

# Lecture Notes on Group Theory, Lie Algebras and Beyond 群论、李代数及其进阶话题讲义

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## **Abstract**

This is an article based on my lecture notes on Groups and Lie algebras.

这是我关于群与李代数的课程讲义。

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## Preface 前言

本讲义为作者为苏州大学物理科学与技术学院物理学专业研究生一年级新生的课程,“群论及其在物理学中的应用”所准备的课程讲义。

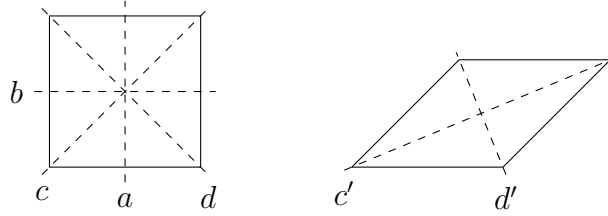


Figure 1: Square vs Rhombus

## 0 Introduction and Preparation

### 0.1 A rough idea about group and symmetries

Let us think about a simple question: why is a square (正方形) more special than a rhombus (菱形)? (See Figure 1) A naive way to answer this question is that a square is a special case of the rhombi with four angles equal to  $90^\circ$ . An equivalent but more sophisticated answer is that a square is a special rhombus that is invariant under a  $90^\circ$  rotation around its center. That is to say the square is more symmetric.

There are eight symmetry transformations that keep the square invariant. They are rotations of  $0^\circ$  (doing nothing!),  $90^\circ$ ,  $180^\circ$  and  $270^\circ$ , and reflections about four axes shown as dashed lines ( $a$ ,  $b$ ,  $c$  and  $d$ ) in Figure 1. *One can easily confirm that any two such transformations combined together again act as another symmetry transformation of the square listed here.* This (the set of all these eight transformations) is a typical example of the group, and it is known as the dihedral group (二面体群)  $\mathbf{Dih}_8$ .

The set of symmetry transformations of the rhombus, on the other hand, has only four elements. Rotations of  $0^\circ$  and  $180^\circ$ , and two reflections about the dashed lines  $c'$  and  $d'$  in Figure 1. This is in fact also a dihedral group,  $\mathbf{Dih}_4$ .

When we deform the rhombus to be a square at the end, the symmetry group  $\mathbf{Dih}_4$  of the rhombus will be a subset of the full symmetry group  $\mathbf{Dih}_8$  of the square. However, as mentioned in the beginning, by adding a  $90^\circ$  rotation to  $\mathbf{Dih}_4$ , we should be able to recover  $\mathbf{Dih}_8$ . Indeed, for example, the reflection about  $b$  can be generated as a (clockwise) rotation of  $90^\circ$  followed by the reflection about  $d$ . In this way, we see that by simply adding one element in the set of symmetry transformation, the whole set is enlarged by four elements (in this case). This is associated to the closedness (闭合性) of the mathematical concept, group. In this article, we wish to present various different (both fundamental and applicational) aspects of the group.

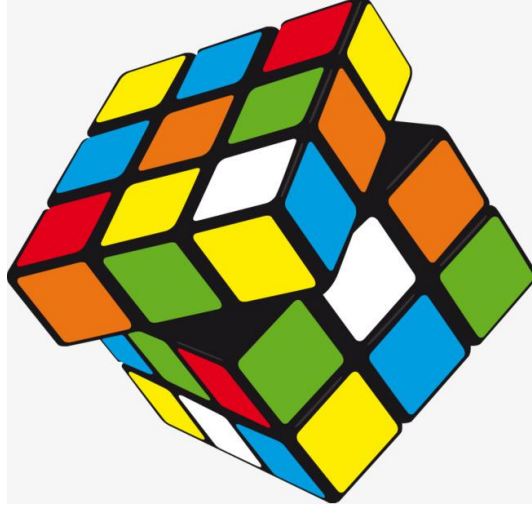


Figure 2: A typical Rubik's cube.

## 0.2 An Example for Fun: Rubik's Cube

A Rubik's cube is usually a cube that allows one to rotate its parts (see Figure 2). The goal of playing Rubik's cube is to “solve” it, i.e. to restore its original state (with each surface in only one color, see Figure 3).

Of course, as shown in Figure 2, one can rotate a face of the Rubik's cube by any degree, but to achieve the final goal of solving the Rubik's cube, we only consider operations that keep its cubic shape. These operations also form a group, and this group is characterized by keeping the shape of the cube invariant. It is said that there are about  $4 \times 10^{19}$  different patterns in Rubik's cube, so without any knowledge of the group theory, it is almost impossible to understand how to solve Rubik's cube. In particular, if we accidentally break a Rubik's cube, and after putting all the pieces back, we want to confirm if the cube can be solved to the original form (Figure 3). In fact, if we assemble pieces completely in a random way, then we will only have a 1/12 chance to obtain a solvable Rubik's cube. The confirmation of the solvability of a random cube involves a lot of knowledge of representation theory, and we wish to explain this idea through this lecture course. This example of fun will be repeatedly used in this course to explain various kinds of basic concepts.

## 0.3 Preliminary 1: Symmetry and its use in Physics

**Example 0.1** (Hydrogen atom, 氢原子) Let us recall how we solve the spectrum of a hydrogen atom, or more precisely the electron moving around a proton. The Hamiltonian of this system reads

$$\mathcal{H} = -\frac{\hbar^2}{2m}\nabla^2 - \frac{e^2}{4\pi\epsilon_0|\vec{x}|}, \quad (0.1)$$



Figure 3: A solved Rubik's cube.

where  $m = \frac{m_e m_p}{m_e + m_p} \simeq 0.9995 m_e$  is the reduced mass of the system. After changing to the spherical coordinates  $(r, \theta, \varphi)$ , we obtain the following equation

$$\left[ -\frac{\hbar^2}{2m} \left( \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right) - \frac{e^2}{4\pi\epsilon_0 r} \right] \psi(r, \theta, \varphi) = E\psi(r, \theta, \varphi), \quad (0.2)$$

to solve for the energy eigenstate. This equation can be solved by separating the variables,

$$\psi(r, \theta, \varphi) = P(r)Y(\theta, \varphi), \quad (0.3)$$

where  $Y(\theta, \varphi)$  satisfies

$$\left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right) Y(\theta, \varphi) = \lambda Y(\theta, \varphi). \quad (0.4)$$

The eigenvalue  $\lambda$  is known to take a discrete value

$$\lambda = -L(L+1), \quad \text{for } L \in \mathbb{N}. \quad (0.5)$$

A detailed derivation of this constraint can be found in Appendix A. The function  $Y(\theta, \varphi)$  here is known as the spherical harmonic, which is labeled by two integers,  $L$  and  $m \in [-L, L] \subset \mathbb{Z}$  and is denoted by  $Y_L^m(\theta, \varphi)$ . It further satisfies

$$-i \frac{\partial}{\partial \varphi} Y_L^m(\theta, \varphi) = m Y_L^m(\theta, \varphi). \quad (0.6)$$

Let us define

$$L_z = -i \frac{\partial}{\partial \varphi}, \quad L_{\pm} = e^{\pm i\varphi} \left( \pm \frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \varphi} \right). \quad (0.7)$$

It is not hard to check that  $L_z, L_{\pm}$  satisfy the following commutation relations,

$$[L_z, L_{\pm}] = \pm L_{\pm}, \quad [L_+, L_-] = 2L_z, \quad (0.8)$$

and that (0.4) is translated to

$$\left( \frac{1}{2} (L_+ L_- + L_- L_+) + L_z^2 \right) |\lambda, m\rangle = -\lambda |\lambda, m\rangle, \quad (0.9)$$

by identifying  $Y_L^m(\theta, \varphi)$  as the eigenstate  $|\lambda, m\rangle$  of the Casimir operator  $C := \frac{1}{2} (L_+ L_- + L_- L_+) + L_z^2$ . We also have

$$L_z |\lambda, m\rangle = m |\lambda, m\rangle. \quad (0.10)$$

One can instead define

$$L_x = \frac{L_+ + L_-}{2}, \quad L_y = \frac{L_- - L_+}{2}, \quad (0.11)$$

then the commutation relations for  $L_x, L_y$  and  $L_z$  read

$$[L_x, L_y] = L_z, \quad [L_z, L_x] = L_y, \quad [L_y, L_z] = L_x. \quad (0.12)$$

The above algebraic relations define an algebra known as  $\mathfrak{so}(3)$ , and the exponential of these operators generates rotations in 3D Euclidean space. We will discuss more details on this later. Let us derive the constraint on  $\lambda$  in this algebraic approach. We note that

$$L_z (L_{\pm} |\lambda, m\rangle) = (m \pm 1) (L_{\pm} |\lambda, m\rangle), \quad [L_{\pm}, C] = 0, \quad (0.13)$$

and thus  $L_{\pm}$  connect different states in a single multiplet (多重态; 能量相等的简并态的集合). A multiplet labeled by discrete parameters in Physics usually only has a finite number of states. Let us consider a multiplet with largest value of  $L_z$  denoted as  $m_{max} \in \mathbb{Z}$  and its minimal value denoted as  $m_{min} \in \mathbb{Z}$ . By definition,

$$L_+ |\lambda, m_{max}\rangle = 0, \quad L_- |\lambda, m_{min}\rangle = 0. \quad (0.14)$$

One can evaluate that

$$\lambda = -m_{max}(m_{max} + 1) = -m_{min}(m_{min} - 1). \quad (0.15)$$

We easily see from the above equalities that  $L = m_{max}$  and  $m_{min} = -m_{max}$  (following from  $m_{min} \leq m_{max}$ ).

The spectrum of the hydrogen atom can then be solved from the equation

$$\left[ -\frac{\hbar^2}{2m} \left( \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) - \frac{L(L+1)}{r^2} \right) - \frac{e^2}{4\pi\epsilon_0 r} \right] P(r) = EP(r). \quad (0.16)$$

We remark that the algebraic approach simplified the problem a lot compared to the analysis done in Appendix A.

**Q:** What will happen if  $L$  is not an integer? Why  $m$  has to be an integer?

**Example 0.2** (3D harmonic oscillator, 三维谐振子) The Hamiltonian of a 3D harmonic oscillator is given by

$$\mathcal{H}_{3D} = -\frac{\hbar^2}{2m}\nabla^2 + \frac{m\omega^2}{2}(x^2 + y^2 + z^2). \quad (0.17)$$

It can be decomposed into three copies of the 1D harmonic oscillator

$$\mathcal{H}_{1D} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m\omega^2}{2}x^2. \quad (0.18)$$

A well-known method to solve the 1D oscillator is to define the generation and annihilation operator as

$$a = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}x + \sqrt{\frac{\hbar}{m\omega}}\partial_x\right), \quad a^\dagger = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}x - \sqrt{\frac{\hbar}{m\omega}}\partial_x\right), \quad (0.19)$$

and then the 1D Hamiltonian can be rewritten into

$$\mathcal{H}_{1D} = \hbar\omega a^\dagger a + \frac{1}{2}\hbar\omega. \quad (0.20)$$

The creation and annihilation operator satisfy

$$[a, a^\dagger] = 1. \quad (0.21)$$

The algebra spanned by them is known as the Heisenberg algebra. Energy eigenstates of the 1D oscillator system take the form

$$|n\rangle = (a^\dagger)^n |0\rangle, \quad n \in \mathbb{N}, \quad (0.22)$$

with energy eigenvalue

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega. \quad (0.23)$$

The spectrum of the 3D harmonic oscillator is labeled by three independent integers,  $(n, j, k)$ ,

$$E_{(n,j,k)} = \left(n + j + k + \frac{3}{2}\right)\hbar\omega. \quad (0.24)$$

Now we take an alternative way to solve the problem, i.e. by rewriting the Hamiltonian in the spherical coordinates.

$$\left[-\frac{\hbar^2}{2m}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{L(L+1)}{r^2}\right) + \frac{m\omega^2}{2}r^2\right]P(r) = EP(r). \quad (0.25)$$

Changing the variable to  $u = 2\nu r^2$ , and setting  $P(r) = (2\nu r^2)^{L/2}e^{-\nu r^2}\tilde{P}(r)$ , we obtain

$$\left(8\nu u\partial_u^2 + (12\nu + 8\nu L - 8\nu u)\partial_u - \frac{m^2\omega^2}{\hbar^2 2\nu}u + \frac{2mE}{\hbar^2} + 2\nu u - 4\nu L - 6\nu\right)\tilde{P}(u) = 0, \quad (0.26)$$

and by further choosing  $\nu^2 = \frac{m^2\omega^2}{4\hbar^2}$ , the above equation is simplified to

$$\left(u\partial_u^2 + \left(L + \frac{3}{2} - u\right)\partial_u + \frac{E}{2\hbar\omega} - \frac{L}{2} - \frac{3}{4}\right)\tilde{P}(u) = 0. \quad (0.27)$$

This is a differential equation known as the generalized Laguerre (拉盖尔) equation, whose solution is non-singular only when

$$\frac{E}{2\hbar\omega} - \frac{L}{2} - \frac{3}{4} = i \in \mathbb{N}. \quad (0.28)$$

We thus see

$$E_{(i,L,m)} = \left(L + 2i + \frac{3}{2}\right)\hbar\omega. \quad (0.29)$$

Let us count the number of states in these two different coordinate systems. In the Cartesian coordinates, three non-negative integers sum to a given number  $N$ , and the number of states is given by the partition of  $N$  into 3 parts,  $p_3(N+3)$ . While in the spherical coordinates, it can be computed as

$$\sum_{i=0}^{\lfloor N/2 \rfloor} (2(N-2i)+1) = (2N+1-2\lfloor N/2 \rfloor)(\lfloor N/2 \rfloor + 1). \quad (0.30)$$

In both case of  $N = 2M$  and  $N = 2M + 1$  for  $M \in \mathbb{N}$ , the above expression is simplified to

$$\sum_{i=0}^{\lfloor N/2 \rfloor} (2(N-2i)+1) = \frac{(N+1)(N+2)}{2}. \quad (0.31)$$

The number of parts  $p_k(N)$  in general is computed using the recursive relation

$$p_k(N) = p_k(N-k) + p_{k-1}(N-1), \quad (0.32)$$

together with  $p_k(n) = 0$  if  $n < 0$ . We have

$$p_2(N+2) = N+1. \quad (0.33)$$

The generating function of  $p_3(N+3)$  is

$$\sum_{N=0}^{\infty} p_3(N+3)q^N = \frac{1}{(1-q)^3}, \quad (0.34)$$

and we see that

$$p_3(N+3) = \frac{(N+1)(N+2)}{2}. \quad (0.35)$$

**Q:** Any different proof?

*One main goal of this lecture is to understand this decomposition in a more convenient and algebraic way.*

**Example 0.3** (3D rotation as SO(3), 三维旋转) As mentioned before, rotations in 3D are generated by the product of  $\exp(\theta_1 L_z)$ ,  $\exp(\theta_2 L_x)$  and  $\exp(\theta_3 L_y)$ , where  $L_{x,y,z}$  satisfy the commutation relation (0.12). Let us express these operators more explicitly as an action on the Cartesian coordinates  $(x, y, z)^t$ . On such a three-vector,  $L_{x,y,z}$  can be written (represented) as  $3 \times 3$  matrices, e.g.

$$L_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad L_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (0.36)$$

**Q:** Confirm the commutation relations (0.12) for the above matrices.

More precisely, we have

$$e^{\theta_1 L_z} = \begin{pmatrix} \cos \theta_1 & -\sin \theta_1 & 0 \\ \sin \theta_1 & \cos \theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad e^{\theta_2 L_x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_2 & -\sin \theta_2 \\ 0 & \sin \theta_2 & \cos \theta_2 \end{pmatrix}, \quad (0.37)$$

$$e^{\theta_3 L_y} = \begin{pmatrix} \cos \theta_1 & 0 & \sin \theta_1 \\ 0 & 1 & 0 \\ -\sin \theta_1 & 0 & \cos \theta_1 \end{pmatrix}. \quad (0.38)$$

They respectively correspond to a clockwise rotation around the  $z$ ,  $x$  and  $y$  axes. We observe that

$$(e^{\theta L_{x,y,z}})^t = e^{-\theta L_{x,y,z}}. \quad (0.39)$$

Since an arbitrary 3D rotation can be expressed as a product of these matrices, we conclude that any 3D rotation  $O$  expressed in the form of a  $3 \times 3$  matrix satisfies

$$O^t = O^{-1}. \quad (0.40)$$

Similarly, we see that

$$\det O = 1. \quad (0.41)$$

These two conditions (0.40) and (0.41) turned out to be the characteristic properties of the 3D rotation group, which is known in mathematics as the SO(3) group. When the condition (0.41) is removed, then the group will also contain reflection operations, and this 3D rotation & reflection group is called O(3) group in mathematics.

## 0.4 Preliminary 2: Important Facts in Linear Algebra

The linear algebra (线性代数) deals with linear problems in the nature. Something linear means it is compatible with the addition and scalar product operation. For example, a linear map(线性映射),

$$T : V \mapsto W, \quad (0.42)$$

satisfies

$$T(u + v) = T(u) + T(v), \quad T(a \cdot u) = aT(u), \quad (0.43)$$

for  $\forall u, v \in V$ , and  $\forall a \in \mathbb{R}$ . Of course, to consider such a linear map, we need a linear structure, i.e. the addition and scalar product, defined on the space  $V$  and  $W$ . This leads to the concept of a linear space, or a vector space:

**Definition 0.4** (Vector space, 向量空间) A vector space,  $V = \{\vec{v}\}$ , is a set of elements, called vectors, equipped with the addition operation  $+$  and real (or resp. complex) scalar product  $\cdot$ . These two operations (binary maps) satisfy (for  $\forall \vec{u}, \vec{v}, \vec{w} \in V$  and  $a, b \in \mathbb{R}$  (resp.  $a, b \in \mathbb{C}$ )):

- Associativity(结合律):  $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ ,
- Comuutativity(交换律):  $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ ,
- Identity element(单位元): there exists a (unique) element, called the zero vector  $\vec{0}$  in this case, s.t.  $\vec{0} + \vec{u} = \vec{u} + \vec{0} = \vec{u}$ ,
- Inverse element(逆元): for each vector  $\vec{u}$ , there exists an element to be its inverse element denoted as  $-\vec{u}$ , s.t.  $\vec{u} + (-\vec{u}) = \vec{0}$ ,
- Distributivity(分配律):  $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$  and  $(a + b)\vec{u} = a\vec{u} + b\vec{u}$ ,
- Compatibility of scalar product:  $a(b\vec{u}) = (ab)\vec{u}$ ,
- Identity in the scalar product:  $1 \cdot \vec{u} = \vec{u}$ .

In this lecture, we mainly consider vector spaces of the form  $V = \mathbb{R}^n$  and  $V = \mathbb{C}^n$ . Therefore, vectors take the form

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{pmatrix}. \quad (0.44)$$

We want to further consider linear maps between two vector spaces, say from  $V = \mathbb{F}^n$  to  $W = \mathbb{F}^m$ , where we denoted the field (域) as  $\mathbb{F}$ , which can be chosen to be  $\mathbb{R}$  or  $\mathbb{C}$  accordingly. These linear maps are realized as matrices, which are denoted as

$$M = (M_{i,j})_{1 \leq i \leq m, 1 \leq j \leq n} = \begin{pmatrix} M_{1,1} & M_{1,2} & \dots & M_{1,n} \\ M_{2,1} & M_{2,2} & \dots & M_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{m,1} & M_{m,2} & \dots & M_{m,n} \end{pmatrix}. \quad (0.45)$$

It acts on the vector as

$$\vec{w} = M \cdot \vec{v}, \quad (0.46)$$

with

$$w_i = \sum_{j=1}^n M_{i,j} v_j. \quad (0.47)$$

It is straightforward to check the linearity of this map, since

$$M \cdot (\vec{v}_1 + \vec{v}_2) = M \cdot \vec{v}_1 + M \cdot \vec{v}_2, \quad M \cdot (a\vec{v}) = aM \cdot \vec{v}. \quad (0.48)$$

Similarly, we can extend this definition of multiplication to that between matrices,

$$M \cdot N = ((M \cdot N)_{i,j}), \quad (M \cdot N)_{i,j} = \sum_{k=1}^n M_{i,k} N_{k,j}, \quad (0.49)$$

where only the multiplication of  $m \times n$  matrix onto  $n \times n'$  matrix is allowed. The addition between two matrices with the same size can be defined as

$$(M_1 + M_2)_{i,j} = (M_1)_{i,j} + (M_2)_{i,j}. \quad (0.50)$$

These operations again are linear, i.e.

$$(M_1 + M_2) \cdot N = M_1 \cdot N + M_2 \cdot N, \quad M \cdot (N_1 + N_2) = M \cdot N_1 + M \cdot N_2, \quad (0.51)$$

$$M \cdot (aN) = aM \cdot N, \quad (0.52)$$

for  $\forall a \in \mathbb{F}$ .

Let us denote the space of all  $m \times n$  matrices as  $\mathbb{M}_{m,n}$ , and it is easy to see that this is also a vector space (but equipped with an additional operation, matrix multiplication, if  $m = n$ ). The vector space  $V = \mathbb{F}^n$  can be viewed as  $\mathbb{M}_{n,1}$  or  $\mathbb{M}_{1,n}$ .

Squared matrices, i.e.  $\mathbb{M}_{n,n}$ , are more interesting, as it sometimes is possible to define the inverse to the matrix multiplication, i.e. matrix satisfying

$$M^{-1} \cdot M = M \cdot M^{-1} = \mathbb{I}_{n \times n}, \quad (0.53)$$

where  $\mathbb{I}_{n \times n}$  stands for the  $n \times n$  identity matrix. It is easy to show the uniqueness of the inverse element: suppose there are two different inverse elements to  $M$  denoted as  $M^{-1}$  and  $M'^{-1}$ , then we have

$$M^{-1} = M^{-1} \cdot M \cdot M'^{-1} = M'^{-1}. \quad (0.54)$$

To see the condition for the existence of the inverse matrix, we define the concept of the determinant. In order to define it, we need to consider the permutation of the set of numbers  $(1, 2, 3, \dots, n)$ . A permutation(置换) is a shuffle of these numbers, for example, the map

$$\mathfrak{s} : (1, 2, 3, 4) \mapsto (2, 4, 1, 3), \quad (0.55)$$

is a typical permutation. These permutations, as we will see later, actually form a group, called the symmetry group  $\mathfrak{S}_n$ , and they can be generated by combining the permutations interchanging the  $i$ -th and the  $i + 1$ -th number,  $\sigma_i$ 's. e.g.  $\mathfrak{s}$  can be realized as

$$\mathfrak{s} = \sigma_2 \sigma_3 \sigma_1. \quad (0.56)$$

We can assign a signature  $\text{sgn}(\sigma) = (-1)^{\sum_i \# \sigma_i}$  to each permutation operator,  $\sigma$ . The determinant of an  $n \times n$  matrix is then defined as

$$\det M = \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma) M_{i, \sigma(i)}. \quad (0.57)$$

The most frequently used cases are  $n = 2$  and  $n = 3$ , where we have for  $n = 2$ ,

$$\det(M) = M_{11}M_{22} - M_{12}M_{21}, \quad (0.58)$$

and for  $n = 3$ ,

$$\begin{aligned} \det(M) = & M_{11}M_{22}M_{33} - M_{11}M_{32}M_{23} - M_{12}M_{21}M_{32} + M_{12}M_{31}M_{22} \\ & + M_{13}M_{21}M_{32} - M_{13}M_{31}M_{22}. \end{aligned} \quad (0.59)$$

In general, we can define a smaller (sub)-matrix of  $M$  with its  $i$ -th row and  $j$ -th column knocked out,  $M^{\vee i, j}$ ,

$$(M^{\vee i, j})_{a, b} = \begin{cases} M_{a, b} & 1 \leq a < i, 1 \leq b < j, \\ M_{a+1, b} & i < a \leq n, 1 \leq b < j, \\ M_{a, b+1} & 1 \leq a < i, j < b \leq n, \\ M_{a+1, b+1} & i < a \leq n, j < b \leq n. \end{cases} \quad (0.60)$$

The determinant of an  $n \times n$  matrix can recursively be determined from those of  $(n - 1) \times (n - 1)$  matrices,

$$\det(M) = \sum_{1 \leq i, j \leq n} (-1)^{i+j} M_{i, j} \det(M^{\vee i, j}). \quad (0.61)$$

An extremely interesting property of the determinant is that

$$\det(M_1 M_2) = \det(M_1) \det(M_2). \quad (0.62)$$

Let us try to check the above identity for  $n = 2$ : note that

$$(M_1 M_2)_{11} = (M_1)_{11}(M_2)_{11} + (M_1)_{12}(M_2)_{21}, \quad (0.63)$$

$$(M_1 M_2)_{12} = (M_1)_{11}(M_2)_{12} + (M_1)_{12}(M_2)_{22}, \quad (0.64)$$

$$(M_1 M_2)_{21} = (M_1)_{21}(M_2)_{11} + (M_1)_{22}(M_2)_{21}, \quad (0.65)$$

$$(M_1 M_2)_{22} = (M_1)_{21}(M_2)_{12} + (M_1)_{22}(M_2)_{22}, \quad (0.66)$$

then we have

$$\begin{aligned} \det(M_1 M_2) &= ((M_1)_{11}(M_2)_{11} + (M_1)_{12}(M_2)_{21})((M_1)_{21}(M_2)_{12} + (M_1)_{22}(M_2)_{22}) \\ &\quad - ((M_1)_{11}(M_2)_{12} + (M_1)_{12}(M_2)_{22})((M_1)_{21}(M_2)_{11} + (M_1)_{22}(M_2)_{21}) \\ &= (M_1)_{11}(M_1)_{22}((M_2)_{11}(M_2)_{22} - (M_2)_{12}(M_2)_{21}) \\ &\quad + (M_1)_{12}(M_1)_{21}((M_2)_{12}(M_2)_{21} - (M_2)_{11}(M_2)_{22}) \\ &= \det(M_1)\det(M_2). \end{aligned} \quad (0.67)$$

Another important quantity characterizing the matrices is called the trace(迹). It is defined as the sum of all diagonal elements,

$$\text{tr}(M) = \sum_{i=1}^n M_{i,i}. \quad (0.68)$$

One may further define functions of variables in matrix value,  $f(M)$ . We need to utilize the Taylor expansion (泰勒展开) of  $f(x)$ ,

$$f(x) = \sum_n a_n x^n, \quad (0.69)$$

then we define

$$f(M) = \sum_n a_n M^n. \quad (0.70)$$

The most useful functions are

$$\exp(M) = \sum_{n=0}^{\infty} \frac{1}{n!} M^n, \quad (0.71)$$

and

$$\sin(M) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} M^{2n+1}, \quad \cos(M) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} M^{2n}. \quad (0.72)$$

We remark that properties of the corresponding number-valued function do not necessarily become the properties of the matrix-valued function. For example, for  $x, y \in \mathbb{C}$  we have

$$\exp(x+y) = \exp(x)\exp(y), \quad (0.73)$$

while for  $M_1, M_2 \in \mathbb{M}_{n,n}$ ,

$$\exp(M_1 + M_2) \neq \exp(M_1) \exp(M_2). \quad (0.74)$$

Instead we have the Baker-Campbell-Hausdorff formula:

$$\exp(M_1) \exp(M_2) = \exp\left(M_1 + M_2 + \frac{1}{2} [M_1, M_2] + \frac{1}{12} [M_1, [M_1, M_2]] - \frac{1}{12} [M_2, [M_1, M_2]] + \dots\right). \quad (0.75)$$

The following identity connects the determinant and the trace of the matrices,

$$\det(\exp(M)) = \exp(\operatorname{tr}(M)). \quad (0.76)$$

It directly follows that if the determinant of  $\exp(M)$  is 1, then  $M$  itself is traceless, i.e. its trace is zero.

Let us work out the exponential of

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (0.77)$$

Note that  $\sigma_x^2 = \mathbb{I}_{2 \times 2}$ , so we have

$$\begin{aligned} \exp(u\sigma_x) &= \sum_{n=0}^{\infty} \frac{1}{(2n)!} u^{2n} \mathbb{I}_{2 \times 2} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} u^{2n+1} \sigma_x \\ &= \begin{pmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \end{pmatrix}. \end{aligned} \quad (0.78)$$

We also remark that the logarithm can be defined through,

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n, \quad (0.79)$$

therefore

$$\log(M) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (M - \mathbb{I})^n. \quad (0.80)$$

**direct sum (直和)** The direct product of two matrices  $M_1$  and  $M_2$ , denoted as  $M_1 \oplus M_2$ , is schematically given by

$$M_1 \oplus M_2 = \begin{pmatrix} M_1 & \mathbb{O} \\ \mathbb{O} & M_2 \end{pmatrix}, \quad (0.81)$$

where  $\mathbb{O}$  is zero matrix with proper size. When both  $M_1$  and  $M_2$  are square matrices (say respectively  $n \times n$  and  $m \times m$  matrices), we certainly have

$$\text{tr}(M_1 \oplus M_2) = \text{tr}(M_1) + \text{tr}(M_2). \quad (0.82)$$

The direct sum of matrices acts on the direct sum of two vectors  $\vec{v}_1$  and  $\vec{v}_2$  as

$$(M_1 \oplus M_2) (\vec{v}_1 \oplus \vec{v}_2) := (M_1 \vec{v}_1) \oplus (M_2 \vec{v}_2). \quad (0.83)$$

If the components of  $\vec{v}_1$  and  $\vec{v}_2$  are given by

$$\vec{v}_1 = \begin{pmatrix} v_{11} \\ v_{12} \\ \vdots \\ v_{1n} \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} v_{21} \\ v_{22} \\ \vdots \\ v_{2m} \end{pmatrix}, \quad (0.84)$$

we can write

$$\vec{v}_1 \oplus \vec{v}_2 = \begin{pmatrix} v_{11} \\ v_{12} \\ \vdots \\ v_{1n} \\ v_{21} \\ v_{22} \\ \vdots \\ v_{2m} \end{pmatrix}. \quad (0.85)$$

**direct product (直积)** In the context of physics, the direct product is sometimes more useful. It also satisfies

$$(M_1 \otimes M_2) (\vec{v}_1 \otimes \vec{v}_2) = (M_1 \vec{v}_1) \otimes (M_2 \vec{v}_2). \quad (0.86)$$

We define

$$(A \otimes B)_{ij;kl} := A_{ij} B_{kl}, \quad (0.87)$$

$$(\vec{a} \otimes \vec{b})_{i;j} = a_i b_j. \quad (0.88)$$

We can relabel the set of  $(i, j)$  indices into a single index, and re-express  $A \otimes B$  as a matrix and  $\vec{a} \otimes \vec{b}$  as a vector. For example, if both  $A$  and  $B$  are  $2 \times 2$  matrices, and  $\vec{a}, \vec{b}$  being 2-component

vectors, then

$$\begin{aligned}
A \otimes B &= \left( \begin{array}{cc|cc} A_{11}B & A_{12}B & & \\ \hline A_{21}B & A_{22}B & & \end{array} \right) \\
&= \left( \begin{array}{cc|cc} A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\ A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\ \hline A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\ A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22} \end{array} \right). \tag{0.89}
\end{aligned}$$

In particular, we see that

$$\text{tr}(A \otimes B) = \text{tr}(A)\text{tr}(B). \tag{0.90}$$

# 1 Abstract Algebra

## 1.1 Basic concepts

**Definition 1.1** (Group, 群) A group  $G$  refers to a set of elements equipped with an operation (we denote it as  $\cdot$  here) that combines two elements and map them to another element in the set. That is to say,

$$\forall a, \forall b \in G \Rightarrow a \cdot b \in G. \tag{1.1}$$

The operation is required to satisfy the following properties:

- $(a \cdot b) \cdot c = a \cdot (b \cdot c)$  (associativity, 结合律)
- There exists an element  $e \in G$  s.t.  $\forall a \in G, (e \cdot a) = (a \cdot e) = a$ .  $e$  is usually called the unit element or identity element (单位元) .
- For  $\forall a \in G$ , there exists an element, usually denoted as  $a^{-1}$ , s.t.  $a \cdot a^{-1} = a^{-1} \cdot a = e$  (inverse element, 逆元) .

**Remark** Uniqueness of the identity element: say there are two identity elements  $e$  and  $e'$ , then  $e = e \cdot e' = e'$ .

**Example 1.1.1** The set of all integers,  $\mathbb{Z}$ , form a group under the action of addition  $+$ . The identity element is 0, and the inverse element of  $a$  is usually denoted as  $-a$ .

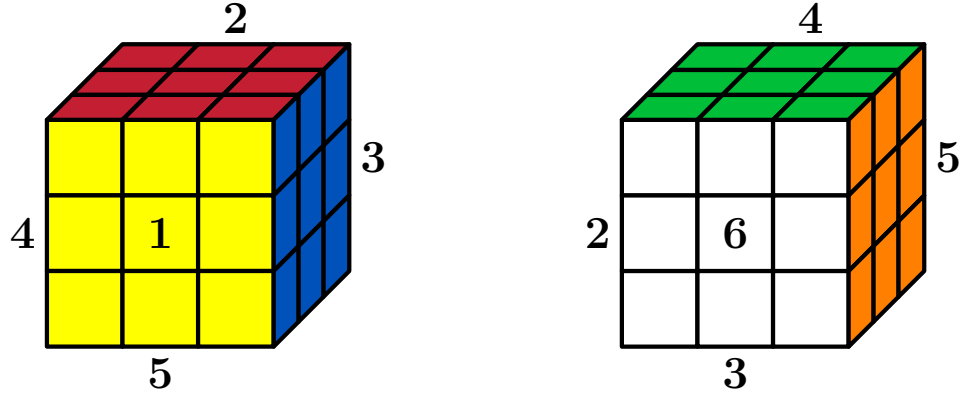


Figure 4: The labeling of all the faces of Rubik's cube.

**Example 1.1.2** All the bijections from any set  $X$  to itself form a group under the action of composition of such maps. This group is usually called the symmetric group (对称群) of  $X$ . The symmetric group of degree  $n$  is defined for  $X = \{1, 2, 3, \dots, n\}$ , and is denoted as  $\mathfrak{S}_n$ .

**Example 1.1.3 (魔方)** Operations rotating one of the faces of the Rubik's cube by  $90^\circ$ ,  $180^\circ$ ,  $270^\circ$  or  $360^\circ$  form a group that keeps the cube invariant. Let us call this group Rubik's group in this article. The identity element is doing nothing to the cube, and the inverse element is a rotation with the same angle but in the opposite direction.

Figure 4 shows how we label the faces of Rubik's cube, and Figure 5 tells us the relative positions of surfaces with labels on them. We adopt the traditional coloring of Rubik's cube in our figures. The rotations (with  $90^\circ$  clockwise) will be named after the related surface in this lecture (we will always start from the solved cube and fix the angle we look), for example the rotation of the 3-rd surface acts on the solved cube as in Figure 6.

**Example 1.1.4** (A toy model of Rubik's cube) Figure 7 shows a toy model of Rubik's cube that we want to consider in this article. It is similar to a Rubik's cube projected to 2D. We are allowed to rotate it by  $\frac{\pi}{2}$  and we can also perform a cyclic permutation on three right-hand-side boxes, whose action from the solved puzzle (Figure 7) is shown in Figure 8.

**Definition 1.2** (Abelian Group, 阿贝尔群) An Abelian group is a group satisfying the following commutativity (交换律) .

$$a \cdot b = b \cdot a. \tag{1.2}$$

**Example 1.2.1** The group of integers  $\mathbb{Z}$  is an Abelian group.

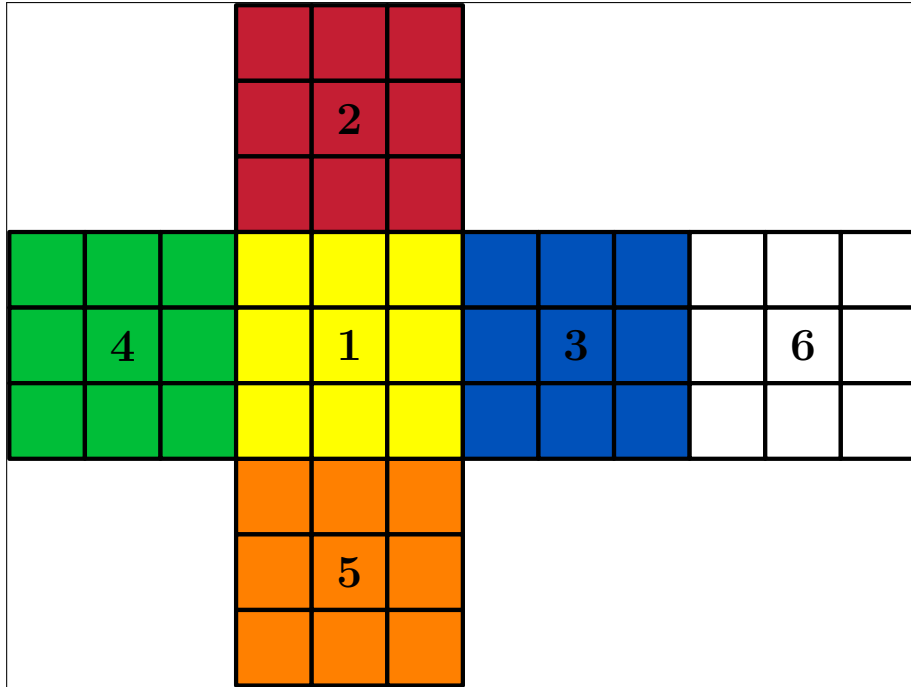


Figure 5: An developed representation (展开图) of Rubik's cube with labels on each surface.

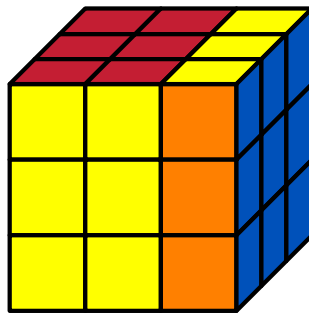


Figure 6: Rubik's cube after the rotation of the 3-rd surface acted on the solved cube.

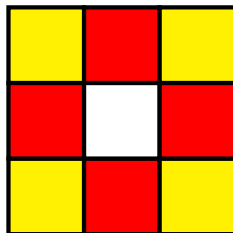


Figure 7: A toy puzzle.

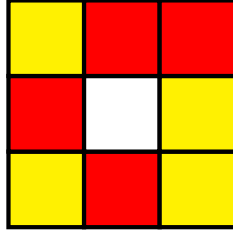
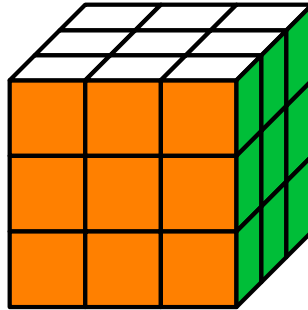


Figure 8: The cyclic move (permutation) allowed in our toy.

**Example 1.2.2** The set of all  $2 \times 2$  matrices with real entries and determinant 1,  $SL_2(\mathbb{R})$ , forms a group under the multiplication of matrices. However it is not an Abelian group, as we can see

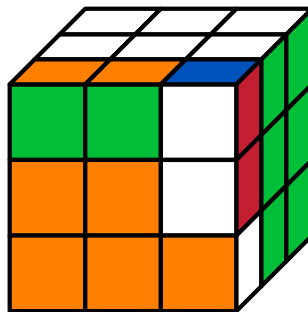
$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (1.3)$$

**Example 1.2.3** Let us examine if the Rubik's group is Abelian. We start from the following solved shape:



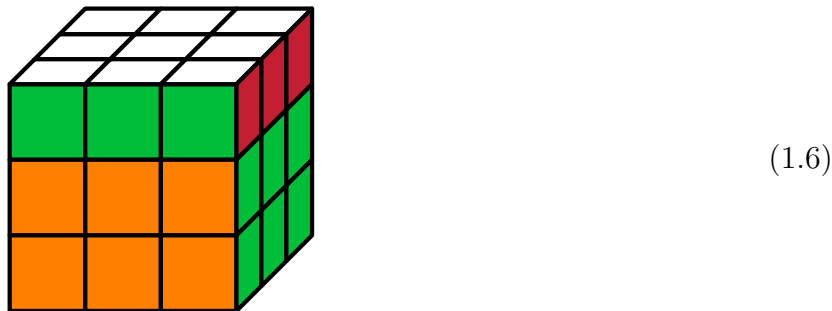
(1.4)

and perform a  $90^\circ$  (clockwise) rotation (denoted as  $R$ ) on the green face (the 4-th face), a  $90^\circ$  (clockwise) rotation (denoted as  $U$ ) on the white face (the 6-th face), and a  $90^\circ$  (counter-clockwise) rotation on the 4-th face (corresponding to  $R^{-1}$  to obtain



(1.5)

We see that it is far different from the shape obtained from the rotation on the 6-th face,



That is to say,

$$R^{-1}UR \neq U, \tag{1.7}$$

so Rubik's group is not Abelian.

**Definition 1.3** (Semigroup, 半群) A semigroup  $G$  is a set of elements closed under the action  $\cdot$ , and satisfies the associativity. i.e.

$$a, b \in G \Rightarrow a \cdot b \in G, \quad (a \cdot b) \cdot c = a \cdot (b \cdot c) \text{ for } \forall a, b, c \in G. \tag{1.8}$$

**Example 1.3.1** The set of all natural numbers,  $\mathbb{N}$ , is a semigroup under the action of addition  $+$ . The set of all positive integers,  $\mathbb{Z}_+$ , is also a semigroup under the same action. They are also semigroups under the operation of multiplication  $\times$ .

**Remark** As we have seen in the above example, a semigroup can have identity element(s), or it can also be consistent to have no identity. In general, we can have one-sided identity elements in a semigroup, that is element  $e$  satisfying  $e \cdot a = a$  for  $\forall a \in G$ , or  $a \cdot e = a$  for  $\forall a \in G$ . The element satisfying both of these properties is called a two-sided identity element. The two-sided element, if exists, will be unique in the semigroup, on the other hand, one can have more than one one-sided identity elements.

**Example 1.3.2** All functions (say in the form  $f : \mathbb{R} \rightarrow \mathbb{R}$ ),  $\{f\}$ , form a semigroup with the composition of functions,  $f \circ g \in \{f\}$  for  $\forall f, g \in \{f\}$ .

**Example 1.3.3** All constant functions,  $f_c$  that is the function mapping all variables into a constant  $c$ , form a semigroup. One can further add an element, say  $f_*(x) = \delta_{x,0}$ , into this semigroup, and it becomes a new semigroup. We can see that  $f_*$  is a right identity element,  $f_c \circ f_* = f_c$  for  $\forall c \in \mathbb{R}$ , while  $f_* \circ f_c = \delta_{c,0}$  is a constant function depending on the input of  $c$ . Of course we can easily think of many such right identities in this example.

**Definition 1.4** (Monoid, 么半群、单群) A monoid  $M$  is a semigroup equipped with a two-sided identity element.

**Example 1.4.1** The set of all natural numbers,  $\mathbb{N}$ , is a monoid both in the sense of addition  $+$  and multiplication  $\times$ . The identity for the multiplication  $\times$  is 1.

**Example 1.4.2** The set of all  $2 \times 2$  matrices with real entries,  $\mathbb{M}_{2,2}(\mathbb{R})$ , is a monoid under the multiplication of matrices. It is, however, a group under the addition of matrices.

**Definition 1.5** (Ring, 环) A ring  $R$  is a set of elements equipped with two (binary) operations, symbolically denoted as  $+$  and  $\cdot$ .  $R$  is an Abelian group under the action of  $+$ , and a monoid under the action of  $\cdot$ . Furthermore, we require the distributivity (分配律) of the multiplication.

$$a \cdot (b + c) = a \cdot b + a \cdot c, \quad (a + b) \cdot c = a \cdot c + b \cdot c. \quad (1.9)$$

That is to say, the multiplication  $\cdot$  is bilinear.

**Example 1.5.1** The set of all integers,  $\mathbb{Z}$ , forms a ring.

**Example 1.5.2** The set of all  $2 \times 2$  matrices with real entries,  $\mathbb{M}_{2,2}(\mathbb{R})$ , is a ring under the action of addition of matrices and multiplication of matrices.

**Example 1.5.3** (Field, 域) A field, also mentioned before as an abstract concept of the sets such as  $\mathbb{R}$  and  $\mathbb{C}$ , is a ring but also Abelian under the multiplication  $\cdot$ , and has identity element and inverse elements of the multiplication. A famous field beyond the real and complex numbers is the quaternion (四元数), usually denoted as  $\mathbb{H}$ . It was originally discovered by an Irish mathematician, Hamilton, and similar to  $\mathbb{C}$  that can be thought as a two-dimensional space,  $\mathbb{H}$  is a four-dimensional space spanned by four basis vectors, 1,  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$ . A general quaternion number,  $h \in \mathbb{H}$ , is parameterized by four real numbers,  $a, b, c, d \in \mathbb{R}$ ,

$$h = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}. \quad (1.10)$$

These basis satisfy

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1. \quad (1.11)$$

From this, we have

$$\mathbf{ij} = -\mathbf{ijk}^2 = -(\mathbf{ijk})\mathbf{k} = \mathbf{k}. \quad (1.12)$$

**Definition 1.6** (Subgroup, 子群) A subgroup,  $H$ , of a given group  $G$  is a subset (子集) of  $G$  that also forms a group itself.

**Example 1.6.1**  $H_0 = \{e\}$  and  $H_{full} = G$  are all subgroups of  $G$ .  $H_0$  is often called the trivial (平凡) subgroup of  $G$ .

**Proposition 1.6.2** The unit element in a subgroup,  $H$ , of  $G$  is the same as the unit element in  $G$ . The inverse of any element in  $H$  is also equal to the inverse of that element in  $G$ .

**Proof:** Denote the unit element in  $H$  as  $e_H$  and the unit element in  $G$  as  $e$ . Then we have

$$e_H \cdot e = e_H, \quad e_H \cdot e_H = e_H, \tag{1.13}$$

respectively following from the property of the identity of  $e$  (in  $G$ ) and  $e_H$  (in  $H$ ). That is to say

$$e_H \cdot e = e_H \cdot e_H. \tag{1.14}$$

By multiplying the inverse element  $e_H^{-1}$  in  $G$ , we obtain  $e \cdot e = e \cdot e_H \Rightarrow e = e_H$ . Let us further denote the inverse element of  $a \in H$  in  $H$  as  $\tilde{a}^{-1}$  to distinguish it with the inverse element  $a^{-1}$  in  $G$ . Since  $e_H = e$ , now we know that

$$\tilde{a}^{-1} \cdot a = a^{-1} \cdot a = e. \tag{1.15}$$

By further using  $a \cdot a^{-1} = e$ , we see that  $\tilde{a}^{-1} = a^{-1}$ .  $\square$

**Example 1.6.3 (魔方)** Rotating only one face of Rubik's cube gives a subgroup of Rubik's group.

**Definition 1.7** (Subring, 子环) A subring is a subset of the ring that preserves the ring structure.

**Definition 1.8** (Generator, 生成元) The set of generators is defined as a subset of a given group (or resp. ring) that is not contained in any subgroup (resp. subring) other than the entire group (resp. ring). The number of elements in the set of generators is called the rank of the group (resp. ring).

**Definition 1.9** (Conjugate, 共轭) For two elements  $h$  and  $h'$  in the group  $G$  satisfying

$$h' = ghg^{-1}, \quad \text{for } \exists g \in G, \tag{1.16}$$

then we call them conjugate to each other.

**Proposition 1.9.1** One can define classes of elements such that all elements related by the conjugate relation are sorted into one class. We usually call it a conjugacy class, and denote it as  $\text{Cl}(g)$ .

**Definition 1.10** (Homomorphism, 同态) Given two groups,  $G$  and  $H$ . If there exists a map from  $G$  to  $H$ ,  $\varphi : G \rightarrow H$ , which preserves the group structure, then we call this map a homomorphism between  $G$  and  $H$ . More precisely, it satisfies (for  $\forall g_1, g_2 \in G$ )

$$\varphi(g_1 \cdot g_2) = \varphi(g_1) \cdot \varphi(g_2), \quad (1.17)$$

and  $\varphi(e)$  is the unit element in  $H$ .

**Example 1.10.1** The trivial (embedding) map from the integer numbers  $\mathbb{Z}$  to real numbers  $\mathbb{R}$  is a homomorphism, while the embedding of the nature numbers  $\mathbb{N}$  in the integers  $\mathbb{Z}$  is **NOT** a homomorphism, as  $\mathbb{N}$  is not a group under  $+/ \cdot$  operation.

**Example 1.10.2** It is also possible to construct a homomorphism from the symmetry group of a rhombus, i.e.  $\mathbf{Dih}_4$ , to the symmetry group of a square, i.e.  $\mathbf{Dih}_8$ . We simply map the reflection about the  $c'$  (resp.  $d'$ ) axis to the reflection about  $c$  (resp.  $d$ ) axis in the square.

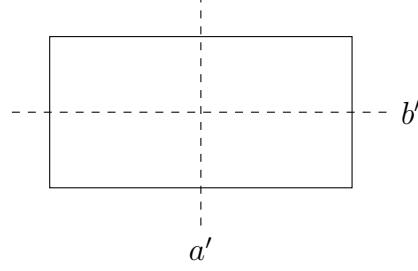
**Definition 1.11** (Kernel, 同态核、核) In a homomorphism  $\varphi : G \rightarrow H$ , the set of all elements in  $G$  that are mapped to the identity element  $e_H$  in  $H$  is called the kernel of this map  $\varphi$ , and is usually denoted as  $\text{Ker}\varphi$ .

Note that the kernel  $\text{Ker}\varphi$  forms a subgroup in  $G$ .

**Definition 1.12** (Isomorphism, 同构) If a homomorphism,  $\varphi : G \rightarrow H$  is also a bijection (双射) map (that is to say, there is a one-to-one correspondence between all the elements in  $G$  and  $H$ ), then we call  $\varphi$  an isomorphism between  $G$  and  $H$ . With such a bijection map, we say that  $G$  is isomorphic to  $H$  and denote this relation as  $G \simeq H$ .

**Example 1.12.1** The group  $\{e, \sigma\}$  satisfying  $e \cdot \sigma = \sigma \cdot e = \sigma$ ,  $e^2 = e$ ,  $\sigma^2 = e$  is isomorphic to  $\mathbb{Z}_2 = \{0, 1\}$  equipped with the addition mod 2.

**Example 1.12.2** The symmetry of a rectangle (长方形) is isomorphic to that of the rhombus.



More precisely, we can list the group action of these two groups respectively consisted of  $\{e, c_\pi, r_{a'}, r_{b'}\}$  and  $\{e, c_\pi, r_{c'}, r_{d'}\}$ .

	$e$	$c_\pi$	$r_{a'}$	$r_{b'}$
$e$	$e$	$c_\pi$	$r_{a'}$	$r_{b'}$
$c_\pi$	$c_\pi$	$e$	$r_{b'}$	$r_{a'}$
$r_{a'}$	$r_{a'}$	$r_{b'}$	$e$	$c_\pi$
$r_{b'}$	$r_{b'}$	$r_{a'}$	$c_\pi$	$e$

	$e$	$c_\pi$	$r_{c'}$	$r_{d'}$
$e$	$e$	$c_\pi$	$r_{c'}$	$r_{d'}$
$c_\pi$	$c_\pi$	$e$	$r_{d'}$	$r_{c'}$
$r_{c'}$	$r_{c'}$	$r_{d'}$	$e$	$c_\pi$
$r_{d'}$	$r_{d'}$	$r_{c'}$	$c_\pi$	$e$

The isomorphism map can be constructed as

$$e \mapsto e, \quad c_\pi \mapsto c_\pi, \quad r_{a'} \mapsto r_{c'}, \quad r_{b'} \mapsto r_{d'}. \quad (1.18)$$

**Definition 1.13** (Normal subgroup, 正规子群 or Invariant subgroup, 不变子群) Given a subgroup  $H$  of  $G$ , we denote  $gHg^{-1} = \{ghg^{-1} | h \in H\}$ . When  $H = gHg^{-1}$ , for  $\forall g \in G$ , we call such a subgroup  $H$  a normal subgroup (or an invariant subgroup) of  $G$ .

**Corollary 1.13.1** The kernel of  $H$ ,  $\text{Ker}\varphi$ , is a normal subgroup in  $G$  (with  $\varphi : G \rightarrow H$ ).

**Definition 1.14** (Quotient group, 商群) Let  $H$  be a normal subgroup of  $G$ . The quotient of  $G$  by  $H$ , denoted as  $G/H$ , is defined as the set  $\{gH | g \in G\}$ . A group structure can be defined on this set as

$$(gH) \circ (g'H) = (g \cdot g')H. \quad (1.19)$$

**Remark** The representative  $g$  for  $gH$  is of course not unique. Say  $g_1H = g_2H$  and  $g'_1H = g'_2H$ , then we have

$$(g_1 \cdot g_2)H = g_1(g_2H) = g_1(g'_2H). \quad (1.20)$$

Due to the invariant property of  $H$ ,  $g'_2H = Hg'_2$ , and thus

$$(g_1 \cdot g_2)H = g_1Hg'_2 = (g_1H)g'_2 = g'_1Hg'_2 = (g'_1 \cdot g'_2)H. \quad (1.21)$$

We thus see that the group structure does not depend on the choice of the representatives of the quotient group.

**Remark** Do not confuse the notation of the quotient group  $G/H$  with the complement set (補集) of  $H$  in  $G$ ,  $G \setminus H$ . The latter represents the set of elements that are in  $G$  but not in  $H$ , i.e. if  $g \in G \setminus H$ , then  $g \in G$ , but  $g \notin H$ .

**Proposition 1.14.1** Given a homomorphism map  $\varphi : G \rightarrow H$ ,  $\varphi(G) \simeq G/\text{Ker}\varphi$ .  $\varphi(G)$  is often called the image of  $\varphi$ , and denoted as  $\text{Im}\varphi$ .

**Proof** Let us construct the following map  $\pi : G/\text{Ker}\varphi \rightarrow \varphi(G)$ ,

$$\pi(g\text{Ker}\varphi) = \varphi(g). \quad (1.22)$$

We need to show that this map preserves the group structure and it is one-to-one. The first property is almost obvious,

$$\pi(g\text{Ker}\varphi) \cdot \pi(g'\text{Ker}\varphi) = \varphi(g) \cdot \varphi(g') = \varphi(g \cdot g') = \pi((g \cdot g')\text{Ker}\varphi) = \pi((g\text{Ker}\varphi)\varphi \circ (g'\text{Ker}\varphi)). \quad (1.23)$$

Now let us show the second property and given  $g\text{Ker}\varphi = g'\text{Ker}\varphi$ , then  $g' = g \cdot h$  for  $\exists h \in \text{Ker}\varphi$ . We thus have

$$\varphi(g') = \varphi(g \cdot h) = \varphi(g) \cdot \varphi(h) = \varphi(g) \cdot e_H = \varphi(g), \quad (1.24)$$

and vice versa. That is the map is independent of the choice of the representatives, and is an injection. It is also obviously a surjection, so the map  $\pi$  is a bijection, and therefore is an isomorphism.

**Example 1.14.2** Let us take  $G = \mathbb{Z}$  and  $H = 2\mathbb{Z} = \{2i\}_{i \in \mathbb{Z}}$ . Of course,  $H$  is a subgroup of  $G$  and since  $G$  is abelian (under the action of addition), any subgroup of  $G$  becomes a normal subgroup. The quotient  $G/H$  has elements in the form  $\{g + 2i | g, i \in \mathbb{Z}\}$ , and it only has two different elements, i.e. when  $g = 2n$  for  $n \in \mathbb{Z}$ ,  $g + 2\mathbb{Z} = 2\mathbb{Z}$ , and when  $g = 2n + 1$  for  $n \in \mathbb{Z}$ ,  $g + 2\mathbb{Z} = \{2\mathbb{Z} + 1\}$  (the set of all odd integers). We thus see that  $G/H = \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2 = \{0, 1\}$ .

**Example 1.14.3** Let us consider the projection map,  $\varphi : G \otimes H \rightarrow H$  with  $g \otimes h \mapsto h$ . It is easy to see that the kernel of this map is given by  $\text{Ker}\varphi = G \otimes e_H := \{g \otimes e_H | g \in G\} \simeq G$ . Then we can define the quotient group  $G \otimes H/\text{Ker}\varphi = \{h(G \otimes e_H) = G \otimes h | h \in H\} \simeq H = \text{Im}\varphi$ .

**Definition 1.15** (Left action on a set, 在集合上的左方作用) We say a group  $G$  acts on a set  $X$  from the left when for each element  $g \in G$ , there exists a map,

$$\varphi_g : X \rightarrow X, \quad (1.25)$$

s.t. for  $\forall g, h \in G$ ,

$$\varphi_g \circ \varphi_h = \varphi_{(g \cdot h)}. \quad (1.26)$$

For  $\forall x \in X$ , we usually use the following short notation to denote  $\varphi_g(x)$ ,

$$gx := \varphi_g(x). \quad (1.27)$$

**Example 1.15.1** Consider the set of all column vectors,  $\mathbb{R}^2$ , whose typical element looks like

$$\begin{pmatrix} a \\ b \end{pmatrix}, \quad a, b \in \mathbb{R}, \quad (1.28)$$

then the set of all real  $2 \times 2$  matrices,  $\mathbb{M}_{2,2}(\mathbb{R})$ , naturally acts on  $\mathbb{R}^2$  from the left, i.e. for  $\forall v \in \mathbb{R}^2$  and  $M \in \mathbb{M}_{2,2}(\mathbb{R})$ , we have  $Mv \in \mathbb{R}^2$ .

**Definition 1.16** (automorphism, 自同构) When we have an isomorphism  $\varphi : G \rightarrow G$  from a group  $G$  to itself, we call it an automorphism.

## 1.2 Representation Theory (表示论)

Going beyond the defining set of an algebraic concept, such as a group or a ring etc., we may extract out the abstract algebraic relations that characterize the underlying group or ring, and realize these relations in a more general space. This is basically the idea of the representation theory. As we will see later in this lecture, there exists many different ways to realize the same algebraic relations. Very often, there are some central elements behind the algebra, and taking different representations of the algebra gives different values to the central elements.

**Definition 1.17** (representation, 表示) A representation of a group  $G$  is a homomorphism from  $G$  to (usually) a group of square matrices  $M$ , denoted as  $R : G \rightarrow M$ . It follows from the property of the homomorphism that  $R(g \cdot h) = R(g)R(h)$  for  $\forall g, h \in G$ .

**Remark:** From the above definition, we always use a finite-size matrix to represent each element in the group  $G$ . However, this is not always necessary or even possible for any given  $G$ . Strictly speaking, the representations defined here are mathematically called finite-dimensional representations. Indeed it is known that there are groups without any non-trivial finite-dimensional representation.

**Remark:** First note that the identity element  $e$  of  $G$  is mapped the identity matrix,  $\mathbb{I}$ , in  $M$ . Then note that from the very definition of group, there is always an inverse element  $g^{-1}$  for any  $g \in G$  and it satisfies

$$R(g^{-1})R(g) = R(e) = \mathbb{I} = R(g)R(g^{-1}). \quad (1.29)$$

That is to say,  $R(g)$  is invertible for  $\forall g \in G$ . The set of all  $n \times n$  invertible matrices is called the general linear group (一般线性群) of degree  $n$ , and is denoted as  $GL(n)$ . One can set  $M$  in the above definition to be  $GL(n)$  for  $\exists n$  without losing any generality. Furthermore, depending on the real or complex nature of all the elements in the matrices, one may further specify the general linear group to be  $GL(n, \mathbb{R})$  or  $GL(n, \mathbb{C})$ .

**Example 1.17.1** (魔方) In the case of Rubik's cube, we first need to construct the representation space (表示空间), which is spanned by all the allowed configurations in Rubik's cube:

$$\left\{ \begin{array}{c} \text{[Rubik's cube configurations]} \end{array} \right\} \quad (1.30)$$

We assign to each configuration a vector, say  $(1, 0, 0, \dots)$ ,  $(0, 1, 0, \dots)$ ,  $(0, 0, 1, \dots)$  and so on, of dimension equal to the number of configurations (in this case roughly  $4 \times 10^{19}$ ).

**Definition 1.18** (trivial representation, 平凡表示) A trivial representation maps  $I$  all the elements of a group  $G$  to the identity matrix, so  $I(g \cdot h) = \mathbb{I} \cdot \mathbb{I} = \mathbb{I} = I(g)I(h)$  is trivially satisfied for  $\forall g, h \in G$ .

**Remark:** Of course, it is always possible to construct a trivial representation (with arbitrary matrix size) for any group  $G$ .

**Example 1.18.1** (魔方) The trivial representation of Rubik's group is constructed on the following cube (without distinguishing different colors in the original Rubik's cube).

$$\text{[A 3x3x3 cube with all faces shaded gray]} \quad (1.31)$$

**Definition 1.19** (dimension, 维数) The dimension of a representation is defined as the order of matrices used in the representation.

**Definition 1.20** (faithful representation, 忠实表示) When the representation is isomorphic, then we call it a faithful representation of the group  $G$ .

Let us depict the core idea of the representation theory with some simple examples.

**Example 1.20.1** The first example is  $\mathbb{Z}_2 = \{0, 1\}$ . 0 is the identity element for the addition operation, and we also have  $1 + 1 = 0$ . This relation can be represented by a map  $R_2$  to two  $2 \times 2$  matrices, i.e.

$$0 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad 1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (1.32)$$

and the multiplication of matrices is identified as the group action. The dimension of this representation  $R_2$  is two, and one can confirm that it is also a faithful representation. We can also construct a one-dimensional representation,  $R_1 : \mathbb{Z}_2 \rightarrow \mathbb{R}$ , that maps both 0 and 1 into the real number 1.  $R_1$  clearly is not a faithful representation, as it is a trivial representation.

**Example 1.20.2\*** The second example we look into, which is more interesting, is the symmetric group. The symmetric group can be generated by the set of adjacent transpositions  $\sigma_i = (i, i + 1)$  that permutes the  $i$ -th and  $(i + 1)$ -th elements (with  $N + 1$  identified with 1). These generators satisfy

$$\sigma_i^2 = 1, \quad \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i - j| > 1, \quad (\sigma_i \sigma_{i+1})^3 = 1. \quad (1.33)$$

The last relation can be confirmed with the explicit action,

$$\sigma_i \sigma_{i+1} = (i + 2, i, i + 1), \quad (1.34)$$

that is a cyclic permutation of three elements. The relations (1.33) are known as special cases of the Coxeter group, where several constraints of the form

$$(\sigma_i \sigma_j)^{m_{ij}} = 1, \quad (1.35)$$

are imposed on the set  $\{\sigma_i\}$  and  $m_{ii} = 1$ , and if  $m_{ij} \neq 0$  for some  $i \neq j$ , then  $m_{ij} \geq 2$ . We denote the symmetric group acting on  $n$  labels as  $\mathfrak{S}_n$ .

A very important concept in the context of the symmetric group is the partition. A partition (整数拆分) is a set of positive integer numbers, say  $\lambda := \{\lambda_i\}_{i=1}^N$ , satisfying  $\lambda_i \geq \lambda_{i+1}$  for  $i = 1, 2, \dots, N - 1$ . A partition is often graphically expressed as a Young diagram, with the number of boxes in each row of the diagram equal to the integer  $\lambda_i$ . An example for  $\lambda = \{5, 4, 4, 2, 2, 1\}$  is shown in Figure 9. We further define the object named  $\lambda$ -tableau, which is defined by assigning the number  $1, 2, \dots, |\lambda| = \sum_{i=1}^N \lambda_i$  to each box of  $\lambda$  (see Figure 10). One can symmetrize the numbers

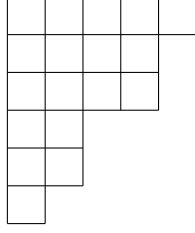


Figure 9: The Young diagram for  $\lambda = \{5, 4, 4, 2, 2, 1\}$ .

1	2	4	5
3	7	6	
8			

Figure 10: A  $\lambda$ -tableau associated to  $\lambda = \{4, 3, 1\}$ .

assigned to each line, that is to say, when we consider the symmetric group  $\mathfrak{S}_{|\lambda|}$  acting on the set of numbers  $\{1, 2, \dots, |\lambda|\}$ , for example the subgroup that only shuffles among  $\{1, 2, 4, 5\}$  does not change the  $\lambda$ -tableau shown in Figure 10. One can easily see that the set of all  $\lambda$ -tableaux associated to a given Young diagram  $\lambda$  forms a vector space that the symmetric group naturally acts on. More practically, since there are  $|\lambda|!$  ways to assign  $\{1, 2, \dots, |\lambda|\}$  into a Young diagram  $\lambda$ , we can use a  $|\lambda|!$ -dimensional vector space for the representation space, and then we can express the symmetric group  $\mathfrak{S}_{|\lambda|}$  in the form of matrices.

The conjugacy classes of the symmetric group is also labeled by Young diagrams. Let us see how it works in several examples. In the simplest case of  $\mathfrak{S}_2 = \mathbb{Z}_2$ , there are only two elements and thus two conjugacy classes. Let us proceed to the case of  $\mathfrak{S}_3$  generated by  $\{\sigma_1, \sigma_2\}$ . We have  $6 = 3!$  elements in total,  $1, \sigma_1, \sigma_2, \sigma_1\sigma_2, \sigma_2\sigma_1, \sigma_1\sigma_2\sigma_1$ . Note that  $\sigma_2\sigma_1\sigma_2 = \sigma_1\sigma_2\sigma_1$  as we have

$$\sigma_1(\sigma_2\sigma_1\sigma_2)\sigma_1\sigma_2 = (\sigma_1\sigma_2)^3 = 1. \quad (1.36)$$

We can write down a table to show the group action in  $\mathfrak{S}_3$ .

	1	$\sigma_1$	$\sigma_2$	$\sigma_1\sigma_2$	$\sigma_2\sigma_1$	$\sigma_1\sigma_2\sigma_1$
1	1	$\sigma_1$	$\sigma_2$	$\sigma_1\sigma_2$	$\sigma_2\sigma_1$	$\sigma_1\sigma_2\sigma_1$
$\sigma_1$	$\sigma_1$	1	$\sigma_1\sigma_2$	$\sigma_2$	$\sigma_1\sigma_2\sigma_1$	$\sigma_2\sigma_1$
$\sigma_2$	$\sigma_2$	$\sigma_2\sigma_1$	1	$\sigma_1\sigma_2\sigma_1$	$\sigma_1$	$\sigma_1\sigma_2$
$\sigma_1\sigma_2$	$\sigma_1\sigma_2$	$\sigma_1\sigma_2\sigma_1$	$\sigma_1$	$\sigma_2\sigma_1$	1	$\sigma_2$
$\sigma_2\sigma_1$	$\sigma_2\sigma_1$	$\sigma_2$	$\sigma_1\sigma_2\sigma_1$	1	$\sigma_1\sigma_2$	$\sigma_1$
$\sigma_1\sigma_2\sigma_1$	$\sigma_1\sigma_2\sigma_1$	$\sigma_1\sigma_2$	$\sigma_2\sigma_1$	$\sigma_1$	$\sigma_2$	1

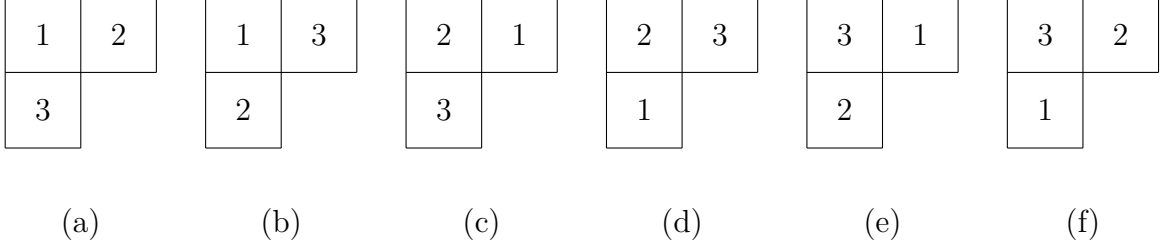


Figure 11: Six different Young tableaux, but (a) is equivalent to (c), (b) is equivalent to (e), and (d) is equivalent to (f).

This kind of table is known as the Clayley table (群表). The identity operator always forms a single conjugacy class by itself. We note that e.g.  $(\sigma_1\sigma_2\sigma_1)\sigma_2(\sigma_2\sigma_1\sigma_2) = \sigma_1$ , so  $\{\sigma_1, \sigma_2, \sigma_1\sigma_2\sigma_1\}$  form a conjugacy class. We also have  $\sigma_1(\sigma_1\sigma_2)\sigma_1 = \sigma_2\sigma_1$  and  $\sigma_1\sigma_2\sigma_1(\sigma_1\sigma_2)\sigma_2\sigma_1\sigma_2 = \sigma_1\sigma_2$ , so the remaining conjugacy class is  $\{\sigma_1\sigma_2, \sigma_2\sigma_1\}$ . As we have two conjugacy classes for  $\mathfrak{S}_2$  and three for  $\mathfrak{S}_3$ , it is not hard to map the conjugacy classes to the partitions of  $n$ . One can further check this claim for larger  $n$ 's.

Let us give a concrete example for  $\lambda = (2, 1)$ . There are  $3! = 6$  different assignments of the three numbers, but there are only three inequivalent  $\lambda$ -tableaux (see Figure 11).

We construct the representation on the 6-dim vector space with

$$\begin{aligned}
 (a) &\mapsto (1, 0, 0, 0, 0, 0)^t, & (b) &\mapsto (0, 1, 0, 0, 0, 0)^t, \\
 (c) &\mapsto (0, 0, 1, 0, 0, 0)^t, & (d) &\mapsto (0, 0, 0, 1, 0, 0)^t, \\
 (e) &\mapsto (0, 0, 0, 0, 1, 0)^t, & (f) &\mapsto (0, 0, 0, 0, 0, 1)^t.
 \end{aligned} \tag{1.37}$$

Let us denote  $e_{\{t\}}$  for each of the above basis vector. For example we can use the notation  $\{(1, 2), (3)\}$  for the Young tableau (a) in Figure 11. We further define

$$e_t = \sum_{\pi \in C_\lambda} \text{sgn}(\pi) e_{\pi(\{t\})}, \tag{1.38}$$

where  $C_\lambda$  is the stabilization subgroup of  $\mathfrak{S}_{|\lambda|}$  that keeps each  $\lambda$ -tableau invariant, i.e. the numbers in each row of the  $\lambda$ -tableau are kept. In the case of  $\{(1, 2), (3)\}$ ,  $C_\lambda = \{\sigma_1\}$  consists only one element that exchanges 1 and 2. On the subspace spanned by  $\{e_t\}$ , we obtain an irreducible representation of the symmetric group. Let us change the basis of the subspace to

$$e_{(1,2),(3)} \mapsto (1, 0, 0)^t, \quad e_{(1,3),(2)} \mapsto (0, 1, 0)^t, \quad e_{(2,3),(1)} \mapsto (0, 0, 1)^t, \tag{1.39}$$

and write down the action of  $\sigma_1$  as

$$\sigma_1 \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{1.40}$$

and that of  $\sigma_2$  as

$$\sigma_2 \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (1.41)$$

One can easily check that

$$\sigma_{(3,1,2)} = \sigma_1 \sigma_2 \mapsto \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}, \quad (1.42)$$

which satisfies

$$\sigma_{(3,1,2)}^3 = \text{id}. \quad (1.43)$$

We remark that  $(1, 2), (3)$  for example is not mapped to  $(3, 1), (2)$  under the action of  $\sigma_{(3,1,2)}$ . The number has different meaning the these two different labeling.

**Remark:** The representation constructed here labeled by  $\lambda = (|\lambda|)$  of  $\mathfrak{S}_{|\lambda|}$  is always a trivial representation.

**Remark:** (similarity transformation) Given a representation  $R$  of  $G$ , one can define a class of representations via

$$R_M(g) := M^{-1}R(g)M, \quad (1.44)$$

as long as  $M$  is an invertible matrix with the same size as the representation  $R$ . We call this the similarity transformation of the original representation  $R$ . Since we are not interested in such similarity representations, we will only consider a representative of such a class of representations in what follows.

**Definition 1.21** (irreducible representation, 不可约表示) An irreducible representation is defined as a representation that does not have a subspace of the representation space on which the action of the group closes.

**Example 1.21.1** Let us start with  $\mathbb{Z}_2 = \{0, 1\}$ . It is not difficult to construct a three-dimensional representation,

$$0 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad 1 \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (1.45)$$

If we perform a similarity transformation with

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (1.46)$$

we will have

$$0 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad 1 \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (1.47)$$

We see that this representation is nothing but a direct sum of a trivial representation and the faithful representation (1.32), so it is not irreducible.

In fact the faithful representation  $R_2$  (1.32) is also reducible, since we can diagonalize

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (1.48)$$

without changing the identity operator by using the matrix

$$M = \begin{pmatrix} \frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix}. \quad (1.49)$$

Therefore, after a similarity transformation, we obtain a new 2-dimensional representation,  $R'_2$ ,

$$0 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad 1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (1.50)$$

We see that there are two irreducible representations in the representation theory of  $\mathfrak{S}_2 = \mathbb{Z}_2$ : one is the trivial representation (we denote it by  $R_{(2)}$  in terms of the Young diagram), and another is  $R_{(1,1)}$  that maps

$$0 \mapsto 1, \quad 1 \mapsto -1, \quad (1.51)$$

with the group action mapped to the multiplication of integers.

We note that the representation labeled by  $\lambda = (1, 1)$  constructed in the basis of (1.38) is simply the 2-dim representation (1.32), and we need to further decompose it to the direct sum of lower-dimensional representations.

**Example 1.21.3\***  $\mathfrak{S}_3$  represented on (1.39) is an irreducible representation, but on the 6-dim vector space (1.37) it is not an irreducible representation, since the action of  $\mathfrak{S}_3$  closes on the subspace of (1.39). It is well-known that all the irreducible representations of the symmetric group can be mapped to such an irreducible representation labeled by some  $\lambda$ -tableau.

**Decomposition of reducible representation** One of the main purposes to study group theory in physics is to decompose a composite representation of into irreducible representations. A well-known example is the decomposition of the (tensor) product of an up spin and a down spin,

$$|\uparrow\rangle \otimes |\downarrow\rangle = \frac{1}{2} (|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle) + \frac{1}{2} (|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle), \quad (1.52)$$

where the original state is a reducible representation belongs to the composite of two spin-1/2 representations, and the first term on the right-hand side belongs to the spin-1 representation, while the last term belongs to the spin-0 representation.

In Example 1.21.1 described above, we constructed a three-dimensional representation, and let us denote it as  $\mathbf{3}_{\mathbb{Z}_2}$ . Similarly if we denote the trivial representation and the two-dimensional faithful representation (1.32) respectively as  $\mathbf{1}_{\mathbb{Z}_2}$  and  $\mathbf{2}_{\mathbb{Z}_2}$ , then we know that

$$\mathbf{3}_{\mathbb{Z}_2} = \mathbf{2}_{\mathbb{Z}_2} \oplus \mathbf{1}_{\mathbb{Z}_2}. \quad (1.53)$$

## 2 Finite (Discrete) Groups

**Definition 2.1** (order, 阶) The number of elements in a group,  $G$ , is called the order of this group, and is usually denoted as  $|G|$ .

**Definition 2.2** (finite group, 有限群) A group with finite order is called a finite group. Consequently, its elements can be labeled by a discrete parameter.

**Example 2.2.1**  $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$  equipped with the addition  $+$  mod  $n$  for  $\forall n \in \mathbb{Z}_+$  is one of the most known finite group. It is obviously also an Abelian group. This group is usually called the cyclic group (循环群).

**Example 2.2.2** The symmetric groups are also finite groups.  $\mathfrak{S}_N$  is of order  $N - 1$ .

**Example 2.2.3** The rotation group of degree  $0^\circ$ ,  $90^\circ$ ,  $180^\circ$  and  $270^\circ$  appearing in the very beginning of this article is a finite group. Of course, the dihedral group  $\mathbf{Dih}_8$  that contains the above rotation group as a subgroup is also a finite group. The rotation group mentioned here is isomorphic to  $\mathbb{Z}_4$ .

**Example 2.2.4** The rotation group on a 2D plane with arbitrary rotation angle can be represented as  $\{e^{i\theta} | \theta \in \mathbb{R}\}$  and the multiplication of this group is simply given by the product of the complex numbers,  $e^{i\theta}$ , i.e.  $e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$ . It is apparently not a finite group, and it is usually named

the  $U(1)$  group, or equivalently the  $SO(2)$  group. However, the sets of rotations by  $\theta = 2\pi k/n$  (for  $k = 0, 1, \dots, n-1$ ) form subgroups of  $U(1)$ . They are isomorphic to  $\mathbb{Z}_n$  and are thus finite groups.

**Example 2.2.5: Abelian finite groups** In an Abelian finite group  $G$ , since it is finite, we must have  $g^n = 1$  for some  $n \in \mathbb{N}$  and for each  $g \in G$ .  $n$  usually differs for different  $g \in G$ . Furthermore, for a given  $g$ ,  $\{e, g, g^2, \dots, g^{n-1}\}$  forms a subgroup of  $G$ , and is isomorphic to  $\mathbb{Z}_n$ . In general, Abelian finite groups can be decomposed as a direct sum  $G \simeq \bigoplus_i^{|G|} \mathbb{Z}_{n_i}$ .

**Definition 2.3** (stabilizer subgroup, 固定子群) Let  $G$  be a group act from the left on the set  $X$ . Given an element  $x \in X$ , we define the stabilizer subgroup of  $x$  in  $G$  as

$$G_x := \{g \in G | gx = x\}. \quad (2.1)$$

When  $G_x = G$ , we call  $x$  the fixed point (不动点) under the action of  $G$ .

**Definition 2.4** (setwise stabilizer, 集合固定子群 or little group, 小群) The subgroup that preserves a subset  $S$  of  $X$  is called the setwise stabilizer of  $S$ , i.e.

$$G_S := \{g \in G | gx \in S, \forall x \in S\}. \quad (2.2)$$

In the context of physics, this subgroup is often called the little group.

Let us describe the finite subgroups of  $O(3)$  (3D rotation group + reflections) here. They can be classified by listing the subsets of 3-vectors that have non-trivial stabilizers.

**Example 2.4.1** Let us consider the set of points

$$\left( \cos \frac{2\pi k}{n}, \sin \frac{2\pi k}{n}, 0 \right)^t, \quad \text{for } k = 0, 1, \dots, n-1. \quad (2.3)$$

Connecting the neighboring points with straight lines, we obtain a regular  $n$ -gon (正  $n$  边形) on the  $x$ - $y$  plane. The stabilizer subgroup of a regular  $n$ -gon is called the dihedral group (二面体群)  $\mathbf{Dih}_{2n}$ . There are  $2n$  elements in this group:  $n$  of them are rotations and  $n$  of them are reflections. Let us give the explicit matrix expressions of the elements in  $\mathbf{Dih}_6$  as an example. There are three rotations,

$$L_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad L_{\frac{2\pi}{3}} = \begin{pmatrix} \cos \frac{2\pi}{3} & -\sin \frac{2\pi}{3} & 0 \\ \sin \frac{2\pi}{3} & \cos \frac{2\pi}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad L_{\frac{4\pi}{3}} = \begin{pmatrix} \cos \frac{4\pi}{3} & -\sin \frac{4\pi}{3} & 0 \\ \sin \frac{4\pi}{3} & \cos \frac{4\pi}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (2.4)$$

and three reflections

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} \cos \frac{2\pi}{3} & \sin \frac{2\pi}{3} & 0 \\ \sin \frac{2\pi}{3} & -\cos \frac{2\pi}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_3 = \begin{pmatrix} \cos \frac{4\pi}{3} & \sin \frac{4\pi}{3} & 0 \\ \sin \frac{4\pi}{3} & -\cos \frac{4\pi}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (2.5)$$

The dihedral group in general is generated by a rotation operator and a reflection operator. We remark that the  $n$ -gon can be put on any 2D sub-plane (containing the origin) of the 3D space. However, since an arbitrary sub-plane can be obtained from the  $x$ - $y$  plane via a proper rotation  $g$ , the symmetry group of an  $n$ -gon on that plane is a conjugate of the dihedral group explicitly written down above, i.e.

$$g(\mathbf{Dih}_{2n})g^{-1}. \quad (2.6)$$

Our classification will be up to this kind of conjugacy classes.

**Example 2.4.2**  $\mathbf{Dih}_6$ . The symmetry group of a regular triangle is known as the dihedral group of order 6,  $\mathbf{Dih}_6$ . It contains the rotation of  $\frac{2\pi}{3}$  and  $\frac{4\pi}{3}$ , and the reflection with respect to the axes  $S_0, S_1$  and  $S_2$  (see Figure 12). By denoting the counter-clockwise rotation by  $\frac{2\pi}{3}$  and  $\frac{4\pi}{3}$  respectively as  $r_1$  and  $r_2$ , and the reflections as  $s_{0,1,2}$ , we obtain the following Cayley table:

	$e$	$r_1$	$r_2$	$s_0$	$s_1$	$s_2$
$e$	$e$	$r_1$	$r_2$	$s_0$	$s_1$	$s_2$
$r_1$	$r_1$	$r_2$	$e$	$s_1$	$s_2$	$s_0$
$r_2$	$r_2$	$e$	$r_1$	$s_2$	$s_0$	$s_1$
$s_0$	$s_0$	$s_2$	$s_1$	$r_0$	$r_2$	$r_1$
$s_1$	$s_1$	$s_0$	$s_2$	$r_1$	$r_0$	$r_2$
$s_2$	$s_2$	$s_1$	$s_0$	$r_2$	$r_1$	$r_0$

(2.7)

As we have

$$s_0 r_1 s_0 = r_2, \quad r_1 s_0 r_2 = s_2, \quad r_1 s_1 r_2 = s_0, \quad (2.8)$$

we see that the conjugacy classes in  $\mathbf{Dih}_6$  are

$$\{e\}, \quad \{r_1, r_2\}, \quad \{s_0, s_1, s_2\}. \quad (2.9)$$

It is also not hard to construct a representation of  $\mathbf{Dih}_6$  from its graphic realization (12). By identifying  $e \equiv r_0$ , we obtain

$$r_k = \begin{pmatrix} \cos \frac{2\pi k}{3} & -\sin \frac{2\pi k}{3} \\ \sin \frac{2\pi k}{3} & \cos \frac{2\pi k}{3} \end{pmatrix}, \quad s_k = \begin{pmatrix} \cos \frac{2\pi k}{3} & \sin \frac{2\pi k}{3} \\ \sin \frac{2\pi k}{3} & -\cos \frac{2\pi k}{3} \end{pmatrix}, \quad (2.10)$$

for  $k = 0, 1, 2$ . We note that

$$\det r_k = 1, \quad \det s_k = -1. \quad (2.11)$$

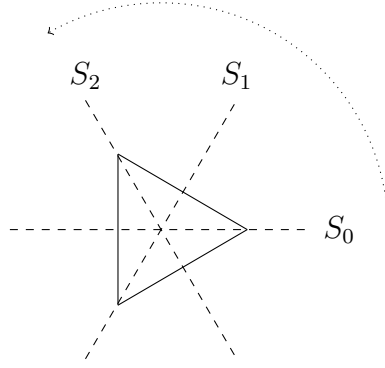


Figure 12: A regular triangle.

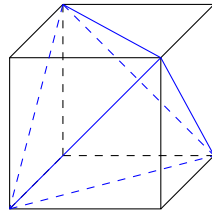


Figure 13: A regular tetrahedron embedded in a cube.

**Example 2.4.3** Let us consider the following set of points,

$$(\pm 1, 0, -\frac{1}{\sqrt{2}})^t, \quad (0, \pm 1, \frac{1}{\sqrt{2}})^t. \quad (2.12)$$

These points form a regular tetrahedron (正四面体). The stabilizer of the tetrahedron can be understood by embedding it in a cube (or a hexahedron, 正六面体、立方体). See Figure 13. Around the axis penetrating perpendicularly through the center of each triangular face of the tetrahedron, there exist respectively a  $120^\circ$  and a  $240^\circ$  rotation that keep the tetrahedron invariant. There are in addition three  $180^\circ$  rotations with their axes perpendicular to the faces of the framing cube. Together with the identity operator and the reflections, we obtain a symmetry group of order  $12 \times 2 = 24$ , and this group is usually called the tetrahedral group (正四面体群),  $\mathcal{T}$ .

**Example 2.4.4** Now we consider a regular octahedron (正八面体). It can also be embedded into a cube (see Figure 14). We remark that the octahedron is obtained by connecting the center of each face of the cube, and in fact connecting the centers of faces of an octahedron also produces a cube. When this kind of relation is satisfied, we say that the cube and the (regular) octahedron are dual to each other. This is the analogue of the relation between a rectangle and a rhombus (see Figure 15). The symmetry groups of the dual geometric shapes are obviously isomorphic to each other. We can see that there are  $3 \times 3$  ( $90^\circ, 180^\circ, 270^\circ$  rotations on each surface of cube) +  $2 \times 4$  ( $120^\circ$  and  $240^\circ$  rotations on each surface of octahedron) +  $1 \times 6$  ( $180^\circ$  rotation around each axis connecting the

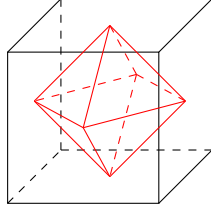


Figure 14: A regular octahedron embedded in a cube.

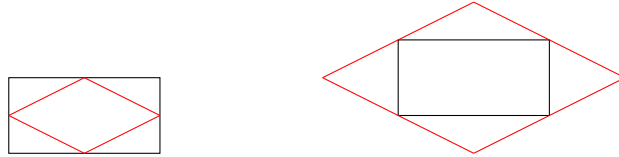


Figure 15: The dual relation between the rectangle and the rhombus.

centers of opposite sides of cube)+1 (identity)= 24 rotational generators in the group. Combined with the reflection operator, the whole symmetry group is doubled to have 48 elements. This group is called the Octahedral group (正八面体群),  $\mathcal{O}$ .

As a well-known fact, the (convex) regular polyhedra can be classified into 5 simple types: tetrahedron, cube, octahedron, dodecahedron (正十二面体) and icosahedron (正二十面体). The dodecahedron and the icosahedron are again dual to each other, and their symmetry group is called the icosahedral group (正二十面体群)  $\mathcal{I}$ .

Correspondingly, it was shown that up to the conjugacy classes, finite subgroups of  $O(3)$  are classified as  $\mathbb{Z}_n$ ,  $\mathbf{Dih}_{2n}$ ,  $\mathcal{T}$ ,  $\mathcal{O}$  and  $\mathcal{I}$ . This is known as McKay's ADE classification. We will see its similarity with the classification in Lie algebras later, and here to help memorizing, we may simply think of three classes as A(belian), D(ihedral) and E(xceptional) (including  $\mathcal{T}$ ,  $\mathcal{O}$  and  $\mathcal{I}$ ). Their direct connection with the ADE classes in Lie algebra can be most clearly seen in the context of the algebraic geometry (代数几何).

## 2.1 $O(N)$ and $SO(N)$

As has been described in (0.40) and (0.41), all  $3 \times 3$  matrices with unit determinant and satisfying

$$O^t O = \mathbb{I}, \quad (2.13)$$

for the  $SO(3)$  group. Similarly, one can extend such a definition to the set of all  $N \times N$  matrices, and the obtained group is called the  $SO(N)$  group. As mentioned before, we can remove the restriction of unit determinant (0.41) in the case of  $SO(3)$  to get an  $O(3)$  group. This also extends the  $SO(N)$

group to an  $O(N)$  group. Back to the case of  $O(3)$ , we first see that

$$-\mathbb{I} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in O(3). \quad (2.14)$$

As the operation of the transpose does not change the determinant, we have from (2.13) that

$$(\det O)^2 = 1 \Rightarrow \det O = \pm 1. \quad (2.15)$$

Given an arbitrary element  $O \in SO(3)$ , then we obtain an element

$$-O \notin SO(3), \quad \text{but} \quad -O \in O(3). \quad (2.16)$$

We thus see a  $\mathbb{Z}_2$  structure of the  $O(3)$  group. Let us define the following concept of direct product (直积).

**Definition 2.5** (direct product, 直积) Given two groups,  $G$  and  $H$ , when their product  $G \times H := \{(g, h) | g \in G, h \in H\}$  has the following multiplication operation

$$(g, h) \circ (g', h') := (g \cdot g', h \cdot h'), \quad (2.17)$$

preserving a group structure, we say that  $G \times H$  is the direct product of  $G$  and  $H$ .

It is indeed as a well-known fact that for  $N$  odd,

$$O(N) = SO(N) \times \mathbb{Z}_2, \quad (2.18)$$

where more explicitly  $\mathbb{Z}_2 = \{\pm \mathbb{I}\}$ .

When  $N$  is even, something more interesting happens. We need a new notion of the product of groups called the semidirect product (直积).

**Definition 2.6** (semidirect product, 半直积) Given two groups  $G$  and  $H$ , if we have an automorphism of  $G$   $\varphi_h : g \mapsto \varphi_h(g)$  labeled by elements  $h \in H$ , then one can define a product on the product set  $G \times H = \{(g, h) | g \in G, h \in H\}$  by

$$(g, h) \circ (g', h') = (g \cdot \varphi_h(g'), h \cdot h'). \quad (2.19)$$

This structure on  $G \times H$  is called the semidirect product, and the obtained group is often denoted as

$$G \rtimes H. \quad (2.20)$$

An important fact can then be written as

$$O(2N) = SO(2N) \rtimes \mathbb{Z}_2. \quad (2.21)$$

**Example 2.6.1** As  $\mathbf{Dih}_6$  is a subgroup of  $O(2)$ , let us study the  $\mathbb{Z}_2$  structure of this finite subgroup. Ignoring the label of  $r_i$  and  $s_j$ , we observe that the group action can be schematically written as

$$r \cdot r = r, \quad r \cdot s = s, \quad s \cdot r = s, \quad s \cdot s = r, \quad (2.22)$$

which is exactly a  $\mathbb{Z}_2$  action. More precisely, the labels obey

$$\begin{cases} r_i \cdot r_j = r_{i+j} \\ r_i \cdot s_j = s_{i+j} \\ s_i \cdot r_j = s_{i-j} \\ s_i \cdot s_j = r_{i-j} \end{cases}, \quad (2.23)$$

where  $i, j \in \{0, 1, 2\}$  take value in  $\mathbb{Z}_3$ . We see that the group action in  $\mathbf{Dih}_6$  depends on the first element being  $r$  or  $s$ . Therefore we have

$$\mathbf{Dih}_6 = \mathbb{Z}_3 \rtimes \mathbb{Z}_2. \quad (2.24)$$

## 2.2 Point Group

In condensed matter physics, what we are most interested in is the symmetry of crystals. A crystal is constituted of a set of points, usually periodic and has a unit cell spanned by  $N$  vectors,  $v_{1,2,\dots,N}$ . Then the set of points that describe the crystal are given by

$$n_1 v_1 + n_2 v_2 + \dots + n_N v_N, \quad (2.25)$$

for all integers  $n_{1,2,\dots,N}$ . The symmetry group that preserves such set of crystal points is called the point group (点群), and point groups are the most frequently used groups in condensed matter physics. When stated more mathematically, we can say that the point group is the setwise stabilizer of the three-dimensional Euclidean group (symmetry of the Euclidean space including the translation) with the choice of the set as (2.25).

There are several types of point groups considerable here. The first type consists of only rotations of  $\frac{2\pi k}{n}$  for  $k = 0, 1, \dots, n-1$ . This group is called the cyclic group, denoted as  $C_n$ . One can further add a mirror reflection into the group as a new generating element. Depending on the mirror plane being perpendicular or parallel to the rotation axis, we say the new group a horizontal cyclic group or a vertical cyclic group, respectively denoted as  $C_{nh}$  and  $C_{nv}$ . The reason we use the word horizontal and vertical is that we usually depict the axis of rotation in the  $z$ -direction, and the mirror plane that perpendicular to this axis is in the horizontal direction (see Figure 16). A new kind of symmetry operation, rotation-reflection, can be introduced as the combination of a rotation of  $\frac{2\pi}{n}$  and a horizontal mirror reflection. The group generated by such an operation is denoted as  $S_{2n}$ . The dihedral group  $D_n = \mathbf{Dih}_{2n}$  can of course be taken as a symmetry of crystal<sup>1</sup>, together

<sup>1</sup>On two dimensional plane, we described the symmetry of folding of dihedrons as reflections, but in three dimensions, they are in fact  $180^\circ$  rotations.

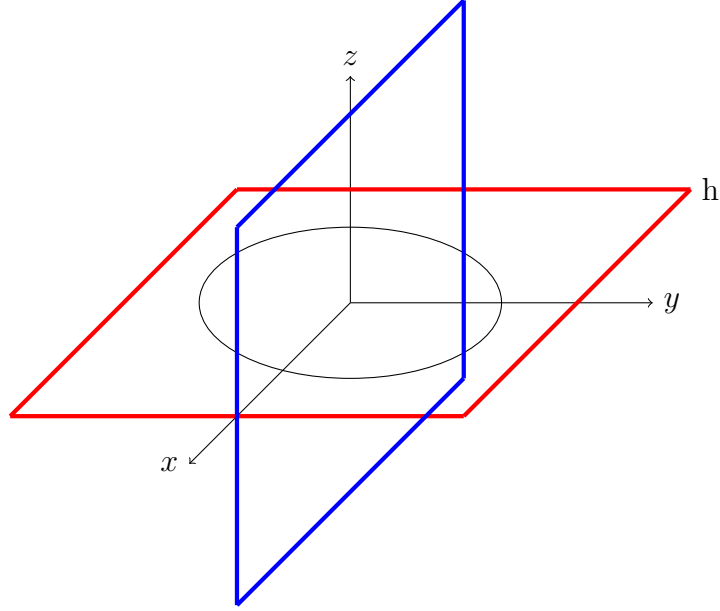


Figure 16: Two different kinds of mirrors, respectively colored in red and blue, considered in point groups.

with an associated group  $D_{nh}$  with a horizontal mirror added and another group  $D_{nd}$  with a vertical mirror included. In addition to these series, we also have the tetrahedral group  $T = \mathcal{T}$ , its rotation-reflection version  $T_d$ , tetrahedral group with inversion added  $T_h$ , octahedral group  $O = \mathcal{O}$  and its rotation-reflection version  $O_h$  as point groups.

According to the crystallographic restriction theorem (晶体学限制定理), only  $n = 1, 2, 3, 4, 6$  are allowed as the label of point groups<sup>2</sup>. Furthermore, there are several isomorphisms (equivalence) among all the points groups:  $C_{1v} \simeq C_{1h}$ ,  $D_1 \simeq C_2$ ,  $D_{1h} \simeq C_{2v}$  and  $D_{1d} \simeq C_{2h}$ , and we note the fact that  $D_{4d}$ ,  $D_{6d}$ ,  $S_8$  and  $S_{12}$  do not exist, so in total, there are only  $5 \times 7 - 4 - 4 = 27$  serial point groups plus 4 (exceptional ones) ( $T$ ,  $T_d$ ,  $T_h$  and  $O$ ,  $O_h$ ), that is 32 point groups.

### 2.3 Wallpaper group

The wallpaper group (墙纸群) is a collection of finite subgroups of the Euclidean group that preserves graphic patterns (see Figure 17 for examples). Mathematically we can say that the wallpaper group is a setwise stabilizer preserving a set of lines instead of a set of points in the case of point groups.

<sup>2</sup>As we may recall that there is no lattice consisted of regular pentagons.



Figure 17: Traditional patterns from Europe, China and Japan.

## 2.4 Examples of Point groups

**Example:**  $C_2$   $C_2$  only consists of the identity operator and the rotation of  $180^\circ$ , so it is isomorphic to  $\mathbb{Z}_2$ .

**Example:**  $C_3$   $C_3$  is isomorphic to  $\mathbb{Z}_3 = \{0, 1, 2\}$ . Let us consider the representations of this group. First we have the trivial representation,

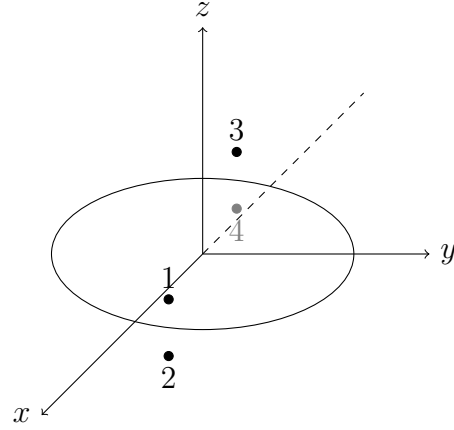
$$R_0^{\mathbb{Z}_3} : 0 \mapsto 1, \quad 1 \mapsto 1, \quad 2 \mapsto 1, \quad (2.26)$$

$$R_1^{\mathbb{Z}_3} : 0 \mapsto 1, \quad 1 \mapsto e^{\frac{2\pi}{3}}, \quad 2 \mapsto e^{\frac{4\pi}{3}}, \quad (2.27)$$

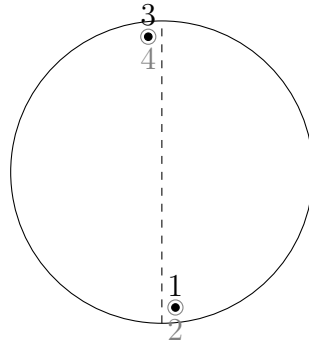
$$R_2^{\mathbb{Z}_3} : 0 \mapsto 1, \quad 1 \mapsto e^{\frac{4\pi}{3}}, \quad 2 \mapsto e^{\frac{2\pi}{3}}. \quad (2.28)$$

We note that for any element  $g$  in  $C_3 = \mathbb{Z}_3$ , it is always true that  $g^3 = 0$ .

**Example:**  $C_{2h}$  This group only has three non-trivial elements,  $c_\pi$  ( $180^\circ$  rotation),  $\sigma_h$  (a reflection with respect to the horizontal plane  $x-y$ ) and their product  $c_\pi\sigma_h$ . One can try to understand the action of  $C_{2h}$  with the help of a 3D graph as (2.29). We consider a disk with four dots around it, labeled by numbers  $1 \sim 4$ . Thus we can represent the action of each element as how these dots are exchanged. Explicitly,  $c_\pi : 1 \leftrightarrow 3, 2 \leftrightarrow 4$ ,  $\sigma_h : 1 \leftrightarrow 2, 3 \leftrightarrow 4$ , and  $c_\pi\sigma_h : 1 \leftrightarrow 4, 2 \leftrightarrow 3$ . For future convenience, let us use a 2D diagram, (2.30), to represent the configuration of (2.29). The dots above the disk are denoted as usual black dots, while those below the disk are represented with gray circles in the 2D diagram.



(2.29)



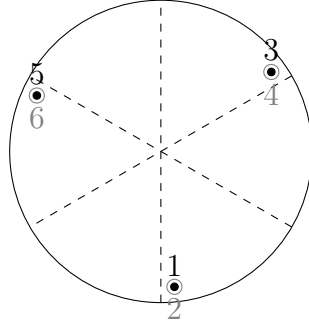
(2.30)

We note that  $c_\pi \sigma_h = \sigma_h c_\pi$ , so this group  $C_{2h}$  is an Abelian group, and thus  $C_{2h} \simeq \mathbb{Z} \otimes \mathbb{Z}_2$ .

It is also straightforward to construct a 4-dim representation based on such an action:

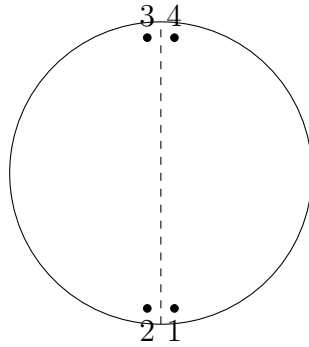
$$c_\pi = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \sigma_h = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad c_\pi \sigma_h = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (2.31)$$

**Example:**  $C_{3h}$  The group is generated by the rotation by  $\frac{2\pi}{3}$ ,  $c_{\frac{2\pi}{3}}$ , and again the reflection  $\sigma_h$ . Again we can use the graph (2.35) to analyze the action of the group.  $c_{\frac{2\pi}{3}} : 1 \rightarrow 3 \rightarrow 5 \rightarrow 1, 2 \rightarrow 4 \rightarrow 6 \rightarrow 2, \sigma_h : 1 \leftrightarrow 2, 3 \leftrightarrow 4, 5 \leftrightarrow 6, c_{\frac{2\pi}{3}}^2 : 1 \rightarrow 5 \rightarrow 3 \rightarrow 1, 2 \rightarrow 6 \rightarrow 4 \rightarrow 2, c_{\frac{2\pi}{3}} \sigma_h = \sigma_h c_{\frac{2\pi}{3}} : 1 \rightarrow 4 \rightarrow 5 \rightarrow 2 \rightarrow 3 \rightarrow 6 \rightarrow 1, c_{\frac{2\pi}{3}}^2 \sigma_h = \sigma_h c_{\frac{2\pi}{3}}^2 = c_{\frac{2\pi}{3}} \sigma_h c_{\frac{2\pi}{3}} : 1 \rightarrow 6 \rightarrow 3 \rightarrow 2 \rightarrow 5 \rightarrow 4 \rightarrow 1$ . Again we see that this group is an Abelian group, and is isomorphic to  $\mathbb{Z}_3 \otimes \mathbb{Z}_2$ .



(2.32)

**Example:**  $C_{2v}$  Similarly we can use a two-dimensional diagram (2.33) to represent the action of  $C_{2v}$ , in which the vertical reflection is the reflection about the  $x$ -axis, i.e. in the  $y$ -direction. There are two non-trivial generators in the group,  $c_\pi : 1 \leftrightarrow 3, 2 \leftrightarrow 4$ , and  $\sigma_v : 1 \leftrightarrow 4, 2 \leftrightarrow 3$ , with  $c_\pi \sigma_v = \sigma_v c_\pi : 1 \leftrightarrow 2, 3 \leftrightarrow 4$ . It is again an Abelian group,  $C_{2v} = \mathbb{Z}_2 \otimes \mathbb{Z}_2$ .

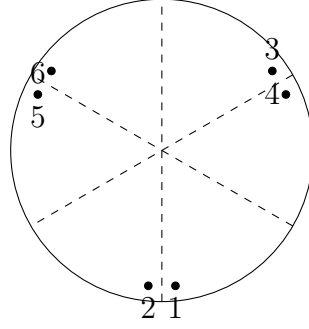


(2.33)

Let us write down the 4-dim representation read from the diagram:

$$c_\pi = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \sigma_v = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad (2.34)$$

**Example:**  $C_{3v}$  It can be graphically represented by the following diagram,



(2.35)

The generators of this group act as  $c_{\frac{2\pi}{3}} : 1 \rightarrow 3 \rightarrow 5 \rightarrow 1, 2 \rightarrow 4 \rightarrow 6 \rightarrow 2$  and  $\sigma_v : 1 \leftrightarrow 2, 3 \leftrightarrow 6, 4 \leftrightarrow 5$ . We have  $c_{\frac{2\pi}{3}}\sigma_v : 1 \leftrightarrow 4, 2 \leftrightarrow 3, 5 \leftrightarrow 6$ , and  $\sigma_v c_{\frac{2\pi}{3}} : 1 \leftrightarrow 6, 2 \leftrightarrow 5, 3 \leftrightarrow 4$ . This group is a non-Abelian group!

We can again work out the matrix representation of all the elements in the group.

$$c_{\frac{2\pi}{3}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad \sigma_v = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad (2.36)$$

$$c_{\frac{4\pi}{3}} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad c_{\frac{2\pi}{3}}\sigma_v = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad (2.37)$$

$$\sigma_v c_{\frac{2\pi}{3}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = c_{\frac{4\pi}{3}}\sigma_v = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (2.38)$$

$$\sigma_v c_{\frac{4\pi}{3}} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} = c_{\frac{2\pi}{3}}\sigma_v. \quad (2.39)$$

We see that there are only 6 independent elements in this group:  $\mathbb{I}$ ,  $c_{\frac{2\pi}{3}}$ ,  $c_{\frac{4\pi}{3}}$ ,  $\sigma_v$ ,  $c_{\frac{2\pi}{3}}\sigma_v$  and  $\sigma_v c_{\frac{2\pi}{3}}$ . As we have

$$\sigma_v c_{\frac{2\pi}{3}} \sigma_v^{-1} = c_{\frac{4\pi}{3}}, \quad (2.40)$$

and

$$c_{\frac{2\pi}{3}} \left( c_{\frac{2\pi}{3}} \sigma_v \right) \left( c_{\frac{2\pi}{3}} \right)^{-1} = c_{\frac{4\pi}{3}} \sigma_v c_{\frac{4\pi}{3}} = c_{\frac{4\pi}{3}} c_{\frac{2\pi}{3}} \sigma_v = \sigma_v, \quad (2.41)$$

$$c_{\frac{4\pi}{3}} \left( \sigma_v c_{\frac{2\pi}{3}} \right) \left( c_{\frac{4\pi}{3}} \right)^{-1} = c_{\frac{4\pi}{3}} \sigma_v c_{\frac{4\pi}{3}} = c_{\frac{4\pi}{3}} c_{\frac{2\pi}{3}} \sigma_v = \sigma_v, \quad (2.42)$$

$c_{\frac{2\pi}{3}}$  and  $c_{\frac{4\pi}{3}}$  belong to the same conjugacy class, while  $\sigma_v$ ,  $c_{\frac{2\pi}{3}}\sigma_v$  and  $\sigma_v c_{\frac{2\pi}{3}}$  are in the same conjugacy class.

The generators of other groups can be found in e.g. [http://www.gernot-katzers-spice-pages.com/character\\_tables/](http://www.gernot-katzers-spice-pages.com/character_tables/).

## 2.5 Characters and their usage

**Definition 2.7** (character, 特征标) Given a (matrix-valued) representation  $R$  of a group  $G$ , then the character of an element  $g \in G$  is given by

$$\chi_R(g) := \text{tr}(R(g)). \quad (2.43)$$

**Proposition 2.7.1** It immediately follows that the character of the identity element is equal to the dimension of the representation, i.e.

$$\chi_R(e) = \dim(R). \quad (2.44)$$

**Remark:** It is also obvious that the character of any element in the trivial representation is 1.

**Proposition 2.7.2** When a representation  $R$  of a group  $G$  is decomposed into two, say  $R_1$  and  $R_2$ , i.e.  $R = R_1 \oplus R_2$ , then we have

$$\chi_R(g) = \chi_{R_1}(g) + \chi_{R_2}(g), \quad (2.45)$$

since the character is defined as a trace.

**Proposition 2.7.3** The character does not change under the similarity transformation, as

$$\chi_R(g) = \text{tr}(R(g)) = \text{tr}(M^{-1}R(g)M). \quad (2.46)$$

Therefore the character keeps to be the same for the whole conjugacy class.

**Example 2.7.4** We start from  $\mathbb{Z}_2$  again. We have

$$\chi_{R(2)}(0) = 1, \quad \chi_{R(2)}(1) = 1, \quad \chi_{R(1,1)}(0) = 1, \quad \chi_{R(1,1)}(1) = -1. \quad (2.47)$$

**Example 2.7.5** Let us consider the  $(2, 1)$  representation constructed in (1.40) and (1.41) for  $\mathfrak{S}_3$ . The characters can be read as

$$\begin{aligned} \chi_{\tilde{R}(2,1)}(1) = 3, \quad \chi_{\tilde{R}(2,1)}(\sigma_1) = \chi_{\tilde{R}(2,1)}(\sigma_2) = -1, \quad \chi_{\tilde{R}(2,1)}(\sigma_1\sigma_2) = \chi_{\tilde{R}(2,1)}(\sigma_2\sigma_1) = 0, \\ \chi_{\tilde{R}(2,1)}(\sigma_1\sigma_2\sigma_1) = -1. \end{aligned} \quad (2.48)$$

One can in fact find another 1-dim representation in addition to the trivial representation,

$$1 \mapsto 1, \quad \sigma_1 \mapsto -1, \quad \sigma_2 \mapsto -1. \quad (2.49)$$

Let us denote this representation as  $R_{(2,1)}$ , and then the characters are found as

$$\chi_{R(2,1)}(1) = \chi_{R(2,1)}(\sigma_1\sigma_2) = \chi_{R(2,1)}(\sigma_2\sigma_1) = 1, \quad \chi_{R(2,1)}(\sigma_1) = \chi_{R(2,1)}(\sigma_2) = \chi_{R(2,1)}(\sigma_1\sigma_2\sigma_1) = -1. \quad (2.50)$$

We can decompose  $\tilde{R}_{(2,1)}$  into  $R_{(2,1)}$  and a 2-dim representation,  $R_{(1,1,1)}$ , with characters

$$\begin{aligned} \chi_{R(1,1,1)}(1) = 2, \quad \chi_{R(1,1,1)}(\sigma_1) = \chi_{R(1,1,1)}(\sigma_2) = \chi_{R(1,1,1)}(\sigma_1\sigma_2\sigma_1) = 0, \\ \chi_{R(1,1,1)}(\sigma_1\sigma_2) = \chi_{R(1,1,1)}(\sigma_2\sigma_1) = -1. \end{aligned} \quad (2.51)$$

The most naive representation (corresponding to  $\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}$  in the construction introduced in **Example 1.20.2**) is a 6-dim representation with

$$\sigma_1 \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \sigma_2 \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \quad (2.52)$$

The characters of this representation is

$$\chi_{\tilde{R}(1,1,1)}(1) = 6, \quad \chi_{\tilde{R}(1,1,1)}(g) = 0, \quad \text{for } \forall g \neq 1. \quad (2.53)$$

**Theorem 2.8** (Orthogonality, 正交性) For two irreducible representations  $R_1$  and  $R_2$  of  $G$ , we have

$$\sum_{g \in G} \chi_{R_1}(g) \chi_{R_2}(g^{-1}) = |G| \delta_{R_1, R_2}, \quad (2.54)$$

$$\sum_{g \in G} \chi_{R_1}(g) \chi_{R_2}(g)^* = |G| \delta_{R_1, R_2}. \quad (2.55)$$

**Remark:** The proof of this theorem requires some tedious mathematical work involving Schur's lemma. We provide such details in Appendix B.

**Example 2.8.1** In  $\mathbb{Z}_2$ ,

$$\chi_{R_{(2)}}(0)\chi_{R_{(2)}}(0) + \chi_{R_{(2)}}(1)\chi_{R_{(2)}}(1) = 2 = |\mathbb{Z}_2|, \quad (2.56)$$

$$\chi_{R_{(1,1)}}(0)\chi_{R_{(1,1)}}(0) + \chi_{R_{(1,1)}}(1)\chi_{R_{(1,1)}}(1) = 2 = |\mathbb{Z}_2|, \quad (2.57)$$

$$\chi_{R_{(2)}}(0)\chi_{R_{(1,1)}}(0) + \chi_{R_{(2)}}(1)\chi_{R_{(1,1)}}(1) = 0. \quad (2.58)$$

We remark that we can determine the characters of  $R_{(1,1)}$  simply from the orthogonality, and note that since  $|\mathbb{Z}_2| = 2$ ,  $R_{(1,1)}$  can only be a one-dimensional representation. In general, however, we need more information to determine the full characters.

**Example 2.8.2** In  $C_3 = \mathbb{Z}_3$ , we constructed three one-dimensional representations, (2.26)-(2.28). Interestingly, in these representations we have again

$$\chi(g^{-1}) = \chi(g)^*. \quad (2.59)$$

Since one dimensional representation further satisfies

$$\chi_R(g) = R(g), \quad (2.60)$$

it is obvious that

$$\chi_R(g^{-1})\chi_R(g) = \chi_R(g)^*\chi_R(g) = 1. \quad (2.61)$$

Therefore representations with the property (2.59) trivially satisfy the orthogonality conditions.

**Corollary 2.8.3** Since  $|\chi_R(g)|^2 = \chi_R(g)\chi_R(g)^*$  is positive-definite, the dimension of the representation, given by  $\chi_R(e)$  is bounded by  $\sqrt{|G|}$ , i.e.

$$\chi_R(e) \leq \sqrt{|G|}. \quad (2.62)$$

**Example 2.8.4** In  $\mathfrak{S}_3$ , we constructed 3 low-dimensional representations, the trivial representation  $R_{(3)}$ , and  $R_{(2,1)}$ ,  $R_{(1,1,1)}$ . It is easy to confirm that the trivial characters and (2.50), (2.51) indeed satisfy the orthogonality condition. As for the 6-dim representation, one can confirm that it is decomposed into

$$\tilde{R}_{(1,1,1)} = R_{(3)} \oplus R_{(2,1)} \oplus 2R_{(1,1,1)}. \quad (2.63)$$

So it is natural to conjecture that  $R_{(3)}$ ,  $R_{(2,1)}$  and  $R_{(1,1,1)}$  are irreducible representations.

Since the character is the same for the whole conjugacy class, the number of independent numbers in the set  $\{\chi(g)\}_{g \in G}$  is equal to the number of conjugacy classes. Therefore we can in general construct as many independent vectors  $\vec{v}_R = (\chi_R(g))_{g_{rep}}$ , where  $g_{rep}$  stands for all the representative elements in each conjugacy class, as the number of conjugacy classes.

**Theorem 2.9** The number of irreducible representations is equal to the number of conjugacy classes in a group.

**Example 2.9.1** We already know that there are two irreducible representations,  $R_{(2)}$  and  $R_{(1,1)}$ , in  $\mathbb{Z}_2$ , and three irreducible representations,  $R_0^{\mathbb{Z}_3}$ ,  $R_1^{\mathbb{Z}_3}$  and  $R_2^{\mathbb{Z}_3}$ .

**Example 2.9.2** Now we proceed to  $\mathfrak{S}_3$ . We know there are three conjugacy classes, and there are three (conjectured-to-be) irreducible representations. Let us compute the characters of the remaining irreducible representation given the trivial one and  $R_{(2,1)}$ . We assume

$$\chi(\text{Cl}(e)) = a, \quad \chi(\text{Cl}(\sigma_1)) = b, \quad \chi(\text{Cl}(\sigma_1\sigma_2)) = c. \quad (2.64)$$

From the orthogonality (2.55), we have

$$a + 3b + 2c = 0, \quad a - 3b + 2c = 0, \quad a^2 + 3|b|^2 + 2|c|^2 = 6. \quad (2.65)$$

Note that  $a$  represents the dimension of the representation, and thus it is a real (and positive) number. It is then easy to find the solution,

$$a = 2, \quad b = 0, \quad c = -1. \quad (2.66)$$

Let us focus on the case of Abelian finite groups for a while. There is a well-known theorem on the classification of Abelian finite groups.

**Theorem 2.10** Given an Abelian finite group  $A$ , one can always find a unique set of integer numbers  $\{n_1, n_2, \dots, n_m\}$ , so that  $A \simeq \mathbb{Z}_{n_1} \otimes \mathbb{Z}_{n_2} \otimes \mathbb{Z}_{n_3} \otimes \dots \otimes \mathbb{Z}_{n_m}$ .

**Example 2.10.1**  $C_{2h}$  and  $C_{2v}$

As we have already argued that both  $C_{2h}$  and  $C_{2v}$  are isomorphic to  $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ , their representations should also be possible to be decomposed into the direct product of irreducible representations of  $\mathbb{Z}_2$ . Since there are two (one-dimensional) irreducible representations in  $\mathbb{Z}_2$ , we have in total four irreducible representations in  $\mathbb{Z}_2 \otimes \mathbb{Z}_2$  with characters,

$$\chi_{0,0}^{\mathbb{Z}_2 \otimes \mathbb{Z}_2}(0 \otimes 0) = 1, \quad \chi_{0,0}^{\mathbb{Z}_2 \otimes \mathbb{Z}_2}(0 \otimes 1) = 1, \quad \chi_{0,0}^{\mathbb{Z}_2 \otimes \mathbb{Z}_2}(1 \otimes 0) = 1, \quad \chi_{0,0}^{\mathbb{Z}_2 \otimes \mathbb{Z}_2}(1 \otimes 1) = 1, \quad (2.67)$$

$$\chi_{0,1}^{\mathbb{Z}_2 \otimes \mathbb{Z}_2}(0 \otimes 0) = 1, \quad \chi_{0,1}^{\mathbb{Z}_2 \otimes \mathbb{Z}_2}(0 \otimes 1) = -1, \quad \chi_{0,1}^{\mathbb{Z}_2 \otimes \mathbb{Z}_2}(1 \otimes 0) = 1, \quad \chi_{0,1}^{\mathbb{Z}_2 \otimes \mathbb{Z}_2}(1 \otimes 1) = -1, \quad (2.68)$$

$$\chi_{1,0}^{\mathbb{Z}_2 \otimes \mathbb{Z}_2}(0 \otimes 0) = 1, \quad \chi_{1,0}^{\mathbb{Z}_2 \otimes \mathbb{Z}_2}(0 \otimes 1) = 1, \quad \chi_{1,0}^{\mathbb{Z}_2 \otimes \mathbb{Z}_2}(1 \otimes 0) = -1, \quad \chi_{1,0}^{\mathbb{Z}_2 \otimes \mathbb{Z}_2}(1 \otimes 1) = -1, \quad (2.69)$$

$$\chi_{1,1}^{\mathbb{Z}_2 \otimes \mathbb{Z}_2}(0 \otimes 0) = 1, \quad \chi_{1,1}^{\mathbb{Z}_2 \otimes \mathbb{Z}_2}(0 \otimes 1) = -1, \quad \chi_{1,1}^{\mathbb{Z}_2 \otimes \mathbb{Z}_2}(1 \otimes 0) = -1, \quad \chi_{1,1}^{\mathbb{Z}_2 \otimes \mathbb{Z}_2}(1 \otimes 1) = 1. \quad (2.70)$$

In  $C_{2h}$ , by identifying

$$\text{id} \leftrightarrow 0 \otimes 0, \quad c_\pi \leftrightarrow 0 \otimes 1, \quad \sigma_h \leftrightarrow 1 \otimes 0, \quad c_\pi \otimes \sigma_h = 1 \otimes 1, \quad (2.71)$$

Rep. \ Class	Class		
	$\mathbb{I}$	$c_{\frac{2\pi}{3}}$	$\sigma_v$
$A_1$	1	1	1
$A_2$	1	1	-1
$E$	2	-1	0

Table 1: The characters of three irreducible representations of  $C_{3v}$ .

we see that the four-dimensional representation (2.31),  $R_4^{C_{2h}}$  can be decomposed into

$$R_4^{C_{2h}} \simeq R_{(0,0)}^{\mathbb{Z}_2 \otimes \mathbb{Z}_2} \oplus R_{(0,1)}^{\mathbb{Z}_2 \otimes \mathbb{Z}_2} \oplus R_{(1,0)}^{\mathbb{Z}_2 \otimes \mathbb{Z}_2} \oplus R_{(1,1)}^{\mathbb{Z}_2 \otimes \mathbb{Z}_2}. \quad (2.72)$$

**Exercise** Repeat the same analysis for  $C_{3h}$  (decomposing the 6-dim representation into irreducible representations of  $\mathbb{Z}_3 \otimes \mathbb{Z}_2$ ).

More generally, one can work out all the irreducible representations of each non-abelian finite group, and perform the decomposition of a given representation into irreducible ones. The characters of irreducible representations are all known in the literature, so one can simply look them up from

**Example 2.11.1**  $C_{3v}$ . The table of all characters in all irreducible representations of  $C_{3v}$  is known as shown in Table 1. Note that the representation (2.36) to (2.37) is traceless except for the identity element, so it is not difficult to write down the following decomposition:

$$R_6^{C_{3v}} = R_{A_1}^{C_{3v}} \oplus R_{A_2}^{C_{3v}} \oplus 2R_E^{C_{3v}}. \quad (2.73)$$

**Example 2.11.2**  $D_3$ . The characters of irreducible representations of  $D_3$  are known as follows.

Rep. \ Class	Class		
	$e$	$r$	$s$
$A_1$	1	1	1
$A_2$	1	1	-1
$E$	2	-1	0

(2.74)

We note that it looks exactly the same as that of  $C_{3v}$ , but they are not isomorphic. Let us check if the 2-dim representation (2.10) is irreducible. Since  $\cos \frac{2\pi}{3} = \cos \frac{4\pi}{3} = -\frac{1}{2}$ , we see that

$$\chi(e) = 2, \quad \chi(r) = -1, \quad \chi(s) = 0. \quad (2.75)$$

It is indeed an irreducible representation, equivalent to  $E$ ! Another interesting problem to consider is whether it is possible to put  $E$  as a one-dimensional representation given the information of  $A_1$

and  $A_2$  representation. Let us assume

$$\chi_{E'}(e) = 1, \quad \chi_{E'}(r) = x, \quad \chi_{E'}(s) = y, \quad (2.76)$$

then

$$1 + 2x + 3y = 0, \quad 1 + 2x - 3y = 0 \Rightarrow x = -\frac{1}{2}, \quad y = 0. \quad (2.77)$$

That is to say

$$R_{E'}(r_1) = R_{E'}(r_2) = -\frac{1}{2}, \quad (2.78)$$

but

$$R_{E'}(r_2) = R_{E'}(r_1) \cdot R_{E'}(r_1) = -\frac{1}{4}, \quad (2.79)$$

so such characters are not consistent.

**isomorphism** We see that  $D_3$  and  $C_{3v}$  have the exactly same character table, so it is natural to conjecture that  $D_3 \simeq C_{3v}$ . Indeed one can construct the following isomorphism between them

$$r_k \mapsto C_{\frac{2\pi k}{3}}, \quad s_k \mapsto C_{\frac{2\pi k}{3}} \sigma_v, \quad (2.80)$$

for  $k = 0, 1, 2$  with the identification  $e = C_0$ . Then it is tempting to further conjecture that for any two finite groups with the same character table, they are isomorphic to each other. However this is not true generally (see [1]), but for Abelian groups, all irreducible representations are one-dimensional and it will be the case.

So far we focused on using the completeness of the basis  $(\chi_{R_i}(\text{Cl}_k))$  to decompose a given reducible representation into the direct sum of irreducible ones. In this context, if we define the following normalized orthogonal basis vectors,  $\{\vec{v}_{R_i}\}$ ,

$$(\vec{v}_{R_i})_k = \sqrt{\frac{n_k}{|G|}} \chi_{R_i}(\text{Cl}_k), \quad (2.81)$$

where  $n_k = |\text{Cl}_k|$  denotes the number of elements in the  $k$ -th conjugacy class, then the orthogonality of this basis is translated to

$$\vec{v}_{R_i}^\dagger \cdot \vec{v}_{R_j} = \delta_{i,j}. \quad (2.82)$$

This is exactly what **Theorem 2.8** states. However in linear algebra, we know that for a complete basis,

$$\sum_{i=1}^{n_{Cl}} \vec{v}_{R_i} \otimes \vec{v}_{R_i} = \mathbb{I}_{n_{Cl} \times n_{Cl}}, \quad (2.83)$$

where  $n_{Cl}$  denotes the number of conjugacy classes in this group. This leads to the following theorem.

**Theorem 2.12** (Completeness, 完备性) Given the set of all irreducible representations,  $\{R_i\}_{i=1}^{n_{Cl}}$ ,

$$\sum_{i=1}^{n_{Cl}} \chi_{R_i}(Cl_k) \chi_{R_i}(Cl_\ell) n_k = |G| \delta_{k,\ell}. \quad (2.84)$$

As an exercise, let us try to complete the character table of  $C_{4v}$  and  $T_d$ .

## 2.6 A bite on Monster Groups\*

We recall that in any group, there are at least two normal subgroups, i.e. the set contains only the identity element and the original group itself. These are called the trivial subgroups.

**Definition 2.13** (Simple group, 单纯群) Groups that do not contain non-trivial normal subgroups are called simple groups.

In fact simple groups with finite order can be completely classified. This great project was initiated since the late 19th century, and got finally done in 2004 by Aschbacher and Smith [2]. In addition to finite groups we have seen so far, classified in the way like  $\mathbb{Z}_n$ ,  $\mathbf{Dih}_{2n}$ ,  $\mathcal{T}$ ,  $\mathcal{O}$  and  $\mathcal{I}$  in  $SO(3)$  (they form infinite families of Lie type, which will be seen later again in our classification of Lie algebras), there are 26 exceptional ones called sporadic groups. The smallest one known as the Mathieu group  $M_{11}$  with 7920 elements, while the biggest sporadic group is called the monster group  $\mathbb{M}$  whose order is

$$|\mathbb{M}| = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \sim 8 \times 10^{53}. \quad (2.85)$$

## 3 Lie Algebra and Lie Group

**Definition 3.1** (Lie algebra, 李代数) A Lie algebra,  $\mathfrak{g}$ , is a ring under the addition operation  $+$  and a bracket  $[\bullet, \bullet]$  (called Lie bracket), further with a scalar multiplication added. For any element  $x \in \mathfrak{g}$ , and  $a \in \mathbb{C}$ , the scalar product is usually denoted as  $ax$ .  $\mathfrak{g}$  forms a vector space under the action of the addition  $+$  and the scalar product, that is, for  $a, b \in \mathbb{C}$  and  $x, y \in \mathfrak{g}$ ,

$$a(bx) = (ab)x, \quad 1x = x, \quad a(x + y) = ax + ay, \quad (a + b)x = ax + bx. \quad (3.1)$$

We further need to impose several conditions on the Lie bracket, ( $x, y, z \in \mathfrak{g}$ )

$$[ax + by, z] = a[x, z] + b[y, z], \quad [z, ax + by] = a[z, x] + b[z, y], \quad (3.2)$$

$$[x, x] = 0, \quad (3.3)$$

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0. \quad (3.4)$$

The last relation is called the Jacobi identity.

**Corollary 3.1.1** For  $\forall x, y \in \mathfrak{g}$ ,  $[x, y] = -[y, x]$ . That is to say, the Lie bracket is an antisymmetric operation.

**Example 3.1.2** General linear Lie algebra,  $\mathfrak{gl}_n(\mathbb{C})$ . This is simply the set of all  $n \times n$  matrices with complex entries, with the addition and scalar product defined in the usual way. The Lie bracket is identified with the commutator of matrices, i.e. for  $X, Y \in \mathfrak{gl}_n(\mathbb{C})$ ,  $[X, Y] = XY - YX$ . The axioms of Lie algebra are automatically satisfied under this identification but the Jacobi identity left non-trivial. Let us evaluate the Jacobi identity explicitly:

$$\begin{aligned} [X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] &= X(YZ - ZY) - (YZ - ZY)X + Z(XY - YX) \\ &\quad - (XY - YX)Z + Y(ZX - XZ) - (ZX - XZ)Y = 0. \end{aligned} \quad (3.5)$$

We can further add constraints to restrict the set of matrices to obtain a lot of interesting examples of Lie algebras.

With this Lie bracket, we can derive the following useful identity,

$$[AB, CD] = ABCD - CDAB = A[B, C]D + [A, C]BD + CA[B, D] + C[A, D]B. \quad (3.6)$$

It plays an important role in the calculation of physics and the enveloping algebra or Lie group discussed later.

**Example 3.1.2**  $\mathfrak{sl}_2(\mathbb{C})$  is the Lie algebra restricted from  $\mathfrak{gl}_2(\mathbb{C})$  by imposing the traceless condition. One can see that all elements in this algebra can be written as a linear combination of

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3.7)$$

If we restrict the scalars to only take values in  $\mathbb{R}$  (mathematically we say we choose the field to be  $\mathbb{R}$ ), then all matrices are in real components, and this new Lie algebra is denoted as  $\mathfrak{sl}_2(\mathbb{R})$ . The algebraic relations characterizing  $\mathfrak{sl}_2(\mathbb{C})$  are

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h. \quad (3.8)$$

**Example 3.1.3** Pauli matrices. We can alternative choose the following basis for  $\mathfrak{sl}_2(\mathbb{C})$ .

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3.9)$$

They are called the Pauli matrices. An interesting feature of these matrices is that they are Hermitian,

$$\sigma_{x,y,z}^\dagger = \sigma_{x,y,z}. \quad (3.10)$$

Note that

$$(xy - yx)^\dagger = y^\dagger x^\dagger - x^\dagger y^\dagger. \quad (3.11)$$

Therefore, if we restrict the field to be  $\mathbb{R}$  and set the Lie bracket to be  $[x, y] := i(xy - yx)$ , then the Pauli matrices generates a Lie algebra (of traceless Hermitian matrices). This algebra is usually called  $\mathfrak{su}(2)$ . The defining relations for this algebra read

$$[\sigma_z, \sigma_x] = -2\sigma_y, \quad [\sigma_z, \sigma_y] = 2\sigma_x, \quad [\sigma_x, \sigma_y] = -2\sigma_z. \quad (3.12)$$

Although  $\mathfrak{su}(2)$  and  $\mathfrak{sl}_2(\mathbb{R})$  are different algebras, by defining

$$\sigma_\pm := \frac{1}{2}(\sigma_x \pm i\sigma_y), \quad (3.13)$$

we can construct a (one-to-one) map,

$$\sigma_+ \mapsto e, \quad \sigma_- \mapsto f, \quad \sigma_z \mapsto h, \quad [\bullet, \bullet]_{\mathfrak{su}(2)} \mapsto i[\bullet, \bullet]_{\mathfrak{sl}_2(\mathbb{R})} \quad (3.14)$$

to identify them. That is to say, if we have a representation of  $\mathfrak{su}(2)$ , we can use the above map to obtain a corresponding representation of  $\mathfrak{sl}_2(\mathbb{R})$ , and vice versa.

**Example 3.1.4** Heisenberg algebra. There are only two elements  $a$  and  $a^\dagger$  in this algebra, and they satisfy

$$[a, a^\dagger] = 1. \quad (3.15)$$

In physics, this algebra is related to the harmonic oscillator system and  $a^\dagger$  is usually called the creation operator, and  $a$  is the annihilation operator. More generally, one can consider creation operators and annihilation operators further with an index,  $a_n$  and  $a_n^\dagger$ , satisfying

$$[a_n, a_m^\dagger] \propto \delta_{n,m}. \quad (3.16)$$

When  $\{n\} = \mathbb{Z}_{>0}$ , we often adopt the convention that  $a_{-n} = a_n^\dagger$ , and set the normalization to be

$$[a_n, a_m] = n\delta_{n+m,0}, \quad \forall n, m \in \mathbb{Z}. \quad (3.17)$$

Note that  $a_0$  is a central element in this algebra.

In the harmonic oscillator system, a number operator

$$N = a^\dagger a, \quad (3.18)$$

which is equal to the Hamiltonian up to some constant, plays an important role. Similarly, we would like to introduce the operator

$$L_0 = \sum_{n \in \mathbb{N}} a_{-n} a_n, \quad (3.19)$$

to extend the Heisenberg algebra. It satisfies

$$[L_0, a_n] = -na_n, \quad (3.20)$$

where we used the identity (3.6).

**Example 3.1.5** Conformal group and Virasoro algebra. ...

**Lie groups** The concept of Lie group is established as a group with its elements parameterized smoothly and continuously. This group should have a differentiable manifold structure. As a manifold, it does not have to be a connected space, but can be divided into several connected parts. Therefore, there exists a discrete group that all elements in the Lie group is continuously connected to an element in the discrete group.

Lie groups are usually constructed as a subspace of  $GL_n(\mathbb{C})$  or  $GL_n(\mathbb{R})$ , that is the set of all  $n \times n$  invertible matrices (with complex or real entries). The scalar product can certainly be defined in the traditional way on this group. More interestingly, around the origin (identity matrix) of the group, we can extract out a Lie algebra on the tangent space.

### 3.1 From Lie group to Lie algebra

**Unitary group  $U(n)$  and  $\mathfrak{u}(n)$**  The unitary group  $U(n)$  is defined as the set of all unitary  $n \times n$  matrices (with complex entries), that is for  $U \in U(n)$ ,

$$U^\dagger U = 1. \quad (3.21)$$

Schematically, one would expect  $U$  to be connected to an element  $u \in \mathfrak{u}(n)$  via

$$U = \exp(i\theta u) \quad \text{for } \exists \theta \in \mathbb{R}. \quad (3.22)$$

Of course, the most generic element in  $U(n)$  is generated by a more complicated orbit in the space of  $n \times n$  matrices. Around  $\theta \sim 0$ , we derive that

$$(1 - i\theta u^\dagger)(1 + i\theta u) + \mathcal{O}(\theta^2) = 1 \Rightarrow u = u^\dagger, \quad (3.23)$$

that is to say  $u$  is a Hermitian matrix. Therefore  $\mathfrak{u}(n)$  is the set of all  $n \times n$  Hermitian matrices. One can further impose a condition that the determinant is one on  $U(n)$ , and the resulting group is called the special unitary group,  $SU(n)$ . The corresponding lie algebra  $\mathfrak{su}(n)$  can be obtained by impose a traceless condition on  $\mathfrak{u}(n)$ .

**Example 3.1.6**  $\mathfrak{su}(2)$  is realized by Hermitian and traceless  $2 \times 2$  matrices. There are only three independent d.o.f. in such matrices, and the most well-known basis was first written down by Pauli, and now is known as Pauli matrices (泡利矩阵):

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (3.24)$$

By relabeling  $\sigma_1 := \sigma_x$ ,  $\sigma_2 := \sigma_y$ ,  $\sigma_3 := \sigma_z$ , we have the following commutation relations and anti-commutation relations,

$$[\sigma_j, \sigma_k] = 2i \sum_l \epsilon_{jkl} \sigma_l, \quad \{\sigma_j, \sigma_k\} = 2\delta_{j,k} \mathbb{I}_{2 \times 2}, \quad (3.25)$$

where  $\{X, Y\} = XY + YX$ .

**Orthogonal group  $\mathbf{O}(n)$  and  $\mathfrak{o}(n)$**  The orthogonal group  $\mathbf{O}(n)$  is defined as the set of all matrices  $O$  satisfying

$$O^t O = 1. \quad (3.26)$$

In the same way we find that the elements  $o$  of  $\mathfrak{o}(n)$  satisfy

$$(1 + i\theta o^t)(1 + i\theta o) + \mathcal{O}(\theta^2) = 1 \Rightarrow o = -o^t, \quad (3.27)$$

that is  $\mathfrak{o}(n)$  is the set of all anti-symmetric  $n \times n$  matrices. We can again impose the determinant equal to one condition to obtain the special orthogonal group,  $\mathbf{SO}(n)$ . However, the traceless condition does not change the  $\mathfrak{o}(n)$  algebra, that is to say,  $\mathfrak{so}(n)$  is the same as  $\mathfrak{o}(n)$ .

**Symplectic group  $\mathbf{Sp}(2n, \mathbb{C})$  and  $\mathfrak{sp}(2n, \mathbb{C})$**  The symplectic group  $\mathbf{Sp}(2n, \mathbb{C})$  is only defined for matrices with even size. For an element  $M$  in this group, it satisfies

$$M^t \Omega M = \Omega, \quad (3.28)$$

where

$$\Omega = \begin{pmatrix} 0 & -I_{n \times n} \\ I_{n \times n} & 0 \end{pmatrix}. \quad (3.29)$$

Therefore we can derive the defining property for the elements  $m$  in the corresponding Lie algebra,  $\mathfrak{sp}(2n, \mathbb{C})$ ,

$$\Omega + i\theta m^t \Omega + i\theta \Omega m + \mathcal{O}(\theta^2) = \Omega, \quad (3.30)$$

that is to say,

$$\Omega m^t \Omega = m. \quad (3.31)$$

We note that the determinant of  $M \in \mathbf{Sp}(2n, \mathbb{C})$  appears to be  $\pm 1$  (but since there are even numbers of diagonal entries,  $-M$  has the same determinant as  $M$ ), so there is no special version of the symplectic Lie algebra.

**Remark: why these groups?** As we have already discussed for several times, the orthogonal group  $O(n)$  represents the symmetry of rotations and reflections in  $n$  dimensions. Therefore, given any two real vectors,  $\vec{v}$  and  $\vec{u}$ , ( $v_i, u_i \in \mathbb{R}$ ,  $i = 1, 2, \dots, n$ ), the orthogonal group preserves the inner product

$$\vec{u} \cdot \vec{v} = (\vec{u}^t)_i \times (\vec{v})_i. \quad (3.32)$$

This is why the orthogonal group that rotates

$$\vec{v} \rightarrow O\vec{v}, \quad (3.33)$$

should satisfy

$$O^t O = \mathbb{I}. \quad (3.34)$$

When we consider complex vectors, say  $\vec{a}$  and  $\vec{b}$  ( $a_i, b_i \in \mathbb{C}$  for  $i = 1, 2, \dots, n$ ), then the inner product is defined as

$$\vec{a}^\dagger \cdot \vec{b} = (\vec{a}^t)_i^* \times (\vec{b})_i, \quad (3.35)$$

and the group that preserves the above inner product satisfies

$$U^\dagger U = \mathbb{I}, \quad (3.36)$$

and it is nothing but the unitary group. How about the symplectic group? One may think of an inner product defined for quaternions. We can define for any  $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{H}$ , its conjugate is given by

$$q^* = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}. \quad (3.37)$$

We can check that

$$q^* q = a^2 + b^2 + c^2 + d^2. \quad (3.38)$$

If we consider matrices with  $\mathbb{H}$  entries, then they will satisfy a similar relation as the unitary group, but we want to represent them in terms of complex matrices with doubled size. The inner product of two quaternions  $q_1^*$  and  $q_2$  is given by

$$\begin{aligned} q_1^* q_2 &= (a_1 a_2 + b_1 b_2 + c_1 c_2 + d_1 d_2) + (a_1 b_2 - b_1 a_2 - c_1 d_2 + d_1 c_2) \mathbf{i} \\ &+ (a_1 c_2 + b_1 d_2 - c_1 a_2 - d_1 b_2) \mathbf{j} + (a_1 d_2 - b_1 c_2 + c_1 b_2 - d_1 a_2) \mathbf{k}. \end{aligned} \quad (3.39)$$

We see that the above product can be realized by the product of  $2 \times 2$  complex matrices, i.e. we represent each quaternion  $q$  with the following matrix,

$$q \mapsto M_q := \begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}. \quad (3.40)$$

Then we have

$$q^* \mapsto M_{q^*} = \begin{pmatrix} a - bi & -c - di \\ c - di & a + bi \end{pmatrix} = \Omega_{2 \times 2}^{-1} M_q^t \Omega_{2 \times 2}, \quad (3.41)$$

where

$$\Omega_{2 \times 2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (3.42)$$

Therefore in one-dimensional quaternion space  $\mathbb{H}$ , the rotational group that acts on  $M_q$  as

$$M_q \rightarrow S M_q, \quad (3.43)$$

satisfies

$$S^t \Omega_{2 \times 2} S = \Omega_{2 \times 2}. \quad (3.44)$$

What happens when we go to higher dimensions? In two dimensions, we may define a  $4 \times 2$  matrix to represent a vector in  $\mathbb{H}^2$ ,

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \mapsto \begin{pmatrix} M_{q_1} \\ M_{q_2} \end{pmatrix}. \quad (3.45)$$

The inner product of such vectors results in a quaternion number,

$$\begin{aligned} \begin{pmatrix} q_3^* & q_4^* \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} &\mapsto \Omega_{2 \times 2}^{-1} (M_{q_3}^t \Omega_{2 \times 2} M_{q_1} + M_{q_4}^t \Omega_{2 \times 2} M_{q_2}) \\ &= \Omega_{2 \times 2}^{-1} \begin{pmatrix} M_{q_3} & M_{q_4} \end{pmatrix} \begin{pmatrix} \Omega_{2 \times 2} & \\ & \Omega_{2 \times 2} \end{pmatrix} \begin{pmatrix} M_{q_1} \\ M_{q_2} \end{pmatrix}. \end{aligned} \quad (3.46)$$

We can perform a basis permutation to convert the matrix

$$\Omega_{4 \times 4}^{(0)} := \begin{pmatrix} \Omega_{2 \times 2} & \\ & \Omega_{2 \times 2} \end{pmatrix}, \quad (3.47)$$

into the standard form  $\Omega$  shown before. For example, to permute the 4-th row (and resp. column) and the 2-nd row (and resp. column), we can perform the following similarity transformation,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \Omega_{4 \times 4}^{(0)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad (3.48)$$

and it can be further transformed to  $\Omega$  by permuting the 2-nd row (and column) and the 3-rd row (and column) by acting

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (3.49)$$

In this basis, the quaternion vector is mapped to

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \mapsto \begin{pmatrix} a_1 + b_1 i & c_1 + d_1 i \\ -c_2 + d_2 i & a_2 - b_2 i \\ a_2 + b_2 i & c_2 + d_2 i \\ -c_1 + d_1 i & a_1 - b_1 i \end{pmatrix}. \quad (3.50)$$

In this way, the rotational group of quaternion vectors can be generalized to arbitrary dimensions,  $\mathbb{H}^n$ .

**Compact Lie groups** Since the Lie group can be identified as a manifold, we can discuss the compactness of the Lie group. A particularly interesting case in physics is to take the field as  $\mathbb{R}$ . It is easy to see that the exponential of Hermitian matrices form a compact Lie group. In particular, when diagonalized, the eigenvalues of a Hermitian matrix are all real. Therefore, the Lie group generated by exponential of  $\mathfrak{su}(n)$  or  $\mathfrak{u}(n)$  is compact. Similarly, antisymmetric matrices with real entries can always be diagonalized and the eigenvalues are paired as  $\pm\lambda_i$  (if the matrix size is odd, then there will be a zero eigenvalue). We then see that the Lie group generated by  $\mathfrak{o}(n, \mathbb{R})$  is also compact. Note that all elements in  $\mathfrak{o}(n, \mathbb{R})$  are always traceless, so it is equivalent to  $\mathfrak{so}(n, \mathbb{R})$ , and the Lie group simply generated from the origin (identity matrix) is in fact  $\text{SO}(n)$ .

Of course, we can consider  $\text{Sp}(2n, \mathbb{R})$  as a Lie group over the field  $\mathbb{R}$ , but it is known to be non-compact. Let us take the example  $n = 1$  to have a more careful look. Let a general element in  $\text{Sp}(2n, \mathbb{R})$  to take the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad (3.51)$$

then the only condition reads

$$ad - bc = 1. \quad (3.52)$$

Even when we consider the subspace  $ad = 1$ , we obtain a non-compact manifold,  $b = 0$  with  $c \in \mathbb{R}$  arbitrary, or  $c = 0$  with  $b \in \mathbb{R}$  arbitrary.

Instead, we consider a compact symplectic group  $\text{Sp}(n)$  defined as  $\text{Sp}(2n, \mathbb{C}) \cap \text{SU}(2n)$ . We remark that in the simplest case of  $n = 1$ , it is nothing but  $\text{SU}(2)$ , as  $\text{Sp}(2, \mathbb{C})$  only imposes a determinant one condition. The corresponding Lie algebra satisfies both (3.31) and the Hermitian condition.

## Geometric picture of SO(3) and SU(2)

### 3.2 $\mathfrak{sl}_2$ and its representations

Let us recall the defining relations for the  $\mathfrak{sl}_2$  algebra.

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h. \quad (3.53)$$

We first try to extend the algebra by adding a new element,

$$M := 2(e f + f e) + h^2, \quad (3.54)$$

whose algebraic relation with other elements can be calculated with (3.6) as

$$[M, h] = 0, \quad [M, e] = 0, \quad [M, f] = 0. \quad (3.55)$$

That is, in the extended algebra,  $M$  is a central element.

The strategy to construct a representation is to map  $h$  to a diagonal matrix, and read out the action of  $e$  and  $f$  in this basis, with the Lie bracket given by  $[X, Y] = XY - YX$ . As the number of basis in this representation is finite, we usually call it a finite (dimensional) representation. Given an eigenstate of  $h$ ,  $|a\rangle$ , which satisfies

$$h |a\rangle = a |a\rangle, \quad a \in \mathbb{C}, \quad (3.56)$$

the states  $e |a\rangle$  and  $f |a\rangle$  are again eigenstates, and their eigenvalues are

$$h(e |a\rangle) = (a + 2)e |a\rangle, \quad h(f |a\rangle) = (a - 2)f |a\rangle. \quad (3.57)$$

We see  $e$  raises the eigenvalue of the basis and  $f$  lowers it, so  $e$  and  $f$  are also respectively called the raising and lowering operators. Eigenvectors with different eigenvalues are independent, so each of these eigenvalues corresponds to a diagonal entry in the matrix representation of  $h$ . As  $h$  is mapped to a finite matrix, we must have a vector  $|w\rangle$  with the “largest” eigenvalue  $w$  s.t.

$$e |w\rangle = 0. \quad (3.58)$$

Similarly, there exists a vector  $|l\rangle$  with the “smallest” eigenvalue  $l$ , and is annihilated by the lowering operator,

$$f |l\rangle = 0. \quad (3.59)$$

$|w\rangle$  is called the highest weight state, and  $w$  is called the highest weight in the representation we consider. Correspondingly,  $|l\rangle$  is the lowest weight state, and  $l$  is the lowest weight. Eigenvalues in the middle of them are called weights in the representation.

In such a matrix representation, the extension to include  $M$  into the algebra is natural and unavoidable (as the matrix multiplication is introduced in the representation space). Since  $M$  is central, its value does not change in the whole representation, that is the eigenvalues of  $M$  acting on both  $|w\rangle$  and  $|l\rangle$  are the same.

$$M|w\rangle = (w^2 + 2w)|w\rangle, \quad M|l\rangle = (l^2 - 2l)|l\rangle. \quad (3.60)$$

This puts a constraint,

$$(w + l)(w - l + 2) = 0, \quad (3.61)$$

on the representation. By definition,  $w \geq l$ , so  $w - l + 2 \neq 0$ , and thus we obtain  $w = -l$  in the representation. Further from the property  $w - 2n = l$  for  $\exists n \in \mathbb{N}$ , we finally see that  $w = n \in \mathbb{N}$ , i.e. the highest weight must be an integer. This strong constraint follows from the finiteness of the matrix representation. As the weights line up at the interval of two, the dimension of representation (or the size of the matrix used in the representation) is given by  $w + 1$ .

**Remark** One can also consider a general highest weight  $w$  to construct a representation, but the weights in the representation will not be bounded from below, i.e. it will be an infinite-dimensional representation.

**Example 3.2.1** Let us write down the explicit representation corresponding to  $w = 1$ . There are two basis with eigenvalues respectively  $\pm 1$  of  $h$ , and thus  $h$  is diagonalized in this basis to be

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3.62)$$

Let us denote the basis with their eigenvalues as  $|\pm 1\rangle$ , and we determine the relative normalization of these vector by  $|1\rangle = e|-1\rangle$ . Then we have  $f|1\rangle = fe|-1\rangle = -h|-1\rangle = 1$ . Therefore,  $e$  and  $f$  are represented by the matrices

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (3.63)$$

We see that it is exactly the defining representation of  $\mathfrak{sl}_2$ .

**Remark** One can compare the above fundamental representation with Pauli matrices in  $\mathfrak{su}(2)$ . We have

$$\sigma_x = e + f, \quad \sigma_y = -ie + if, \quad \sigma_z = h. \quad (3.64)$$

Thus at the level of Lie algebra,  $\mathfrak{sl}_2$  and  $\mathfrak{su}(2)$  are isomorphic, but to generate the corresponding Lie group, we use  $\exp(i\theta X)$  for  $X \in \mathfrak{g}$ , where  $\mathfrak{g}$  is  $\mathfrak{sl}_2$  or  $\mathfrak{su}(2)$  and  $\theta \in \mathbb{R}$ . As we can see that the

isomorphic map (3.64) involves complex numbers, the corresponding Lie group  $SU(2)$  and  $SL(2, \mathbb{R})$  are different. Indeed since the eigenvalues of Hermitian matrices are real,  $SU(2)$  is a compact (紧致的) group, while  $SL(2, \mathbb{R})$  is non-compact.

**Example 3.2.2** Now we construct the representation corresponding to  $w = 2$ .  $h$  is diagonalized as

$$h = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \quad (3.65)$$

We again set the normalization as

$$|2\rangle = e|0\rangle, \quad |0\rangle = e|-2\rangle. \quad (3.66)$$

Then we find

$$f|2\rangle = fe^2|-2\rangle = -h|0\rangle - eh|-2\rangle = 2|0\rangle, \quad (3.67)$$

$$f|0\rangle = -h|-2\rangle = 2|-2\rangle. \quad (3.68)$$

That is to say

$$e = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}. \quad (3.69)$$

### 3.3 General structure of Lie algebras and roots

A general Lie algebra is characterized by the following commutation relation,

$$[J_a, J_b] = \sum_c f_{ab}^c J_c, \quad (3.70)$$

where  $J_a$ 's are the generators of the underlying Lie algebra.

A useful basis of this algebra is to first find the maximal set of simultaneously commuting generators,  $\{H_i\}_{i=1, \dots, r}$ , where  $r$  is usually called the rank of the algebra. The set  $\{H_i\}_{i=1, \dots, r}$  forms a commutative Lie algebra, and is usually called the Cartan subalgebra. Then we build the basis of the remaining generators s.t.

$$[H_i, E_\alpha] = \alpha_i E_\alpha, \quad (3.71)$$

where  $\alpha = (\alpha_1, \dots, \alpha_r)$  is an  $r$ -dimensional vector and is called a root. Let us denote the set of all roots by  $\Delta$ .

**Example 3.3.1** In  $\mathfrak{sl}_2$ , the Cartan subalgebra has only one element,  $h$ . There are only two roots in this algebra,  $\Delta = \{2, -2\}$ , and  $E_2 = e$ ,  $E_{-2} = f$  in (3.53).

**Example 3.3.2**  $\mathfrak{sl}_3$  is defined as a set of  $3 \times 3$  traceless matrices. There are 2 diagonal traceless matrices,

$$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad (3.72)$$

and six off-diagonal ones,

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad (3.73)$$

$$\bar{e}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{e}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \bar{e}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \quad (3.74)$$

Apparently,  $\{h_1, h_2\}$  forms the Cartan subalgebra of  $\mathfrak{sl}_3$ , and we can confirm that

$$e_1 \propto E_{(2,0)}, \quad e_2 \propto E_{(1,3)}, \quad e_3 \propto E_{(-1,3)}, \quad (3.75)$$

$$\bar{e}_1 \propto E_{(-2,0)}, \quad \bar{e}_2 \propto E_{(-1,-3)}, \quad \bar{e}_3 \propto E_{(1,-3)}. \quad (3.76)$$

We remark that there is still a d.o.f. left to normalize the roots properly. We will be dealing with this shortly.  $\square$

Let us try to compute the commutation relation between  $E_\alpha$  and  $E_\beta$ .

$$\begin{aligned} [H_i, [E_\alpha, E_\beta]] &= [E_\alpha, [H_i, E_\beta]] - [E_\beta, [H_i, E_\alpha]] \\ &= (\alpha + \beta) [E_\alpha, E_\beta]. \end{aligned} \quad (3.77)$$

We thus see that if  $\alpha + \beta \notin \Delta$ , then we must have  $[E_\alpha, E_\beta] = 0$ . When  $\alpha + \beta \in \Delta$ , we have

$$[E_\alpha, E_\beta] \propto E_{\alpha+\beta}. \quad (3.78)$$

If  $\alpha + \beta = 0$ , i.e.  $\beta = -\alpha$ , then  $[E_\alpha, E_{-\alpha}]$  commutes with all  $H_i$ 's. This means that  $[E_\alpha, E_{-\alpha}]$  can only be a linear combination of the Cartan generators,  $H_i$ .

Now we need to determine the normalization of  $H_i$ 's and  $E_\alpha$ 's. The Cartan generators are chosen to satisfy

$$\frac{1}{2h^\vee} \text{tr}_{adj} (H_i H_j) = \delta_{i,j}, \quad (3.79)$$

where  $h^\vee$  is the dual Coxeter number (we will give the details on it later) and the trace is taken over the adjoint representation of the generators. We also need to fix the normalization of root generators,  $E_\alpha$ , and it is convenient to set it similarly to

$$\frac{1}{2h^\vee} \text{tr}_{\text{adj}} (E_\alpha E_\beta) = \delta_{\alpha+\beta, 0}. \quad (3.80)$$

The coefficients in  $[E_\alpha, E_{-\alpha}] = \sum_i c_i H_i$  can then be determined through

$$c_i = \frac{1}{2h^\vee} \text{tr}_{\text{adj}} (H_i [E_\alpha, E_{-\alpha}]) = \frac{1}{2h^\vee} \text{tr}_{\text{adj}} (E_{-\alpha} [H_i, E_\alpha]) = \alpha_i \frac{1}{2h^\vee} \text{tr}_{\text{adj}} (E_{-\alpha} E_\alpha) = \alpha_i. \quad (3.81)$$

Let us be more precise about the adjoint representation. It can be constructed as follows. We utilize the commutation relation

$$[J_a, J_b] = \sum_c f_{ab}^c J_c, \quad (3.82)$$

then we define the adjoint map as

$$R_{\text{adj}} : J_a \mapsto M_a = (M_a)_{cb} := f_{ab}^c. \quad (3.83)$$

Following from the Jacobi identity, we have

$$\begin{aligned} & [J_c, [J_a, J_b]] + [J_b, [J_c, J_a]] + [J_a, [J_b, J_c]] \\ &= \sum_{d,e} ((M_c)_{ed}(M_a)_{db} + (M_b)_{ed}(M_c)_{da} + (M_a)_{ed}(M_b)_{dc}) J_e = 0 \end{aligned} \quad (3.84)$$

Note that

$$(M_a)_{cb} = -(M_b)_{ca}, \quad (3.85)$$

we obtain

$$(M_a \cdot M_c)_{be} - (M_c \cdot M_a)_{be} = \sum_d f_{ac}^d (M_d)_{be}, \quad (3.86)$$

which is exactly the matrix realization of the commutation relation (3.82). This map  $R_{\text{adj}}$  is the adjoint representation of the Lie algebra.

We observe that in the adjoint representation, the dimension of the representation is equal to the number of generators in the Lie algebra, so we can establish a one-to-one correspondence between each generator and a basis vector of the representation space,

$$X \in \mathfrak{g} \leftrightarrow |X\rangle. \quad (3.87)$$

These vectors keep the same Lie-algebraic structure,

$$|aX_i + bX_j\rangle = a|X_i\rangle + b|X_j\rangle, \quad (3.88)$$

and

$$X_i |X_j\rangle = |[X_i, X_j]\rangle. \quad (3.89)$$

Then it is not hard to see that all of them are eigenvectors of the Cartan generators,

$$H_i |H_j\rangle = |[H_i, H_j]\rangle = 0, \quad H_i |\alpha\rangle = \alpha_i |\alpha\rangle. \quad (3.90)$$

It is also possible to incorporate a complex structure in the Lie algebra so that

$$H_i^\dagger = H_i, \quad (3.91)$$

then since  $\alpha$ 's are real,

$$([H_i, E_\alpha])^\dagger = \alpha_i E_\alpha^\dagger \Rightarrow -[H_i, E_\alpha^\dagger] = \alpha_i E^\dagger. \quad (3.92)$$

The matrix element in the adjoint representation can be worked out via

$$J_a |J_b\rangle = |[J_a, J_b]\rangle = \sum_c f_{ab}^c |J_c\rangle. \quad (3.93)$$

We see that the representation with weights equal to the roots is the adjoint representation.

**Example 3.3.3** Let us re-construct the adjoint representation from the above basis labeled by the generators. We start from  $\mathfrak{sl}_2$ . Since we have

$$[H, E_2] = 2E_2, \quad [H, E_{-2}] = -2E_{-2}, \quad [E_2, E_{-2}] = H, \quad (3.94)$$

there are three vectors in the adjoint basis,

$$|E_{\pm 2}\rangle, \quad |H\rangle. \quad (3.95)$$

It is easy to obtain

$$H \mapsto \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad E_2 \mapsto \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{-2} \mapsto \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}. \quad (3.96)$$

One can check that it is equivalent to  $w = 2$  representation of  $\mathfrak{sl}_2$  constructed in Example 3.2.2 by using the similarity transformation of

$$U = \text{diag}(-1, 2, 2). \quad (3.97)$$

**Example 3.3.4** In the case of  $\mathfrak{sl}_3$ , the commutation relation in the normalized form reads

$$\begin{aligned}
[E_{\alpha_1}, E_{-\alpha_1}] &= H_1 & [E_{\alpha_2}, E_{-\alpha_2}] &= H_2, \\
[E_{\alpha_1}, E_{\alpha_2}] &= E_{\alpha_1+\alpha_2}, & [E_{-\alpha_1}, E_{-\alpha_2}] &= -E_{-\alpha_1-\alpha_2}, \\
[E_{\alpha_{1,2}}, E_{\alpha_1+\alpha_2}] &= 0, & [E_{-\alpha_1}, E_{\alpha_1+\alpha_2}] &= E_{\alpha_2}, \\
[E_{-\alpha_2}, E_{\alpha_1+\alpha_2}] &= -E_{\alpha_1}, & [E_{-\alpha_{1,2}}, E_{-\alpha_1-\alpha_2}] &= 0, \\
[H_1, E_{\alpha_1}] &= 2E_{\alpha_1}, & [H_2, E_{\alpha_1}] &= -E_{\alpha_1}, \\
[H_1, E_{\alpha_2}] &= -E_{\alpha_2}, & [H_2, E_{\alpha_2}] &= 2E_{\alpha_2}, \\
[H_1, E_{\alpha_1+\alpha_2}] &= E_{\alpha_1+\alpha_2}, & [H_2, E_{\alpha_1+\alpha_2}] &= E_{\alpha_1+\alpha_2}, \\
[E_{\alpha_1}, E_{-\alpha_1-\alpha_2}] &= -E_{-\alpha_2}, & [E_{\alpha_2}, E_{-\alpha_1-\alpha_2}] &= -E_{-\alpha_1},
\end{aligned} \tag{3.98}$$

where we identified

$$\begin{aligned}
H_1 &:= h_1, & H_2 &= -\frac{h_1}{2} + \frac{h_2}{2}, & E_{\alpha_1} &:= e_1, & E_{-\alpha_1} &:= \bar{e}_1, \\
E_{\alpha_2} &:= e_3, & E_{-\alpha_2} &:= \bar{e}_3, & E_{\alpha_1+\alpha_2} &:= e_2, & E_{-\alpha_1-\alpha_2} &:= \bar{e}_2,
\end{aligned} \tag{3.99}$$

with the defining matrices given in Example 3.3.2. The adjoint representation is given by

$$\begin{aligned}
H_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{matrix} |E_{\alpha_1+\alpha_2}\rangle \\ |E_{\alpha_1}\rangle \\ |E_{\alpha_2}\rangle \\ |H_1\rangle \\ |H_2\rangle \\ |E_{-\alpha_2}\rangle \\ |E_{-\alpha_1}\rangle \\ |E_{-\alpha_1-\alpha_2}\rangle \end{matrix}, \\
H_2 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},
\end{aligned}$$

$$E_1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.100)$$

etc. We remark that the normalization chosen in this example is different from the one discussed above, as we know that for  $\mathfrak{sl}_3$ ,  $h^\vee = 3$ , and

$$\frac{1}{2h^\vee} \text{tr}(H_{1,2}^2) = 2, \quad \frac{1}{2h^\vee} \text{tr}(H_1 H_2) = -1, \quad (3.101)$$

yet it is an often-chosen convention in practical computations.  $\square$

The adjoint representation can be understood as a representation built from the highest-weight state  $|E_{\alpha_1}\rangle$ , so the highest weight of this representation is  $w = \alpha_1 = (\alpha_{11}, \alpha_{12}, \dots, \alpha_{1r})$ . A more general representation starts from an arbitrary weight  $w$  and we denote the highest-weight state as

$$|w\rangle, \quad H_i |w\rangle = w_i |w\rangle. \quad (3.102)$$

As

$$H_i E_\alpha |w\rangle = [H_i, E_\alpha] |w\rangle + E_\alpha H_i |w\rangle = (w + \alpha) E_\alpha |w\rangle, \quad (3.103)$$

we see that  $E_\alpha$  raises or lowers the weight of a state.

To work out the highest-weight representation more explicitly, we note that there are (in fact)  $r$  independent  $\mathfrak{sl}_2$  triplets in the Lie algebra we are considering. Let us choose a basis  $\{\alpha_i\}_{i=1}^r$  of the root space, then we see that by defining

$$E_i^\pm := |\alpha_i|^{-1} E_{\pm\alpha_i}, \quad E_i^3 := |\alpha_i|^{-2} \alpha_i \cdot H, \quad (3.104)$$

where  $H = (H_1, H_2, \dots, H_r)$ , they form an  $\mathfrak{sl}_2$  algebra,

$$[E_i^3, E_i^\pm] = |\alpha_i|^{-3} [\alpha_i \cdot H, E_{\pm\alpha_i}] = \pm |\alpha_i|^{-1} E_{\pm\alpha_i} = \pm E_i^\pm, \quad (3.105)$$

$$[E_i^+, E_i^-] = |\alpha_i|^{-2} \alpha_i \cdot H = E_i^3. \quad (3.106)$$

To construct a finite-dimensional representation of the Lie algebra, we need to require each  $\mathfrak{sl}_2$  sub-algebra to be represented finitely. From the discussion in the previous section, we see that the weight of  $h_i := 2E_i^3$ ,<sup>3</sup>

$$\frac{2\alpha_i \cdot w}{|\alpha_i|^2} \in \mathbb{Z}, \quad (3.107)$$

---

<sup>3</sup>Be careful about the normalization!

must be an integer.

Comparing the above expression with the known eigenvalues of  $H_i$ 's in the adjoint representation, we can work out the inner product of the root vectors. For example, in  $\mathfrak{sl}_3$ , we obtain

$$\frac{2\alpha_1 \cdot \alpha_2}{|\alpha_1|^2} = \frac{2\alpha_1 \cdot \alpha_2}{|\alpha_2|^2} = -1. \quad (3.108)$$

Under the normalization condition,

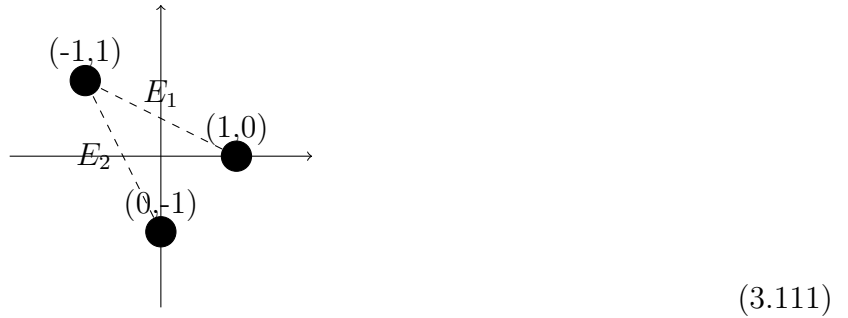
$$|\alpha_1|^2 = |\alpha_2|^2 = 2, \quad (3.109)$$

we have

$$\alpha_1 \cdot \alpha_2 = -1. \quad (3.110)$$

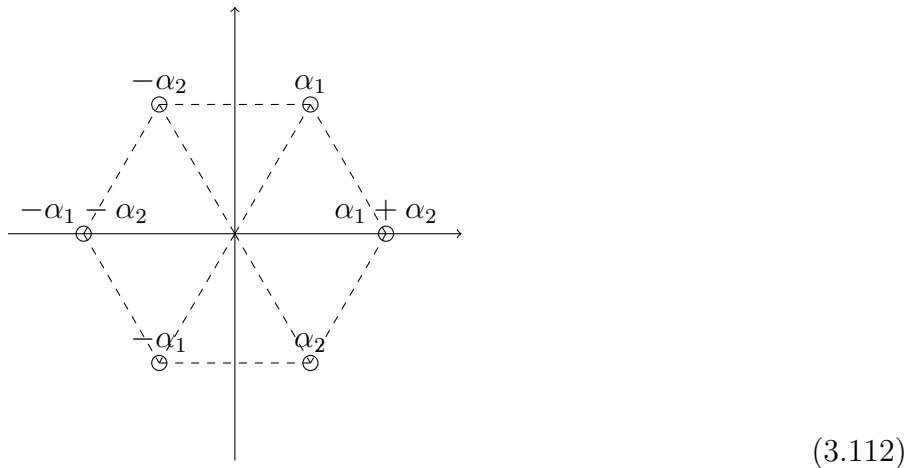
**Remark: Why finite-dimensional representations?**

**Example 3.3.5** Let us construct the representation of  $\mathfrak{sl}_3$  with highest weight  $w = (1, 0)$  (in the basis  $\alpha_1 = (2, -1)$  and  $\alpha_2 = (-1, 2)$ ). The whole structure of the weight space looks like



It is a three-dimensional representation, and usually also called the fundamental representation of  $\mathfrak{sl}_3$ .

**Weyl group** Let us plot the root system of  $\mathfrak{sl}_3$ .



where we employed the expression in the orthogonal basis,

$$\alpha_1 = \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right), \quad \alpha_2 = \left( \frac{1}{2}, -\frac{\sqrt{3}}{2} \right), \quad (3.113)$$

that satisfies (3.109) and (3.110). We see that these roots form a honeycomb (蜂巢) lattice, and therefore we have a finite group standing for the symmetry acting on this lattice. Such a finite group is called the Weyl group.

It is known that the Weyl group can be generated from the following reflection associated to each root  $\alpha$ ,

$$s_\alpha(v) = v - \frac{2v \cdot \alpha}{\alpha \cdot \alpha} \alpha. \quad (3.114)$$

It is easy to see that

$$s_\alpha(\alpha) = -\alpha, \quad (3.115)$$

and when  $\alpha \cdot \alpha = 2$ ,

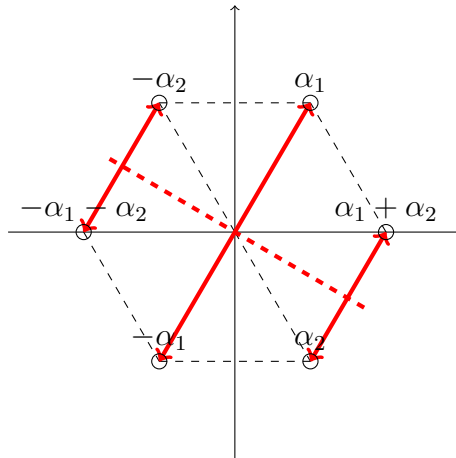
$$s_\alpha(v) = v - (v \cdot \alpha) \alpha. \quad (3.116)$$

In the root system of  $\mathfrak{sl}_3$ , we have

$$s_{\alpha_1}(\alpha_1) = -\alpha_1, \quad s_{\alpha_1}(\alpha_2) = \alpha_1 + \alpha_2, \quad s_{\alpha_1}(\alpha_1 + \alpha_2) = \alpha_2, \quad (3.117)$$

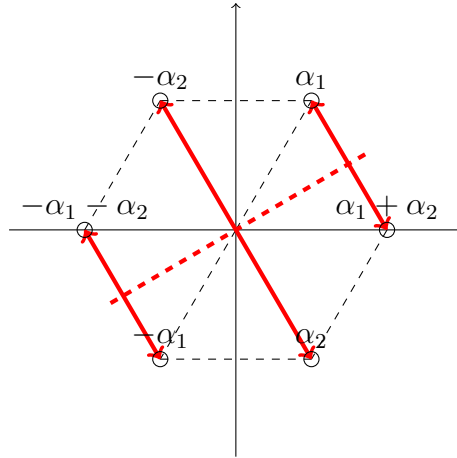
$$s_{\alpha_2}(\alpha_1) = \alpha_1 + \alpha_2, \quad s_{\alpha_2}(\alpha_2) = -\alpha_2, \quad s_{\alpha_2}(\alpha_1 + \alpha_2) = \alpha_1. \quad (3.118)$$

The Weyl reflection associated to  $\alpha_1$  can be graphically understood as



$$(3.119)$$

and similarly the Weyl reflection associated to  $\alpha_2$ ,



(3.120)

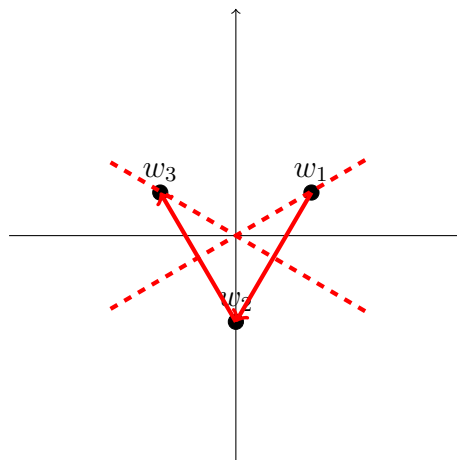
**Example 3.3.6** Note that in fact we have for the fundamental weight  $w_1 = (1, 0)$

$$w_1 = \frac{2\alpha_1 + \alpha_2}{3}, \tag{3.121}$$

thus

$$w_1 \cdot \alpha_1 = 1, \quad w_1 \cdot \alpha_2 = 0. \tag{3.122}$$

This shows why  $w_1$  is lowered only once by  $E_{-\alpha_1}$ , and that maps to  $w_2 = \frac{\alpha_2 - \alpha_1}{3}$ , which can further be mapped to  $w_3 = \frac{-\alpha_1 - 2\alpha_2}{3}$ . These weight can be plotted as the following.



(3.123)

### 3.4 Restrictions on root system and classification of Lie algebras\*

In the last section, we see that for any representation of a given Lie algebra, the weight must satisfy

$$\frac{2\alpha_i \cdot w}{|\alpha_i|^2} \in \mathbb{Z}. \tag{3.124}$$

In particular, if we consider the adjoint representation, as we have all roots as the weights of the representation, we arrive at the conclusion that

$$\frac{2\alpha_i \cdot \alpha_j}{|\alpha_i|^2} \in \mathbb{Z}, \quad \frac{2\alpha_j \cdot \alpha_i}{|\alpha_j|^2} \in \mathbb{Z}, \quad (3.125)$$

for any pair of  $(i, j)$ . Let us denote

$$\frac{2\alpha_i \cdot \alpha_j}{|\alpha_i|^2} = n_1, \quad \frac{2\alpha_j \cdot \alpha_i}{|\alpha_j|^2} = n_2, \quad (3.126)$$

then

$$\cos^2 \theta_{ij} = \frac{(\alpha_i \cdot \alpha_j)^2}{|\alpha_i|^2 |\alpha_j|^2} = \frac{n_1 n_2}{4}. \quad (3.127)$$

We already see that the constraint on the angle between two vectors,  $\alpha_i$  and  $\alpha_j$ , is very strict, and as  $0 \leq \cos^2 \theta_{ij} < 1$  (note that  $\cos \theta = \pm 1$  means  $\theta = 0^\circ$  or  $\theta = 180^\circ$ , and thus  $\alpha_i = \pm \alpha_j$ ), the only choices are  $n_1 n_2 = 0, 1, 2, 3$ . The list of all possible angles is as the following:

$n_1 n_2$	$\theta_{ij}$	
0	90°	
1	60° or 120°	(3.128)
2	45° or 135°	
3	30° or 150°	

As chosen so,  $\{\alpha_1, \dots, \alpha_r\}$  is a basis of the root and the weight space, but there exists some special basis of the root system so that

$$\alpha = \sum_{i=1}^r n_i \alpha_i, \quad (3.129)$$

for  $\forall \alpha$  and  $n_i \in \mathbb{Z}$ . Within the region of integer coefficients, there is a further special set of roots that cannot be expressed as a linear combination (in integer coefficients) of other roots, and we call them *simple roots*. Of course, if  $\alpha$  is a simple root,  $-\alpha$  is also a simple root, It is thus important to set the positivity of roots, and after doing so, the root system will be split into two parts, with positive roots satisfying

$$\alpha = \sum_{i=1}^r n_i \alpha_i, \quad \exists n_i \in \mathbb{N}, \quad (3.130)$$

and negative roots given by all  $-\alpha$ 's.

The set of (positive) simple roots then completely characterizes a Lie algebra. Mathematician Dynkin introduced a visual representation of the simple roots, called Dynkin diagram. First note that for any pair of simple roots,  $\alpha_1$  and  $\alpha_2$ ,  $\alpha_1 - \alpha_2$  is not a root, that means

$$E_{-\alpha_2} |E_{\alpha_1}\rangle = 0, \quad (3.131)$$

i.e.  $|E_{\alpha_1}\rangle$  is the lowest-weight state of the  $\mathfrak{sl}_2$  triplet of  $\alpha_2$ . The corresponding weight

$$\frac{2\alpha_2 \cdot \alpha_1}{|\alpha_2|^2} = -p, \quad (3.132)$$

is a negative integer. Therefore we see that the angle between any two simple roots must be larger than  $90^\circ$ . The only possibilities of the angle  $\theta_{ij}$  are  $90^\circ$ ,  $120^\circ$ ,  $135^\circ$  and  $150^\circ$ . We use a node,



$$(3.133)$$

to represent each simple root. According to the angle between two simple roots, we use the following graphic convention,



$$150^\circ \quad (3.134)$$



$$135^\circ \quad (3.135)$$



$$120^\circ \quad (3.136)$$

and when the angle is  $90^\circ$ , we do not draw any line connecting these nodes,



$$90^\circ \quad (3.137)$$

**Example 3.4.1** In the case of  $\mathfrak{sl}_3$ , as we have already seen that we can choose the simple roots to be  $\alpha_1 = (2, -1)$  and  $\alpha_2 = (-1, 2)$ , but this vector representation is not in the orthonormal basis. Instead it implies

$$\frac{2\alpha_1 \cdot \alpha_2}{|\alpha_1|^2} = \frac{2\alpha_1 \cdot \alpha_2}{|\alpha_2|^2} = -1, \quad (3.138)$$

and thus

$$\cos \theta_{12} = \frac{\alpha_1 \cdot \alpha_2}{|\alpha_1||\alpha_2|} = -\frac{1}{2}. \quad (3.139)$$

The Dynkin diagram of  $\mathfrak{sl}_3$  is thus given by



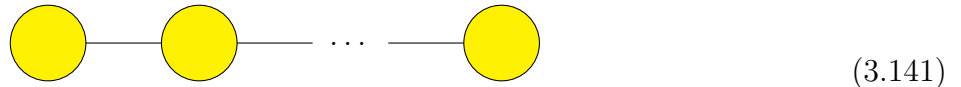
$$(3.140)$$

□

**Example 3.4.2** □

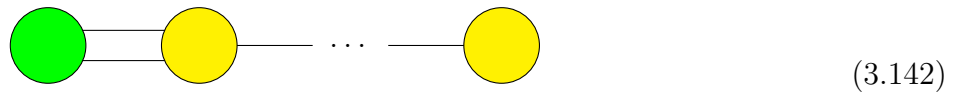
To classify the simple root systems, we further add a condition that in the set of all simple roots, we do not have two subsets that orthogonal to each other. For later convenience, we call such a set of simple roots as simply-connected (单连通) system. In this way, we can classify all Lie algebras (with finite-dimensional representations). Let us first describe the classification result before we sketch the proof of it. There are in total 7 types of Lie algebra, usually called  $A$ ,  $B$ ,  $C$ ,  $D$ ,  $E$ ,  $F$  and  $G$ -type.

The Dynkin diagram of the  $A$ -type Lie algebra is linear:



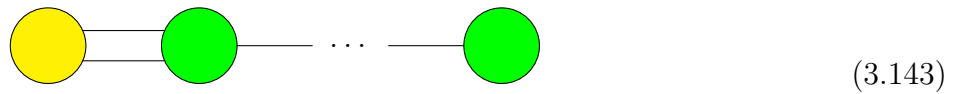
According to the number of nodes, say  $n$  nodes, we call the Lie algebra of type  $A_n$ . As we have already seen,  $n = 1, 2$  respectively corresponds to  $\mathfrak{sl}_2$  and  $\mathfrak{sl}_3$ . More generally, the Lie algebra of  $A_n$ -type is also called  $\mathfrak{sl}_{n+1}$ .

The Dynkin diagram of the  $B$ -type Lie algebra is given by



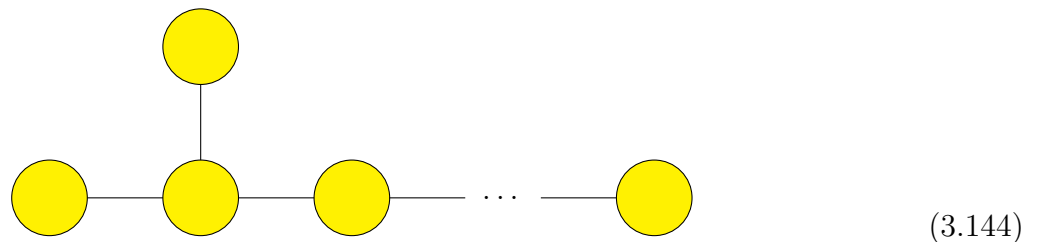
where the green node represents a simple root corresponding to a shorter (-norm) vector. Again they are denoted as  $B_n$  Lie algebra or  $\mathfrak{so}_{2n+1}$ .

The Dynkin diagram of the  $C$ -type Lie algebra is given by



and  $C_n$ -type is alternatively denoted as  $\mathfrak{sp}_{2n}$ .

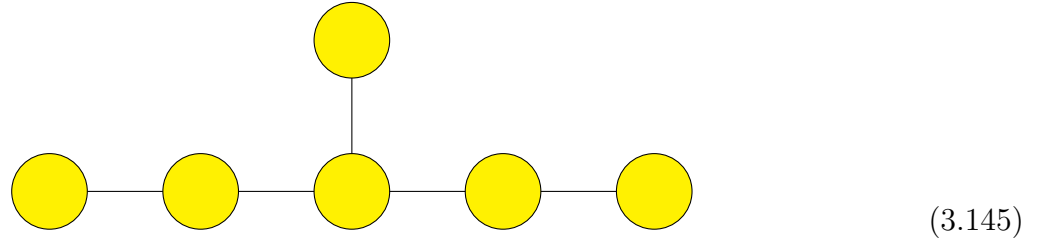
$D$ -type Lie algebra has a branch,



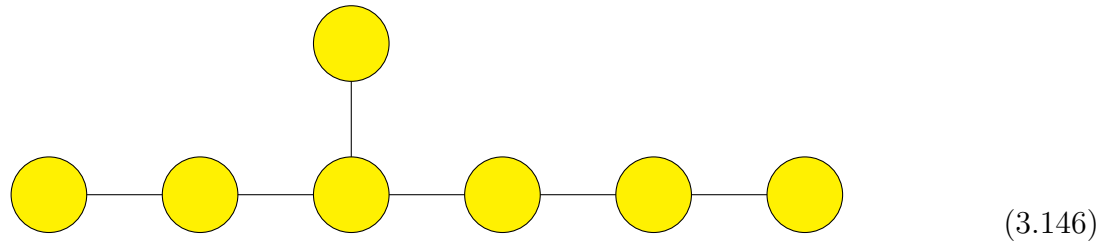
and  $D_n$ -type corresponds to  $\mathfrak{so}_{2n}$ .

$E$ ,  $F$ ,  $G$ -type Lie algebras are usually called exceptional Lie algebras (例外李代数). Unlike the  $ABCD$  siblings having infinite number of family members, there are only 5 exceptional Lie algebras in our classification:  $E_6$ ,  $E_7$ ,  $E_8$ ,  $F_4$  and  $G_2$ .

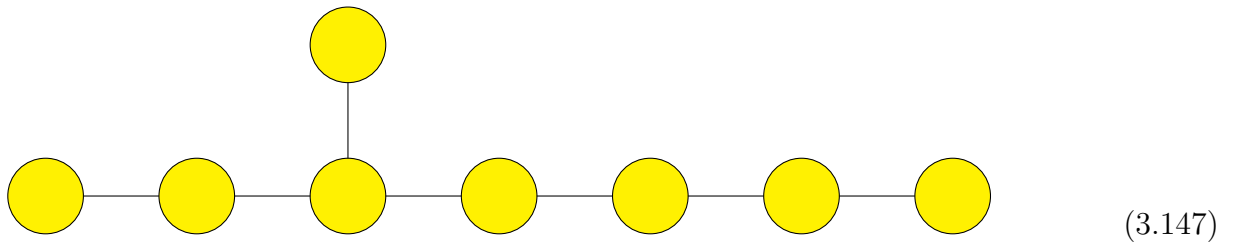
The Dynkin diagram of  $E_6$  looks like



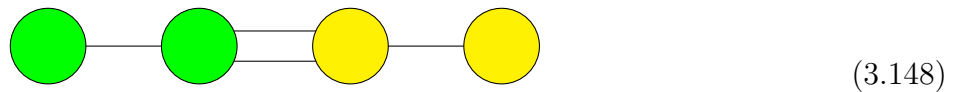
$E_7$  is given by



and the Dynkin diagram of  $E_8$  reads



$F_4$  and  $G_2$  involves double and triple lines respectively. Their Dynkin diagrams are respectively



and



The above is the full list of all Lie algebras. A useful terminology in the context of Lie algebra is simply-laced. It means the Dynkin diagram only contains single lines and no double or triple lines. One can see that simply-laced Lie algebras are of  $A$ ,  $D$  or  $E$ -type.

To prove the classification stated above, we first need several lemmas.

**Lemma 1** The sub-diagram (without any decoupled root) of a system with three vectors can only be



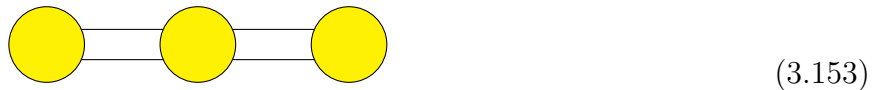
or



**Proof:** The sum of all angles between each two of three vectors,  $\theta_{12} + \theta_{23} + \theta_{31}$  is  $360^\circ$  if and only if three vectors lie on a single plane, i.e. linearly dependent. Therefore the following diagrams,



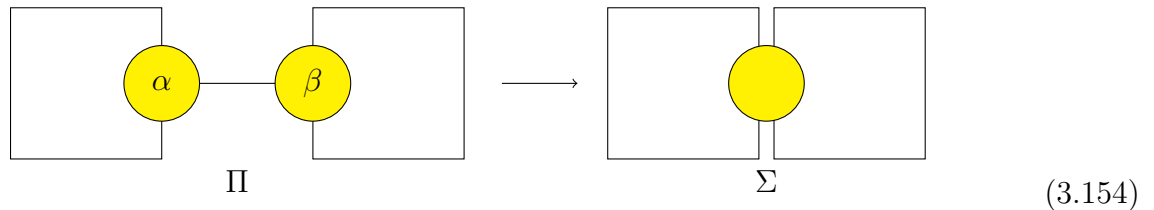
and



etc., are not allowed.

**Lemma 2** If there are two simple roots connected by a single line, then the new system  $\Sigma$  obtained by “shrinking” these two roots into one from the original system  $\Pi$  is also a simply-connected system.

Graphically, the new system  $\Sigma$  is generated from  $\Pi$  as



**Proof:** We remark that graphically, this lemma is almost trivial. To prove this claim mathematically, we need to formulate the “shrinking” operation algebraically. Following from lemma 1, we know that in any three simple roots in  $\Pi$ , there exist at least two root vectors orthogonal to each other. In particular, for  $\alpha$ ,  $\beta$  and another simple root  $\gamma$ , we have

$$\alpha \cdot \gamma = 0, \tag{3.155}$$

or

$$\beta \cdot \gamma = 0, \tag{3.156}$$

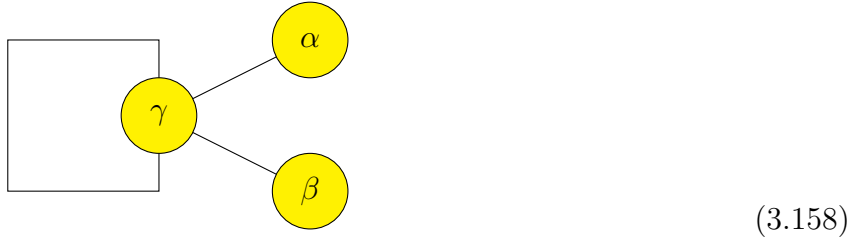
which implies

$$(\alpha + \beta) \cdot \gamma = \alpha \cdot \gamma, \quad \text{or} \quad (\alpha + \beta) \cdot \gamma = \beta \cdot \gamma. \quad (3.157)$$

We see that the shrunk root  $\alpha + \beta$  indeed inherits the positional relations with other simple roots in  $\Pi$ , and thus the new system  $\Sigma$  with this new root  $\alpha + \beta$  is also simply-connected.  $\square$

Using a similar method, we obtain the following lemma.

**Lemma 3** Given a simply-connected system,



(3.158)

then the new system



(3.159)

is also simply-connected.

**Proof:** Let us first confirm that  $\alpha + \beta$  indeed is connected with  $\gamma$  by double-line, i.e the angle between them are  $135^\circ$ . We have  $\alpha \cdot \beta = 0$ , and

$$\frac{\alpha \cdot \gamma}{|\alpha||\gamma|} = \frac{\beta \cdot \gamma}{|\beta||\gamma|} = -\frac{1}{2}. \quad (3.160)$$

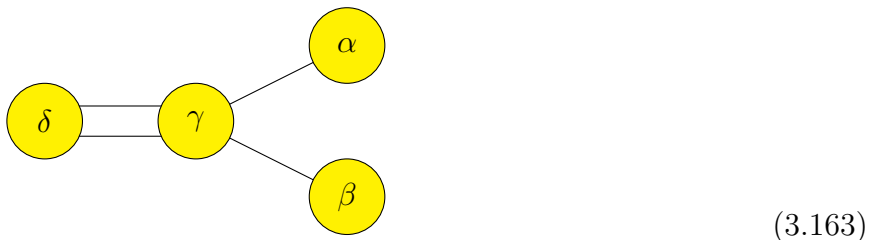
We also have

$$|\alpha + \beta| = \sqrt{|\alpha|^2 + |\beta|^2}. \quad (3.161)$$

Assuming both  $\alpha$  and  $\beta$  are normalized in the same way, i.e.  $|\alpha| = |\beta|$ , then

$$\frac{\alpha \cdot \gamma + \beta \cdot \gamma}{|\alpha + \beta||\gamma|} = -\frac{1}{2} \frac{|\alpha| + |\beta|}{\sqrt{2}|\alpha|} = -\frac{1}{\sqrt{2}}. \quad (3.162)$$

We can rule out the possibility of



(3.163)

as we can construct a system of three vectors as above to obtain

$$\begin{array}{c}
 \delta \text{ --- } \gamma \text{ --- } \alpha + \beta \\
 \hline
 \end{array}
 \tag{3.164}$$

which has already been ruled out by Lemma 1. Therefore all simple roots connected to  $\gamma$  in the black box of the unspecified subdiagram are connected by a single-line. Then we see how the new system is simply-connected.  $\square$

It follows immediately by using Lemma 3 twice that

$$\begin{array}{c}
 \bullet \text{ --- } \bullet \text{ --- } \bullet \\
 \diagdown \quad \diagup \\
 \bullet \text{ --- } \bullet \\
 \hline
 \end{array}
 \tag{3.165}$$

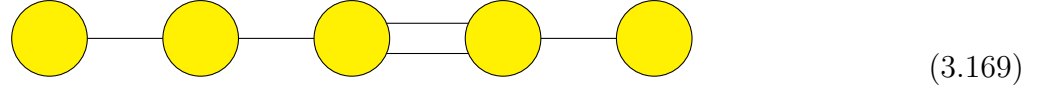
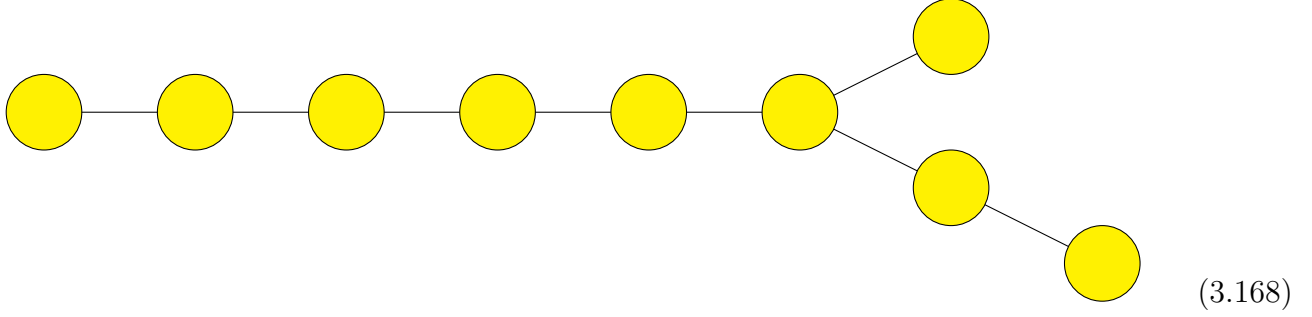
is not allowed.

**Proposition 4** Combining Lemma 2 and Lemma 3 we see that the Dynkin diagram can at most has one branch, which consists of only single-lines.

**Lemma 5** The following diagrams,

$$\begin{array}{c}
 \bullet \text{ --- } \bullet \text{ --- } \bullet \text{ --- } \bullet \text{ --- } \bullet \\
 \diagdown \quad \diagup \\
 \bullet \text{ --- } \bullet \text{ --- } \bullet \\
 \hline
 \end{array}
 \tag{3.166}$$

$$\begin{array}{c}
 \bullet \text{ --- } \bullet \text{ --- } \bullet \text{ --- } \bullet \text{ --- } \bullet \\
 \diagdown \quad \diagup \\
 \bullet \text{ --- } \bullet \text{ --- } \bullet \text{ --- } \bullet \text{ --- } \bullet \\
 \hline
 \end{array}
 \tag{3.167}$$

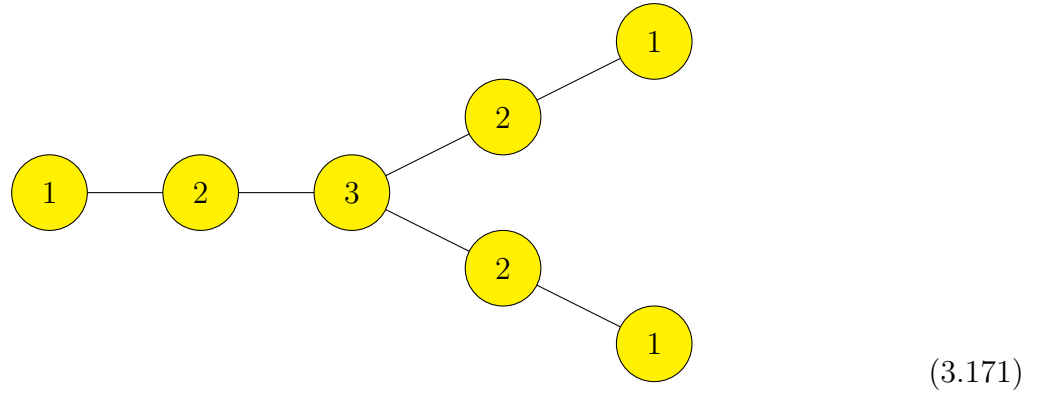


are not allowed as they do not pass the linear independency check.

**Proof:** We assign to each simple root a weight  $\mu_i$  and show that for a proper choice of the such weights, we have

$$\left( \sum_i \mu_i \alpha_i \right)^2 = 0. \quad (3.170)$$

For example, the weight choice for the first diagram is given by



As we have

$$\alpha_1 \cdot \alpha_2 = -1, \quad \alpha_1 \cdot \alpha_{3,4,5,6,7} = 0, \quad (3.172)$$

$$\alpha_2 \cdot \alpha_3 = -1, \quad \alpha_2 \cdot \alpha_{4,5,6,7} = 0, \quad (3.173)$$

$$\alpha_3 \cdot \alpha_4 = \alpha_3 \cdot \alpha_5 = -1, \quad \alpha_3 \cdot \alpha_{6,7} = 0, \quad (3.174)$$

$$\alpha_4 \cdot \alpha_6 = -1, \quad \alpha_5 \cdot \alpha_7 = -1, \quad (3.175)$$

$$\alpha_4 \cdot \alpha_7 = \alpha_5 \cdot \alpha_6 = \alpha_4 \cdot \alpha_5 = \alpha_6 \cdot \alpha_7 = 0, \quad (3.176)$$

then

$$\begin{aligned} \left( \sum_i \mu_i \alpha_i \right)^2 &= -2\mu_1\mu_2 - 2\mu_2\mu_3 - 2\mu_3\mu_4 - 2\mu_3\mu_5 - 2\mu_4\mu_6 - 2\mu_5\mu_7 + 2 \sum_i \mu_i^2 \\ &= -4 - 12 - 12 - 12 - 4 - 4 + 2(3 + 12 + 9) = 0. \end{aligned} \quad (3.177)$$

□

This completes the classification of all possible Dynkin diagrams.

### 3.5 Characters of Lie algebra and symmetric polynomials

In the context of Lie groups, we can again define the characters as

$$\chi_R(g) := \text{tr}(R(g)), \quad (3.178)$$

for each element  $g$  in the Lie group  $G$ . As the Lie group is continuous, there are infinitely many elements in the group. We will focus on the generators of the form  $g = \exp(X)$ , where  $X \in \mathfrak{g}$  is an element in the corresponding Lie algebra  $\mathfrak{g}$ . In this case, we shall evaluate the character in the corresponding representation of the Lie algebra  $\mathfrak{g}$ ,

$$\chi_\rho(X) := \chi_R(\exp(X)) = \text{tr}(\exp(\rho(X))). \quad (3.179)$$

The trace of the exponential of generic matrices is very difficult to evaluate, while for diagonal matrices, this is very simple. Fortunately, the Cartan subalgebra  $\mathfrak{h}$  is always diagonal for all the states in a highest weight representation. Then the character restricted to the Cartan part is given by

$$\chi_\rho(H) = \sum_{\lambda: \text{weight}} m_\lambda e^{\lambda(H)}, \quad (3.180)$$

for  $\forall H \in \mathfrak{h}$ .

**Example 3.5.1**  $SU(2)$ . The Cartan subalgebra of  $\mathfrak{su}(2)$ , or equivalently  $\mathfrak{sl}_2$ , is one-dimensional. So we consider the character associated to the Cartan generator  $H = \theta e_1$ , e.g. for the fundamental representation (with weight  $w = 1$ ),

$$\chi_{w=1}(H) = e^\theta + e^{-\theta} = t + t^{-1}, \quad (3.181)$$

where we further defined  $t := e^\theta$ . More generally, in the representation of weight  $w = m$ , the character reads

$$\begin{aligned} \chi_{w=m}(H) &= e^{m\theta} + e^{(m-2)\theta} + \dots + e^{-(m-2)\theta} + e^{-m\theta} = e^{m\theta} \frac{1 - e^{-2(m+1)\theta}}{1 - e^{-2\theta}} \\ &= \frac{e^{(m+1)\theta} - e^{-(m+1)\theta}}{e^\theta - e^{-\theta}} = \frac{\sinh((m+1)\theta)}{\sinh(\theta)}. \end{aligned} \quad (3.182)$$

**Example 3.5.2** Fundamental representation of  $SU(3)$  or  $\mathfrak{sl}_3$ . As

### 3.6 Noether's theorem

classical version

quantum version

### 3.7 Composition of representations

### 3.8 Invariants in Lie algebras

### 3.9 3D harmonic oscillator revisited

### 3.10 From Lie algebra back to Lie group

## 4 Affine Lie algebras

## A Spherical Harmonics

The spherical harmonics  $Y$  are defined as the eigenfunctions of

$$\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}. \quad (\text{A.1})$$

We can separate the variables to set

$$Y(\theta, \varphi) = \Theta(\theta)\Phi(\varphi), \quad (\text{A.2})$$

satisfying

$$\frac{d^2 \Phi}{d\varphi^2} = -m^2 \Phi, \quad (\text{A.3})$$

$$\sin \theta \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) = m^2 \Theta + \lambda \sin^2 \theta \Theta, \quad (\text{A.4})$$

where  $\lambda$  represents the eigenvalue of  $Y$ . For  $\Phi$  to be a (continuous) periodic function,  $m$  has to be an integer,  $m \in \mathbb{Z}$ , and  $\Phi$  is solved to be  $\Phi = e^{im\varphi}$ . Since  $\theta = 0, \pi$  correspond to two poles of the sphere, which are two points with no speciality, we need to impose a continuous (regular) condition on  $\Theta$  at these two points. That is to say, at  $\theta = 0$ ,

$$\Theta = \theta^\ell \left( c + a\theta + \frac{b}{2}\theta^2 + \frac{e}{3}\theta^3 + \frac{g}{4}\theta^4 + \frac{h}{5}\theta^5 + \dots \right), \quad \ell \geq 0 \quad (\text{A.5})$$

together with

$$\sin \theta = \theta - \frac{\theta^3}{6} + \mathcal{O}(\theta^5), \quad (\text{A.6})$$

and

$$\sin^2 \theta = \theta^2 - \frac{\theta^4}{3} + \mathcal{O}(\theta^6) \quad (\text{A.7})$$

we have

$$\begin{aligned} & (m^2 - \ell^2)c + (m^2 - (\ell + 1)^2) a\theta + \left( \frac{b}{2}(m^2 - (\ell + 2)^2) + \left( \lambda + \frac{\ell(\ell + 1)}{3} \right) c \right) \theta^2 \\ & \quad + \left( \frac{e}{3}(m^2 - (\ell + 3)^2) + \left( \frac{(\ell + 1)(\ell + 2)}{3} + \lambda \right) a \right) \theta^3 \\ & + \left( \left( \frac{1}{2}\lambda + \frac{1}{6}(\ell + 2)(\ell + 3) \right) b + \left( \frac{m^2}{4} - \frac{1}{4}(\ell + 4)^2 \right) g - \left( \frac{\lambda}{3} + \frac{2}{45}\ell(\ell + 2) \right) c \right) \theta^4 + \mathcal{O}(\theta^5) = 0 \end{aligned} \quad (\text{A.8})$$

We have  $\ell = |m|$ , no constraint on  $c$ , but  $a = e = h = 0$ ,

$$b = \frac{\lambda + \frac{|m|(|m|+1)}{3}}{2|m| + 2} c, \quad (\text{A.9})$$

etc. One can solve the series expansion and find the condition for regularity at  $\theta = \pi$ . A more powerful analysis can be done by changing variable to  $x = \cos \theta$ , and then we have

$$(1 - x^2) \frac{d}{dx} \left( (1 - x^2) \frac{d}{dx} \Theta \right) = m^2 \Theta + \lambda(1 - x^2) \Theta, \quad (\text{A.10})$$

which is known as the associated Legendre equation. Let us solve the equation with the series expansion<sup>4</sup>

$$\Theta = x^\ell \sum_{n=0}^{\infty} c_n x^n, \quad (\text{A.11})$$

and the equation is translated to

$$(\lambda + (n + \ell - 2)(n + \ell - 1)) c_{n-2} - (2(n + \ell)^2 + m^2 + \lambda) c_n + (n + \ell + 1)(n + \ell + 2) c_{n+2} = 0. \quad (\text{A.12})$$

One finds

$$\lim_{n \rightarrow 1} \frac{c_{n+2}}{c_n} = 1, \quad (\text{A.13})$$

---

<sup>4</sup>Note that the uniqueness of Taylor expansion ensures the convergence of the series expansion solution on entire  $x \in [-1, 1]$ .

which implies the series is probably not convergent at  $|x| = 1$ , and the only way to have a solution regular at  $x = \pm 1$  is to terminate the series at a finite  $n$ , that is to say for some  $\exists n \in \mathbb{N}$ ,

$$\lambda = -(n + \ell)(n + \ell + 1). \quad (\text{A.14})$$

Note that the equation for  $c_0 \neq 0$  implies

$$\ell = 0, \quad \text{or} \quad \ell = 1, \quad (\text{A.15})$$

so  $\lambda$  is only parameterized by a positive integer  $L$ ,

$$\lambda = -L(L + 1). \quad (\text{A.16})$$

The solution to the associated Legendre equation is a polynomial and is called an associated Legendre polynomial. When  $m = 0$ , the equation is called the Legendre equation, and the solution is the Legendre polynomial. Let us denote the Legendre polynomial labeled by  $L$  as  $P_L(x)$ . We define a new polynomial through  $P_L(x)$  as

$$P_L^m(x) := (-1)^m (1 - x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_L(x). \quad (\text{A.17})$$

Let us act the operator

$$D := (1 - x^2) \frac{d}{dx} \left( (1 - x^2) \frac{d}{dx} \right) = (1 - x^2)^2 \frac{d^2}{dx^2} - 2x(1 - x^2) \frac{d}{dx}, \quad (\text{A.18})$$

on  $P_L^1(x)$ ,

$$DP_L^1(x) = -(1 - x^2)^{\frac{5}{2}} \frac{d^3}{dx^3} P_L(x) + 4x(1 - x^2)^{\frac{3}{2}} \frac{d^2}{dx^2} P_L(x) + (1 - 2x^2)(1 - x^2)^{\frac{1}{2}} \frac{d}{dx} P_L(x), \quad (\text{A.19})$$

while

$$\frac{d}{dx} D = (1 - x^2)^2 \frac{d^3}{dx^3} - 6x(1 - x^2) \frac{d^2}{dx^2} - 2(1 - 3x^2) \frac{d}{dx}, \quad (\text{A.20})$$

and that is to say

$$\begin{aligned} DP_L^1(x) &= -(1 - x^2)^{\frac{1}{2}} \frac{d}{dx} DP_L(x) - 2x(1 - x^2)^{\frac{3}{2}} \frac{d^2}{dx^2} P_L(x) - (1 - x^2)^{\frac{1}{2}} (1 - 4x^2) \frac{d}{dx} P_L(x) \\ &= -(1 - x^2)^{\frac{1}{2}} \frac{d}{dx} DP_L(x) - \frac{2x}{(1 - x^2)^{\frac{1}{2}}} DP_L(x) - (1 - x^2)^{\frac{1}{2}} \frac{d}{dx} P_L(x) \\ &= (1 - x^2)^{\frac{1}{2}} L(L + 1) \frac{d}{dx} (1 - x^2) P_L(x) + 2xL(L + 1)(1 - x^2)^{\frac{1}{2}} P_L(x) - (1 - x^2)^{\frac{1}{2}} \frac{d}{dx} P_L(x) \\ &= (1 - x^2)^{\frac{3}{2}} L(L + 1) \frac{d}{dx} P_L(x) - (1 - x^2)^{\frac{1}{2}} \frac{d}{dx} P_L(x) \\ &= (1 - x^2)(-L(L + 1)) P_L^1(x) + P_L^1, \quad (\text{A.21}) \end{aligned}$$

which implies  $P_L^1$  is a polynomial solution to the associated Legendre equation for  $m = 1$  and  $\lambda = -L(L+1)$ . Now we assume  $P_L^m$  satisfies the associated Legendre equation for  $m$  and  $\lambda = -L(L+1)$ , and from

$$P_L^{m+1}(x) = -(1-x^2)^{\frac{1}{2}} \frac{d}{dx} P_L^m(x) - mx(1-x^2)^{-\frac{1}{2}} P_L^m(x), \quad (\text{A.22})$$

we have

$$\begin{aligned} DP_L^{m+1}(x) &= (1-x^2)^{\frac{1}{2}} \frac{d}{dx} P_L^m(x) + 2x(1-x^2)^{\frac{3}{2}} \frac{d^2}{dx^2} P_L^m(x) - (1-x^2)^{\frac{5}{2}} \frac{d^3}{dx^3} P_L^m(x) \\ &\quad - 3mx(1-x^2)^{-\frac{1}{2}} P_L^m(x) - 2m(1-x^2)^{\frac{1}{2}} \frac{d}{dx} P_L^m(x) - mx(1-x^2)^{\frac{3}{2}} \frac{d^2}{dx^2} P_L^m(x) \\ &\quad - 2x^2(1-x^2)^{\frac{1}{2}} \frac{d}{dx} P_L^m(x) + 2x(1-x^2)^{\frac{3}{2}} \frac{d^2}{dx^2} P_L^m(x) \\ &\quad + 2mx(1-x^2)^{-\frac{1}{2}} P_L^m(x) + 2mx^2(1-x^2)^{\frac{1}{2}} \frac{d}{dx} P_L^m(x) \\ &= -(1-x^2)^{\frac{5}{2}} \frac{d^3}{dx^3} P_L^m(x) + 4x(1-x^2)^{\frac{3}{2}} \frac{d^2}{dx^2} P_L^m(x) + (1-2x^2)(1-x^2)^{\frac{1}{2}} \frac{d}{dx} P_L^m(x) \\ &\quad - mx(1-x^2)^{\frac{3}{2}} \frac{d^2}{dx^2} P_L^m(x) + m(x^2-2)(1-x^2)^{\frac{1}{2}} \frac{d}{dx} P_L^m(x) - mx(1-x^2)^{-\frac{1}{2}} P_L^m(x) \\ &= -(1-x^2)^{\frac{1}{2}} \frac{d}{dx} DP_L^m(x) - \frac{2x}{(1-x^2)^{\frac{1}{2}}} DP_L^m(x) - (1-x^2)^{\frac{1}{2}} \frac{d}{dx} P_L^m(x) \\ &\quad - \frac{mx}{(1-x^2)^{\frac{1}{2}}} DP_L^m(x) - 2m(1-x^2)^{\frac{1}{2}} \frac{d}{dx} P_L^m(x) - mx(1-x^2)^{-\frac{1}{2}} P_L^m(x). \end{aligned} \quad (\text{A.23})$$

Now we use the assumed property of  $P_L^m$ , i.e.

$$DP_L^m(x) = m^2 P_L^m(x) - L(L+1)(1-x^2) P_L^m(x), \quad (\text{A.24})$$

to obtain

$$\begin{aligned} DP_L^{m+1}(x) &= -m^2(1-x^2)^{\frac{1}{2}} \frac{d}{dx} P_L^m(x) + L(L+1)(1-x^2)^{\frac{3}{2}} \frac{d}{dx} P_L^m(x) - 2xL(L+1)(1-x^2)^{\frac{1}{2}} P_L^m(x) \\ &\quad - \frac{2m^2x}{(1-x^2)^{\frac{1}{2}}} P_L^m(x) + 2xL(L+1)(1-x^2)^{\frac{1}{2}} P_L^m(x) - (1-x^2)^{\frac{1}{2}} \frac{d}{dx} P_L^m(x) \\ &\quad - \frac{m^3x}{(1-x^2)^{\frac{1}{2}}} P_L^m(x) + mxL(L+1)(1-x^2)^{\frac{1}{2}} P_L^m(x) - 2m(1-x^2)^{\frac{1}{2}} \frac{d}{dx} P_L^m(x) \\ &\quad - mx(1-x^2)^{-\frac{1}{2}} P_L^m(x) \\ &= m^2 P_L^{m+1}(x) - L(L+1)(1-x^2) P_L^{m+1}(x) - \frac{2m^2x}{(1-x^2)^{\frac{1}{2}}} P_L^m(x) + \frac{2m^2x}{(1-x^2)^{\frac{1}{2}}} P_L^m(x) \\ &\quad + P_L^{m+1}(x) + \frac{mx}{(1-x^2)^{\frac{1}{2}}} P_L^m(x) + 2m P_L^{m+1}(x) - mx(1-x^2)^{-\frac{1}{2}} P_L^m(x) \\ &= (m+1)^2 P_L^{m+1}(x) - L(L+1)(1-x^2) P_L^{m+1}(x). \end{aligned} \quad (\text{A.25})$$

By induction, we see that  $P_L^{m+1}$  provides us a solution to the associated Legendre equation.

What is left to us becomes to solve  $P_L(x)$  itself. The answer is known as

$$P_L(x) = \frac{1}{2^L} \sum_{k=0}^{\lfloor \frac{L}{2} \rfloor} (-1)^k {}_L C_k {}_{2L-2k} C_L x^{L-2k}, \quad (\text{A.26})$$

where  ${}_n C_m$  denotes the binomial coefficient (for simplicity). As we know from the recursion relation that the series terminates at  $n = L$ , where  $n$  is the power of  $x^n$  in the series, which first of all is consistent with the above answer. We have

$$-\frac{{}_L C_k {}_{2L-2k} C_L}{{}_L C_{k+1} {}_{2L-2k-2} C_L} = \frac{(k+1)(2L-2k)(2L-2k-1)}{(L-k)(L-2k)(L-2k-1)} = \frac{c_{L-2k}}{c_{L-2k-2}}, \quad (\text{A.27})$$

and therefore

$$\frac{c_n}{c_{n-2}} = -\frac{(L-n+2)(L+n-1)}{n(n+1)}. \quad (\text{A.28})$$

It is then not hard to confirm the above recursive expression satisfies (A.12) for  $m = 0$ .

Let us list some concrete expressions for the Legendre polynomials

$$\begin{aligned} P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \quad P_3(x) = \frac{1}{2}(5x^3 - 3x), \\ P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3). \end{aligned} \quad (\text{A.29})$$

And for the associated Legendre polynomials, we have

$$\begin{aligned} P_2^2(x) = -3x^2 + 3, \quad P_2^3(x) = 0, \quad P_2^1(x) = -3x(1-x^2)^{\frac{1}{2}}, \\ P_3^2(x) = 15x(1-x^2), \quad P_3^3(x) = -15(1-x^2)^{\frac{3}{2}}. \end{aligned} \quad (\text{A.30})$$

In summary, we have the spherical harmonic function given by

$$Y_L^m(\theta, \varphi) = P_L^{|m|}(\cos \theta) e^{im\varphi}. \quad (\text{A.31})$$

We remark that the associated Legendre equation only depends on  $m$  through  $m^2$ , so we can extend the analysis to negative  $m$  in the same way, and we see that  $|m|$  is bounded by  $L$ .

**Remark:** From the explicit expression of Legendre polynomials, we see that no matter around which point on  $x \in [-1, 1]$  we expand the solution of the Legendre equation, it should truncate at a finite order. Let us analyze the behavior of the associated Legendre differential equation near  $x = 1$ , and we further set  $y = 1 - x$ ,

$$y(2-y) \frac{d}{dx} \left( y(2-y) \frac{d}{dx} \Theta \right) = m^2 \Theta + \lambda y(2-y) \Theta. \quad (\text{A.32})$$

Let us expand

$$\Theta = y^\ell \sum_{n=0}^{\infty} c_n y^n, \quad (\text{A.33})$$

and then the coefficients satisfy

$$((n + \ell)^2 + (n + \ell) + \lambda) c_{n-2} - (4(n + \ell)^2 + 2(n + \ell) + 2\lambda) c_{n-1} + (4(n + \ell)^2 - m^2) c_n = 0. \quad (\text{A.34})$$

For  $n = 0$ , we simply have  $\ell^2 = \frac{m^2}{4}$ , and if the series expansion truncates at certain  $\exists n$ , then we find

$$\lambda = -(n + \ell)(n + \ell + 1). \quad (\text{A.35})$$

For even  $m$ , this constraint is no problem, and indeed agrees with the result from our expansion around  $x = 0$ . However, when  $m$  is odd,  $\lambda$  is no longer an integer, and it is inconsistent with the truncation condition of the expansion around  $x = 0$ . That is to say, the truncation does not happen for  $m$  odd. The series expansion in fact converges in a more delicate manner. Indeed we can see around  $x = 1$ , the series starts from the behavior  $(1 - x)^{\frac{m}{2}}$ , which cannot be expanded in finite terms of  $x^n$  for  $m$  odd. We notice that it is still perfect to work with  $m = 0$  at the very beginning, and impose the truncation condition to solve the Legendre polynomial, and then use the recursion relation (A.22) to generate the associated ones.

## B Schur's Lemma and its Consequences

## C Symmetric Polynomials

Elementary symmetric polynomial:

$$e_r(\{x_i\}_{i=1}^N) := \sum_{i_1 < i_2 < \dots < i_r} x_{i_1} x_{i_2} \dots x_{i_r}. \quad (\text{C.1})$$

Power sum:

$$p_r(\{x_i\}_{i=1}^N) := \sum_{i=1}^N x_i^r. \quad (\text{C.2})$$

For a partition  $\lambda = (\lambda_1, \lambda_2, \dots)$ , we define

$$p_\lambda := \prod_{i=1}^{|\lambda|} p_{\lambda_i}. \quad (\text{C.3})$$

Schur polynomial: for a partition  $\lambda$ ,

$$s_\lambda(\{x_i\}_{i=1}^N) := \det(x_i^{\lambda_j + N - j}) / \det(x_i^{N - j}). \quad (\text{C.4})$$

By definition, we always have  $s_\emptyset = 1$  for any  $N$ .

**Example:**  $N = 2$

$$s_{(1)}(x_1, x_2) = x_1 + x_2, \quad s_{(2)}(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2, \quad s_{(1,1)}(x_1, x_2) = x_1x_2. \quad (\text{C.5})$$

$$p_{(1,1)} = (x_1 + x_2)^2 = x_1^2 + 2x_1x_2 + x_2^2, \quad p_{(2)} = x_1^2 + x_2^2. \quad (\text{C.6})$$

$$p_{(1,1)} = s_{(2)} + s_{(1,1)}, \quad p_{(2)} = s_{(2)} - s_{(1,1)}. \quad (\text{C.7})$$

It is known that

$$p_\lambda(x) = \sum_{\mu} \chi_\lambda^\mu s_\mu(x), \quad (\text{C.8})$$

where  $\chi_\lambda^\mu$  coincides with the character of symmetric group  $\mathfrak{S}_N$  in the representation  $\mu$  and the conjugacy class  $\lambda$ .

From the above, we found that

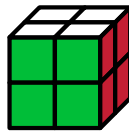
$$\chi_{(2)}^{(2)} = 1, \quad \chi_{(1,1)}^{(2)} = 1, \quad \chi_{(2)}^{(1,1)} = 1, \quad \chi_{(1,1)}^{(1,1)} = -1. \quad (\text{C.9})$$

$$\chi_{(3)}^{(3)} = 1, \quad \chi_{(3)}^{(2,1)} = -1, \quad \chi_{(3)}^{(1,1,1)} = 1, \quad \chi_{(2,1)}^{(3)} = 1, \quad (\text{C.10})$$

$$\chi_{(2,1)}^{(2,1)} = 0, \quad \chi_{(2,1)}^{(1,1,1)} = -1, \quad \chi_{(1,1,1)}^{(3)} = 1, \quad \chi_{(1,1,1)}^{(2,1)} = 2, \quad (\text{C.11})$$

$$\chi_{(1,1,1)}^{(1,1,1)} = 1. \quad (\text{C.12})$$

## D Two-Rubik's Cube



(D.1)

## E 期末报告课题集

请选择任意题数（每节课结束后都可以提交。考虑到老师的批改负担，在期中、期末这样提交较为集中的时期，至多一次提交 2 题）进行解答并提交。题目边括号内为分值。

### 1. 三角形分类及其对称性 (15')

请回忆小学或初中老师教过的不同种类的三角形的分类（如等腰三角形，等边三角形，直角三角形等），从对称性的角度重新对这些分类进行考察，并简述这些对称性对应的群。

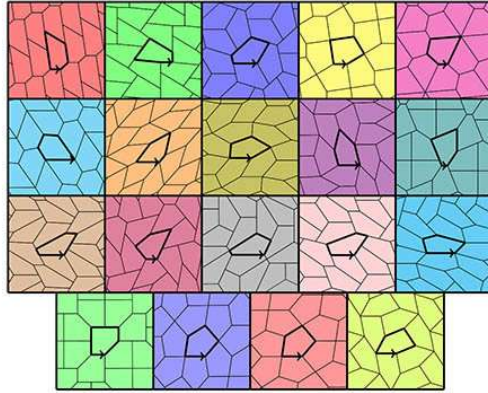


Figure 18: Examples of pentagon tiling.

## 2. 完美五边形 (50')

设计可以用来铺地的五边形瓷砖 (Pentagon tiling) 一直是数学上的一大未解难题。迄今为止人类共发现 15 种这样的五边形 (参照图18)。在一些国内媒体的报道中, 它们被命名为“完美五边形”。但是看上去更加完美的正五边形并不在其中。) 2017 年法国数学家 Rao 宣称证明了不存在第 16 种此类五边形, 但迄今尚未完成同行评审。请自行查找资料考察如何寻找这样的五边形, 它们又与对称性或者群论有何关联? (若能找到第 16 种完美五边形, 则本门课直接满分。)

## 3. 氦原子基态能量 (35)

请通过某些近似计算手法求解氦原子的基态能量。已知实验实测值为  $E_0 \simeq -78.62\text{eV}$ 。请计算至两位有效数字与实验值相符, 并说明你使用的方法与实验值在误差范围内一致。

提示: 微扰法为相对容易并且精度较好的近似法。我们可以把哈密顿量分成可以严格求解的部分  $H_0$  及两个电子之间的相互作用:

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{int}, \quad (\text{E.1})$$

具体地 (氦原子时  $Z = 2$ ),

$$\mathcal{H}_0 = \frac{p_1^2}{2\mu_e} + \frac{p_2^2}{2\mu_e} - \frac{Ze^2}{4\pi\epsilon_0|\vec{r}_1|} - \frac{Ze^2}{4\pi\epsilon_0|\vec{r}_2|}, \quad (\text{E.2})$$

$$\mathcal{H}_{int} = \frac{e^2}{4\pi\epsilon_0|\vec{r}_1 - \vec{r}_2|}. \quad (\text{E.3})$$

我们可以将  $\mathcal{H}_{int}$  作为微扰项进行微扰展开计算。

## 4. SO(3) 的 5 维表示 (25)

请构筑 SO(3) 的 5 维表示, 即用  $5 \times 5$  矩阵表示出 SO(3) 的所有生成元, 并检验他们应该满足的代数关系。同时求解对应的李代数  $\mathfrak{so}(3)$  的五维表示。

## 5. $D_{4d}$ 群与 $S_8$ 群 (25)

请说明  $D_{4d}$  与  $S_8$  不存在的原因。

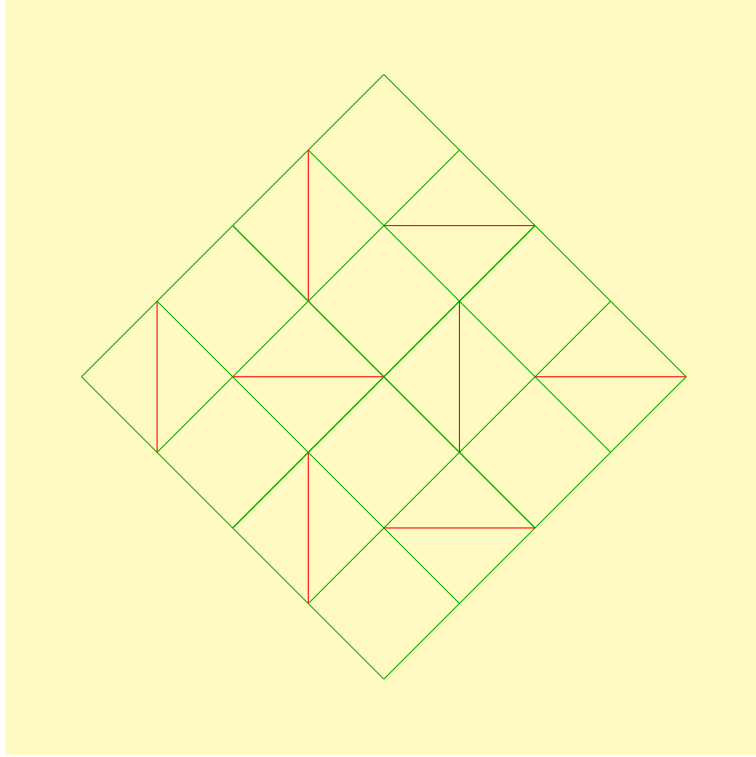


Figure 19: Shastry-Sutherland lattice.

6. Shastry-Sutherland 格子 (25)

请分析 Shastry-Sutherland 格子 (图19) 的对称性, 并写出一个满足该对称性的哈密顿量。

7.  $B$ -型李代数 (25)

请论证  $B_3$  型李代数与  $\mathfrak{so}_7$  等价。

10. 群论的用处 (25)

说说群论与你研究生的研究课题 (或本科毕业设计) 的关系。

## References

- [1] G. H. Cliff and S. K. Sehgal, "On groups having the same character tables," *Communications in Algebra* **9** no. 6, (1981) 627–640, <https://doi.org/10.1080/00927878108822606>.  
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- [2] M. Aschbacher and S. Smith, *THE CLASSIFICATION OF QUASITHIN GROUPS I. STRUCTURE OF STRONGLY QUASITHIN  $K$ -GROUPS*. American Mathematical Society, 2004.