Notes on the upper half-plane model of non-Euclidean geometry

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Abstract

Based on basic facts on non-Euclidean geometry, we construct parabolic coordinate systems for a non-Euclidean plane, and obtain the famous upper half-plane model by a simple change of coordinates. We then determine all non-Euclidean straight lines in the upper half-plane.

1 Some basic facts on non-Euclidean plane geometry

In this section we review some basic facts on non-Euclidean geometry based on Chapters IV-VI in Wolfe's introduction book [1].

Book I of Euclid's *Elements* sets an axiomatic system for Euclidean plane geometry, which consists of 5 postulates on geometry and 5 common notions on logical reasoning, together with some hidden postulates. The fifth postulate asserts that *if a straight line intersects with two other straight lines, and the sum of the two interior angles on one side of the first line is less than two right angles, then the other two lines must intersect on that side of the first line* (see Fig. 1).

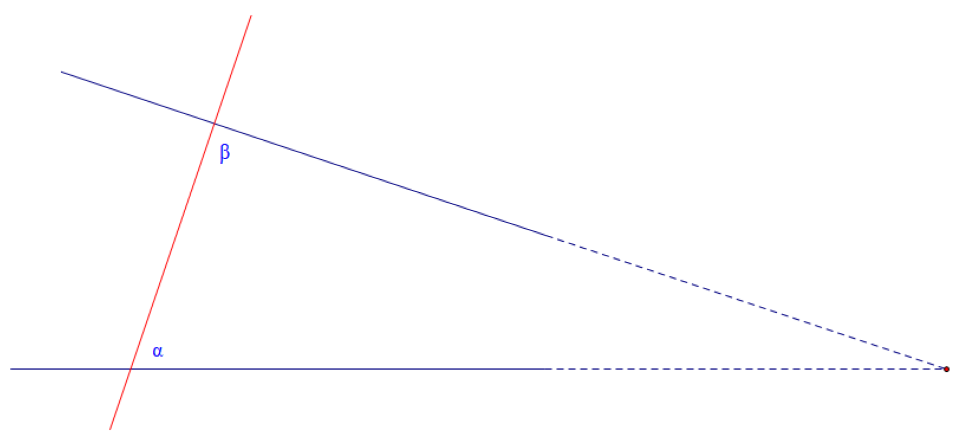

Figure 1: Euclid's fifth postulate

The fifth postulate is equivalent to the so-called *Playfair Axiom* (when all the other postulates including the hidden ones remain unchanged): *In the same plane, given a straight line l and a point P not on l, there is at most one straight line passing P which is parallel to l.*

A non-Euclidean plane geometry is obtained from the Euclidean axiom system by negating the Euclidean fifth postulate or Playfair axiom while preserving all other postulates. Therefore, in a non-Euclidean plane, for a straight line l and a point P not on l, there is at least two straight lines passing P which do not intersect with l (see Fig. 2).

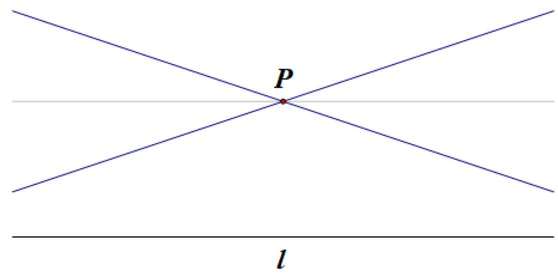

Figure 2: non-Euclidean fifth postulate

Suppose l and m are two directed straight lines in a non-Euclidean plane. We say that l is *asymptotically parallel* to m if (i) l and m do not intersect, (ii) l and m point to the same side of any straight line which intersects with both l and m, and (iii) if l is rotated about any point on l arbitrarily slightly towards m, the resulting straight line will intersect with m (see Fig. 3).

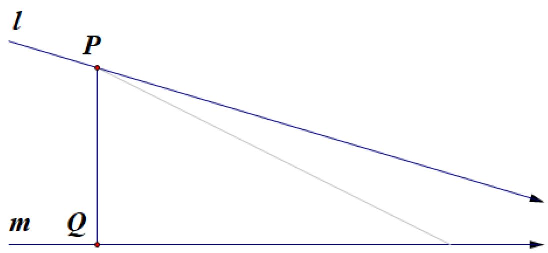

Figure 3: Asymptotically parallel straight lines

The asymptotic parallelism gives rise to an equivalence relation among all directed straight lines in a non-Euclidean plane. Each equivalence class of directed straight lines is called an *ideal point*, which is imagined as a "point" on the boundary of the plane.

It can be shown that, given two distinct ideal points of a non-Euclidean plane, there exists a unique straight line connecting them.

The so-called horocycles are "circles centered at ideal points" which are the limit curves of circles evolving in a special way. Explicitly, let P be a point on a directed straight line l and the ideal point represented by l is denoted Ω. Let C be a moving point on l lying between P and Ω. As C moves towards Ω indefinitely, the circle passing through P and centered at C will limit to a curve passing through P. This curve is called the *horocycle* passing through P and centered at Ω.

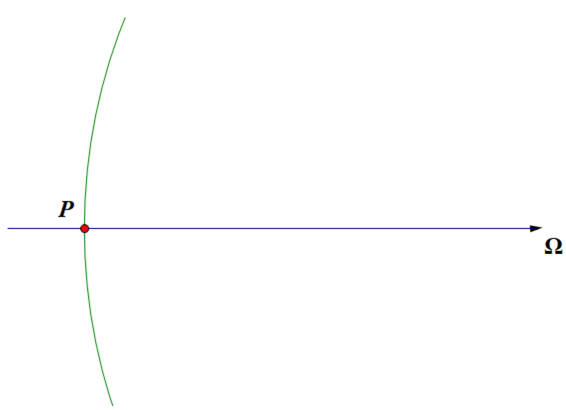

Figure 4: A horocycle centered at an ideal point Ω

The unit length of a non-Euclidean plane is usually chosen as follows: Let l and l' be two directed straight lines which intersect at a point, P. Then there is a unique straight line m connecting the ideal points Ω and Ω' represented respectively by l and l'. The horocycle passing through P and centered at Ω is cut by l and m; the length of the cut-off horocycle arc is chosen as the unit length.

Two horocycles centered at the same ideal point are said to be concentric. It is easy to show that two concentric horocycles are equidistant curves.

There is a very important result on cut-off concentric horocycle arcs: Suppose h_0 and h_x are two concentric horocycles with constant distance $x > 0$, where h_0 is the inner one. Any two straight lines terminating at Ω cut off concentric horocycle arcs of lengths s_x and s_0, respectively. A fundamental formula of non-Euclidean geometry is (see [1, p.135] with different notation)

$$s_x = \mathrm{e}^x s_0. \tag{1}$$

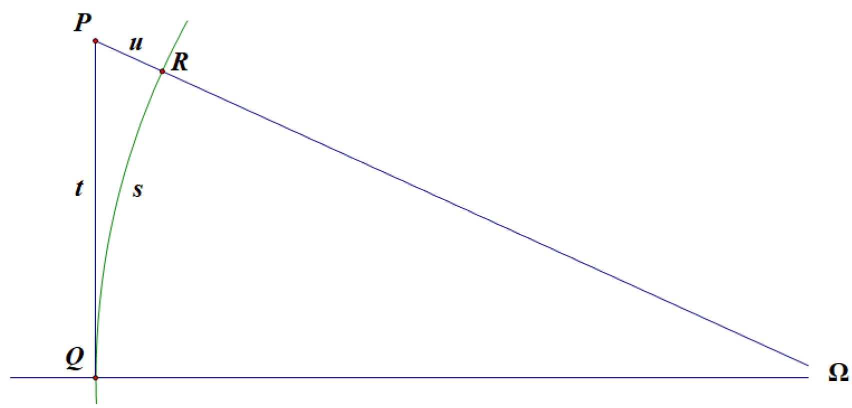

Figure 5: A fundamental triangle in a non-Euclidean plane

For a fundamental triangle PQR in non-Euclidean geometry as shown in Figure 5, where the side QR is a horocycle arc, we have the following important formulae (see [1, p.138]):

$$\mathrm{e}^u = \cosh t, \tag{2}$$

$$s = \tanh t. \tag{3}$$

We also need a fact on perpendicular lines: *Let Ω, Ω_1, Ω_2 be three distinct ideal points of a non-Euclidean plane. Then there exists a unique straight line l terminating at Ω which intersects perpendicularly with the straight line l_{12} connecting Ω_1 and Ω_2.*

To prove this fact, let P be a point on l_{12} which moves from Ω_1 to Ω_2. Then the value of the angle $\angle \Omega P \Omega_2$ is strictly increasing. Since $\lim_{P \to \Omega_1} \angle \Omega P \Omega_2 = 0$ and $\lim_{P \to \Omega_2} \angle \Omega P \Omega_2 = \pi$, there is a unique point P_0 on l_{12} such that $\angle \Omega P_0 \Omega_2 = \frac{\pi}{2}$. This proves the fact.

2 Parabolic coordinate systems for a non-Euclidean plane

Given an ideal point Ω of a non-Euclidean plane, we may construct a parabolic coordinate system for the plane.

Choose a directed straight line l terminating at Ω and a directed horocycle h centered at Ω, and let O the intersection point of h and l.

Given point P on the plane, we obtain the coordinates of P as follows. There is a unique straight line l_P passing through P which terminates at Ω, and a unique horocycle h_P centered at Ω which passes through P. Suppose l_P intersects with h at X, and h_P intersects with l at Y. Let $x \in \mathbf{R}$ be the signed distance from O to X, and let $y \in \mathbf{R}$ be the signed distance from O to Y (which is also the signed distance from X to P). The ordered pair $(x, y) \in \mathbf{R}^2$ gives the coordinates of P.

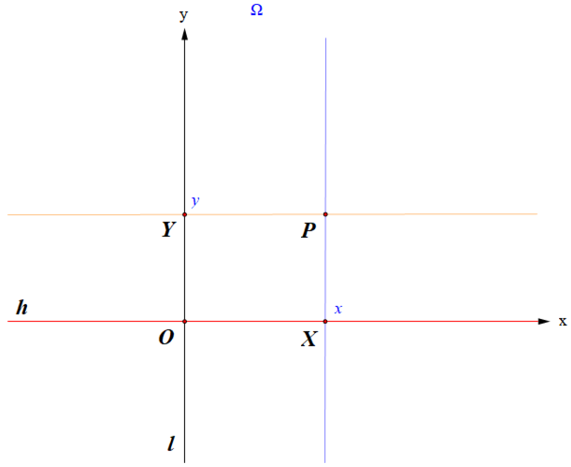

Figure 6: A parabolic coordinate system of a non-Euclidean plane

Conversely, given $(x, y) \in \mathbf{R}^2$, we may determine a unique point on the plane as follows. Let X be the unique point on h such that the signed distance from O to X is x, and let Y be the unique point on l such that the signed distance from O to Y is y. Then there is a unique straight line l_X passing through X which terminates at Ω, and there is a unique horocycle h_Y centered at Ω which passes through Y. Suppose l_X and h_Y intersect at P. It is easily seen that (x, y) is the coordinate pair of P.

In this way we have constructed a parabolic coordinate system for the non-Euclidean plane.

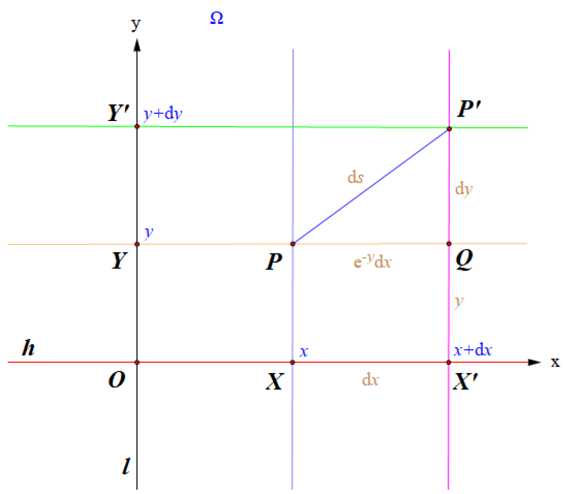

Figure 7: Riemannian metric in parabolic coordinate system

To obtain the Riemannian metric in such a parabolic coordinate system of the non-Euclidean plane, let $P'(x+\mathrm{d}x, y+\mathrm{d}y)$ be a neighbouring point of $P(x, y)$, and let the length of the straight line segment PP' be $\mathrm{d}s$. Let $Q(x+\mathrm{d}x, y)$ be the point of intersection of the horocycle h_P and the straight line $l_{P'}$. Then the length of the straight line segment $P'Q$ is $\mathrm{d}y$, the length of the horocycle arc XX' is $\mathrm{d}x$, and hence the length of the horocycle arc PQ is $\mathrm{e}^{-y}\mathrm{d}x$. By applying Pythagorean Theorem to the infinitesimal triangle PQP', we obtain

$$(\mathrm{d}s)^2 = \mathrm{e}^{-2y}(\mathrm{d}x)^2 + (\mathrm{d}y)^2, \tag{4}$$

which is the formula of the Riemannian metric in the parabolic coordinate system.

3 The upper half-plane model of a non-Euclidean plane

The coordinate region of a parabolic coordinate system for the non-Euclidean plane is the full real plane. Now we do a change of coordinates so that the new coordinate region is the upper-half real plane. The change of coordinates we choose is

$$\tilde{x} = x, \quad \tilde{y} = \mathrm{e}^{y}. \tag{5}$$

Then $\tilde{x} \in \mathbf{R}$ and $\tilde{y} > 0$. It follows that $\mathrm{d}\tilde{x} = \mathrm{d}x$ and $\mathrm{d}\tilde{y} = \mathrm{e}^{y}\mathrm{d}y = \tilde{y}\mathrm{d}y$. Hence we have

$$(\mathrm{d}s)^2 = \frac{(\mathrm{d}\tilde{x})^2 + (\mathrm{d}\tilde{y})^2}{\tilde{y}^2}. \tag{6}$$

We determine all the non-Euclidean straight lines in the upper half-plane model.

Theorem 1. *In the upper half-plane model of a non-Euclidean plane, the non-Euclidean straight lines are the upper vertical half-lines and the upper semi-circle center at the x-axis.*

Proof. Let m be a straight line in a non-Euclidean plane equipped with a parabolic coordinate system centered at an ideal point Ω.

Case 1. m terminates at Ω.

In terms of the parabolic coordinates (x, y), m has equation $x = x_0$, $y \in \mathbf{R}$. In terms of the new coordinates $(\tilde{x}, \tilde{y})$, the straight line m has equation

$$\tilde{x} = x_0, \quad \tilde{y} > 0. \tag{7}$$

Thus m is represented by an upper half-line. Let us denote such a straight line as l_{x_0}.

Case 2. m does not terminate at Ω.

By the fact proven in section 1, there exists a straight line l_{x_0} which intersects m perpendicularly at $Q(x_0, y_0)$. Let P be an arbitrary point on m. Suppose the horocycle h_Q and the straight line l_P intersect at R. Suppose the length of the straight line segment PQ is t, the length of the horocycle arc QR is s, and the length of the straight line segment PR is u. Then $s = \tanh t$, and $\mathrm{e}^{u} = \cosh t$.

It follows that, in the parabolic coordinate system, R and P have coordinates:

$$R(x_0 + s\mathrm{e}^{y_0}, y_0), \quad P(x_0 + s\mathrm{e}^{y_0}, y_0 - u). \tag{8}$$

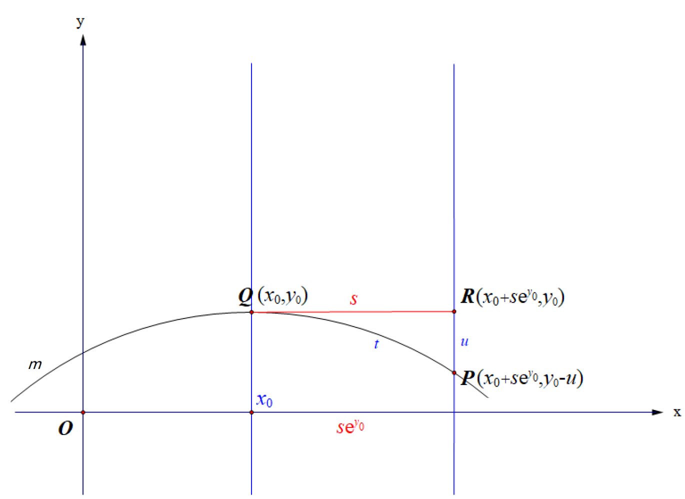

Figure 8: A non-Euclidean straight line m in a parabolic coordinate system

Then, in the upper half-plane model, Q and P have coordinates

$$Q(x_0, \mathrm{e}^{y_0}), \quad P(x_0 + s\mathrm{e}^{y_0}, \mathrm{e}^{y_0 - u}). \tag{9}$$

Let C be the point $(x_0, 0)$ in the upper half-plane model. Then the Euclidean length $|CQ| = \mathrm{e}^{y_0}$, and the the Euclidean length

$$|CP| = \sqrt{(s\mathrm{e}^{y_0})^2 + (\mathrm{e}^{y_0 - u})^2} = \mathrm{e}^{y_0}\sqrt{\tanh^2 t + \cosh^{-2} t} = \mathrm{e}^{y_0}. \tag{10}$$

Therefore the straight line m is represented by the upper semi-circle of radius e^{y_0} centered at $(x_0, 0)$.

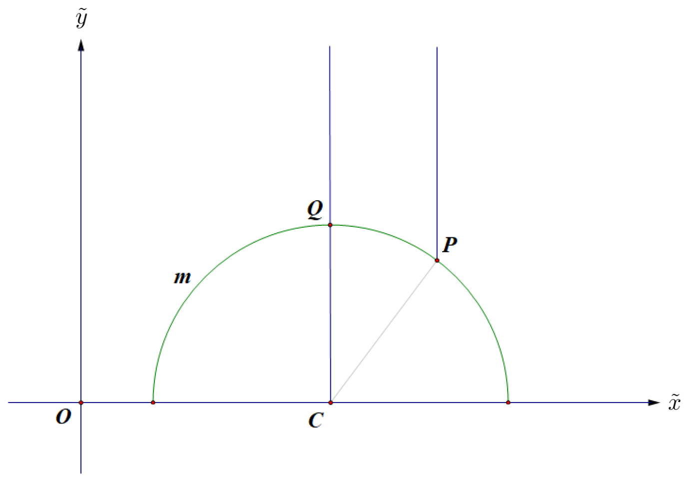

Figure 9: A non-Euclidean straight line m in the upper half-plane model

This finishes the proof of Theorem 1. □

References

[1] H. E. Wolfe, *Introduction to Non-Euclidean Geometry*, The Dryden Press, New York, 1945.