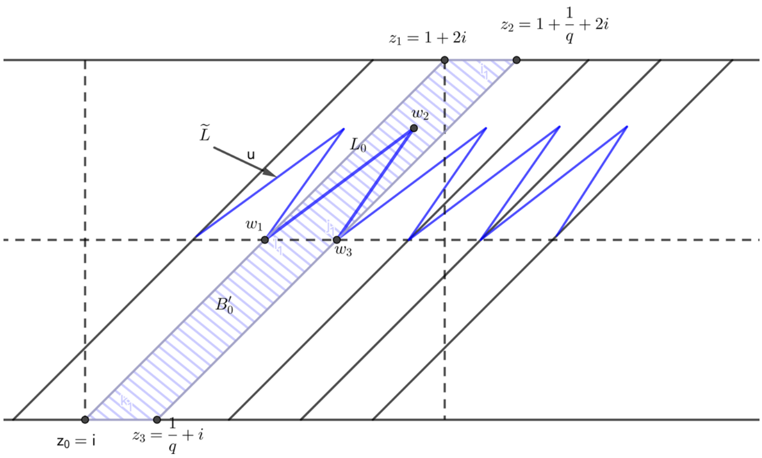

Fig. 4 Definition of $\widetilde{L}$

Let $\gamma : [0, 1] \to \mathbb{A}$ be an essential simple closed curve in $\mathbb{A}$. We define the *folding width* of γ to be the maximal $w > 0$ such that for each $\xi \in [0, 1]$, the circular sector

$$S_\xi := \{z \in \mathbb{C} : 1 \leq |z| \leq 2, 2\pi\xi \leq \arg(z) \leq 2\pi\xi + w\}$$

is separated by at least two disjoint subarcs contained in γ.

8.2 Construction of a periodic folding rotation

We fix $p, q \in \mathbb{N}$ with $(p, q) = 1$ and $q > 6$. We are going to construct a homeomorphism H on $\mathbb{A}$ such that

(1) $HR_{p/q}H^{-1} = R_{p/q}$;
(2) For any $z, z' \in \mathbb{A}$ with $|z| = |z'|$, one has

$$\left|\theta(H(z)) - \theta\left(H\left(z'\right)\right)\right| \leq 5q\left|\theta(z) - \theta\left(z'\right)\right|;$$

(3) The folding width of $H(C_{3/2})$ is not less than $\frac{\pi}{6}$.

Let $z_0 = i, z_1 = 1 + 2i, z_2 = 1 + \frac{1}{q} + 2i, z_3 = \frac{1}{q} + i$. Then let

$$w_1 = \frac{1}{2} + \frac{3}{2}i,\ w_2 = \frac{3}{4} + \frac{1}{2q} + \frac{7}{4}i,\ w_3 = \frac{1}{2} + \frac{1}{q} + \frac{3}{2}i.$$

Define L_0 be the union of the segment between w_1, w_2 and the segment between w_2, w_3. Then define $\tilde{L} = \bigcup_{n\in\mathbb{Z}}(L_0 + \frac{n}{q})$ and let $\widetilde{\Gamma} = \pi(\tilde{L})$ be the simple closed curve in $\mathbb{A}$ (see Fig. 4).

Claim 1. The folding width of $\widetilde{\Gamma}$ is not less than $\frac{\pi}{6}$.

Proof of Claim 1 It suffices to show that there are two disjoint segments in $\widetilde{L}$ linking the boundaries of the rectangle $B_\xi := \{z \in \mathbb{C} : \xi \leq Re(z) \leq \xi + 1/6, 1 \leq Im(z) \leq 2\}$.

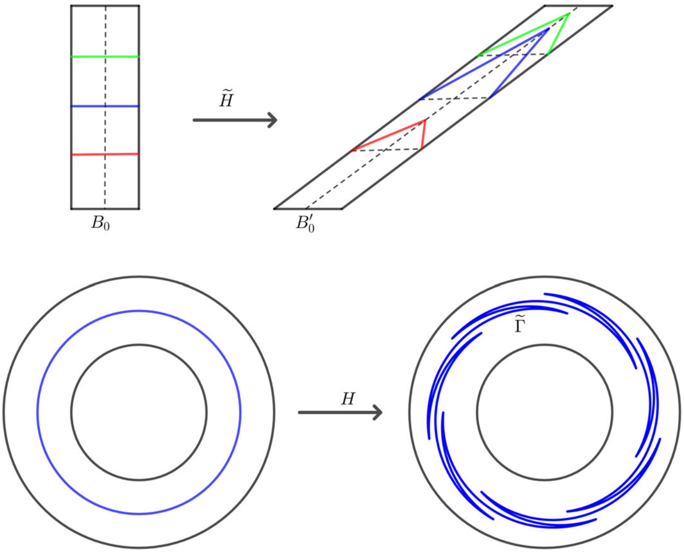

Fig. 5 Constructions of H

We assume that $\xi \in \left(\frac{j}{q}+\frac{1}{2}, \frac{j+1}{q}+\frac{1}{2}\right]$ with $j \in \{0, 1, \ldots, q-1\}$. We claim that $L_0+\frac{j-1}{q}$ and $L_0+\frac{j}{q}$ across B_ξ. Since $q > 6$, we have

$$\frac{1}{2}+\frac{j-1}{q} < \frac{j}{q}+\frac{1}{2} < \xi$$

and

$$\xi+\frac{1}{6} \leq \frac{j+1}{q}+\frac{1}{2}+\frac{1}{6} < \frac{3}{4}+\frac{j-1}{q}+\frac{1}{2q}.$$

Thus there is a segment between $\frac{1}{2}+\frac{j-1}{q}+\frac{3}{2}i$ and $\frac{3}{4}+\frac{j-1}{q}+\frac{1}{2q}+\frac{7}{4}i$ contained in $L_0+\frac{j-1}{q}$, which across B_ξ. Similarly, since

$$\frac{1}{2}+\frac{j}{q} < \xi < \xi+\frac{1}{6} \leq \frac{j+1}{q}+\frac{1}{2}+\frac{1}{6} < \frac{3}{4}+\frac{j}{q}+\frac{1}{2q},$$

the segment between $\frac{1}{2}+\frac{j}{q}+\frac{3}{2}i$ and $\frac{3}{4}+\frac{j}{q}+\frac{1}{2q}+\frac{7}{4}i$ contained in $L_0+\frac{j}{q}$ across B_ξ. $\square$

Now we are going to construct the desired homeomorphism H on $\mathbb{A}$. Let B_0 be the rectangle whose vertices are $i, 2i, \frac{1}{q}+2i, \frac{1}{q}+i$ and B_0' be the parallelogram whose vertices are z_0, z_1, z_2, z_3. First, we define a homeomorphism $\widetilde{H}$ on $\widetilde{\mathbb{A}}$ such that

(i) $\widetilde{H}$ maps B_0 onto B_0';
(ii) $\widetilde{H}$ maps affinely the segment between $\frac{3i}{2}$ and $\frac{1}{q}+\frac{3i}{2}$ onto L_0;
(iii) $\widetilde{H}$ maps affinely each horizontal segment in B_0 to a piecewise linear segment in B_0' such that the images of the end points have the same imaginary part (see Fig. 5);
(iv) $\widetilde{H}$ is extended to $\widetilde{\mathbb{A}}$ by the translation $z \mapsto z+\frac{1}{q}$, precisely,

$$\widetilde{H}\left(z+\frac{n}{q}\right)=\widetilde{H}(z)+\frac{n}{q}, \forall z \in B_0, n \in \mathbb{Z}.$$

Then $\widetilde{H}$ commutes with the integral translations and thus it factors through a homeomorphism H on $\mathbb{A}$. It is clear that such H satisfies our requirements (1) and (3). It remains to show that H obeys (2). Due the periodicity of $\widetilde{H}$, we may assume that $z, z' \in B_0$ with $|z| = |z'|$. But then it follows from our construction that

$$|\theta(H(z)) - \theta(H(z'))| \leq \frac{\sqrt{1+(1+1/q)^2}}{1/2q}|\theta(z) - \theta(z')| \leq 5q|\theta(z) - \theta(z')|,$$

since $q > 6$.

It follows from (1) and (2) that we have

Claim 2. For any $z \in \mathbb{A}$,

$$|\theta(HR_\alpha H^{-1}(z)) - \theta(R_{p/q}(z))| \leq 10q\pi \left|\alpha - \frac{p}{q}\right|.$$

8.3 Final construction

We fix an irrational number α. It follows from Dirichlet's approximation theorem that there is a sequence $(\frac{p_n}{q_n})$ of rational number with $(p_n, q_n) = 1$ and $q_n \to \infty$ such that

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2}, \quad \forall n \in \mathbb{N}.$$

WLOG, we may further assume $q_n > 6$ for each $n \in \mathbb{N}$.

Now for each $\frac{p_n}{q_n}$, the construction above gives us a homeomorphism H_n on $\mathbb{A}$ satisfying (1), (2), (3).

For each $n \in \mathbb{N}$, let

$$A_n = \left\{z \in \mathbb{C} : 2 - \frac{1}{2n} \leq |z| \leq 2 - \frac{1}{2n+1}\right\}.$$

Now we define the desired homeomorphism g on $\mathbb{A}$ as follows.

(a) The restriction of g on A_n is conjugate to $H_n R_\alpha H_n^{-1}$ by squeezing along the radical direction.
(b) The definition of g on $\mathbb{A} \setminus \bigcup_{n=1}^{\infty} A_n$ is the rotation R_α.

It follows from the construction that g is continuous on $\mathbb{A} \setminus C_2$. Thus it suffices to show that g is continuous on $C_2 = \{z \in \mathbb{C} : |z| = 2\}$. For this, let (z_k) be a sequence in $\mathbb{A}$ with $z_k \to z \in C_2$. Since the restriction of g outside $\bigcup_{n=1}^{\infty} A_n$ is the standard rotation R_α. WLOG, we may assume that $\{z_k\} \subset \bigcup_{n=1}^{\infty} A_n$. Further, to show that $g(z_k) \to g(z) = ze^{2\pi i\alpha}$ it suffices to show that

$$\theta(g(z_k)) - \theta(z_k) \to 2\pi\alpha.$$

We assume $z_k \in A_{n_k}$. We denote the conjugation from $g|_{A_n}$ to $H_n R_\alpha H_n^{-1}$ by ϕ_n, which is a squeezing transformation along the radical direction. Then we have

$$\begin{aligned}|\theta\left(g\left(z_k\right)\right) - \theta(z_k) - 2\pi\alpha| &= \left|\theta(H_{n_k} R_\alpha H_{n_k}^{-1}(\phi_{n_k}(z_k))) - \theta((\phi_{n_k}(z_k)) - 2\pi\alpha\right| \\ &\leq |\theta(H_{n_k} R_\alpha H_{n_k}^{-1}((\phi_{n_k}(z_k))) - \theta(R_{p_{n_k}/q_{n_k}}((\phi_{n_k}(z_k)))| \\ &\quad + |2\pi(p_{n_k}/q_{n_k}) - 2\pi\alpha| \\ &\leq 10q_{n_k}\pi\left|\alpha - p_{n_k}/q_{n_k}\right| + 2\pi|p_{n_k}/q_{n_k} - \alpha|\end{aligned}$$

$$\leq \frac{10\pi}{q_{n_k}} + \frac{2\pi}{q_{n_k}^2} \longrightarrow 0.$$

Thus $\theta(g(z_k)) - \theta(z_k) \to 2\pi\alpha$.

It remains to show that g cannot be linearizable. For this, it suffices to show that there is no a transversal. Assume there is a transversal γ.

We denote the conjugate image of $H_n(C_{3/2})$ by D_n, which is clearly invariant under g and its folding width is also no less than $\frac{\pi}{3}$. Precisely, for each $\xi \in [0, 1]$, the circular sector

$$S_\xi := \left\{ z \in \mathbb{C} : 1 \leq |z| \leq 2, 2\pi\xi \leq \arg(z) \leq 2\pi\xi + \frac{\pi}{6} \right\}$$

is separated by at least two disjoint arcs contained in D_n.

Since γ is a curve connecting the boundaries of the annulus, there is some $\xi \in [0, 1]$ such that $\gamma \cap A_n$ is contained in the circular sector

$$S_\xi := \left\{ z \in \mathbb{C} : 1 \leq |z| \leq 2, 2\pi\xi \leq \arg(z) \leq 2\pi\xi + \frac{\pi}{6} \right\}$$

for all sufficiently large n. But then γ must intersect D_n at least twice for sufficiently large n. This contradicts the transversality of γ. Thus g cannot be linearizable.

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